Lie groups and algebras

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Abstract
Lie groups are the mathematical basis for gauge interactions in physics, and gauge symmetries are together with the Lorentz symmetry the fundamental symmetries in our physical universe.
This paper is an introduction to Lie groups and algebras, based on original papers, mathematical and physical literature, and own calculations.
The Flowcharts chapter sums up the important aspects in a graphical form.
In chap. 1, there is a concise summation of the basics of Lie algebras.
In chap. 2, 3, 4 we present resp. Lie groups and manifolds, proper Lie algebras, and fundamental theorems .
In every chapter, there is a concise description, followed by mathematical definitions, theorems and lemmas. This second part is a precise mathematical formulation of the respective subject, and can be skipped by non-mathematicians.
Chap. 5 deals with the important subject of root system, Cartan subalgebra, and Dynkin diagram.
Chap. 6 describes in detail the four classical types and exceptional Lie algebras.
Chap. 7 gives a characterization of low-dimensional Lie algebras for dim=2, 3, 4 .
In chap 8, we describe briefly the spin representations.
In chap. 9 and 10 we give a short introduction to the related subject of Clifford algebras and spin representations

Contents

0 Flowcharts

Structure of Lie algebras

field K=R, C

↓

Lie algebra *g* over K

generators $X_1,...,X_n$, commutator $\left[X_i, X_j\right] = c_{ij}^k X_k$

elements $X \in g$, $X = \sum_i \alpha_i \prod_j X_j^{n_j}$

g=ring with operations + *

additive commutative group

multiplicative non-commutative half-group

representation: kxk matrices over K

g=vector space over K, basis { $X_1,...,X_n$ }

$g = \langle X_1, ..., X_n \rangle$, dim(g)=n

↓

Cartan subalgebra *h*

h maximal commuting subalgebra of g

$h = \langle H_1, ..., H_k \rangle$, rank(g)=dim(h)=k

non-commuting basis $\left\{ \tilde{X}_1, ..., \tilde{X}_{n-k} \right\}$

commutators $\left[h_i, \tilde{X}_j\right] = \sum_k \alpha_{ijk} \tilde{X}_k$

roots $\vec{\alpha}_{ij} = \left(\alpha_{ijk}\right)_k$

↓

Killing matrix κ

$\kappa_{ij} = c_{ib}^a c_{ja}^b$

eigenvalues $\lambda_i\left(\kappa\right) \neq 0 \rightarrow$ g semisimple $g = g_1 \oplus .... \oplus g_m$

$\exists h \subset g$, h Cartan subalgebra, $h \neq 0$

eigenvalues $\lambda_i\left(\kappa\right) = 0 \rightarrow$ g solvable $g \supset \left[g, g\right] \supset \left[g, \left[g, g\right]\right] \supset ... \supset 0$

eigenvalues $\lambda_i\left(\kappa\right) < 0 \rightarrow$ g compact (bounded)

Roots in ei basis

Cartan subalgebra *h*

basis $E^1_{jk} = E_{jk} - E_{k+n,j+n}$

Cartan subalgebra $h = \langle E_{ii} \rangle$, remaining subspace $h_\perp = \langle E^1_{ij} \rangle \;\; i \neq j$

ansatz $H = \sum_{i=1}^{n} \lambda_i E_{ii}$ for $H \in h$

Commutators

simple roots $e_1 = E^1_{12}$, $e_2 = E^1_{23}$, ..., $e_n = f\left(E_{ij}\right)$

with the commutators

$r_1 = [H, e_1] = (\lambda_1 - \lambda_2) e_1$

$r_2 = [H, e_2] = (\lambda_1 - \lambda_2) e_2$

...

$r_n = [H, e_n] = (c_{n-1}\lambda_{n-1} + c_n \lambda_n) e_n$

for su(n+1) $r_n = (\lambda_n - \lambda_{n+1}) E_{n,n+1}$

for so(2n) $r_n = (\lambda_{n-1} + \lambda_n)(E_{2n-1,n} - E_{2n,n-1})$

for so(2n+1) $r_n = \lambda_n (E_{n,2n+1} - E_{2n+1,2n})$

for sp(2n) $r_n = 2\lambda_n (E_{n,2n} - E_{2n,n})$

Roots

simple roots

$\alpha_1 = (\lambda_1 - \lambda_2)$

$\alpha_2 = (\lambda_2 - \lambda_3)$

...

$\alpha_n = (c_{n-1}\lambda_{n-1} + c_n \lambda_n)$

all roots $\pm\alpha_i \pm \alpha_j$

Types of Lie algebras

Lie algebra *g* over K

generators $X_1,...,X_n$, commutator $\left[X_i, X_j\right] = c_{ij}^k X_k$

elements $X \in g$, $X = \sum_i \alpha_i X_i$

g=ring with operations + *

g=vector space over K, basis { $X_1,...,X_n$ }

$g = \langle X_1, ..., X_n \rangle$, dim(g)=n

su(n+1)

$\{x \in gl(n+1, C), \overline{x}^T = -x, tr(x)$

dim=$(n+1)^2-1$, rank=n

generators Ti , i=1...$(n+1)^2-1$

commutator relation

$[T_a, T_b] = c_{ab}^k T_k$

simple roots $\alpha_i = \lambda_i - \lambda_{i+1}$

i=1,...,n+1, n(n+1) roots α_i

so(2n+1) odd orthogonal

$\{x \in gl(2n+1), x^T = -x\}$

dim= n(2n+1), rank=n

generators M_{kl} , $(M_{kl})_{mn} = \delta_{km}\delta_{\ln} - \delta_{kn}\delta_{1m}$,

k,l=1,...2n, k<l

commutator relation

$[M_{mn}, M_{rs}] = \delta_{nr}M_{ms} - \delta_{mr}M_{ns} + \delta_{ms}M_{nr} - \delta_{ns}M_{mr}$

n simple roots $\alpha_i = (\lambda_i - \lambda_{i+1})$ *i=1...n-1,* $\alpha_n = \lambda_n$

$2n^2$ roots: $\pm\lambda_i \pm \lambda_j$, *1<=i<j<=n* , $\pm\lambda_i$ *1<=i<=n*

sp(2n) symplectic

2nx2n matrices of the form

$X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$, with symmetric B, C

dim= n(2n+1), rank=n

generators $A_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}$ i,j=1...n

$B_{ij} = \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix}$, i<=j $C_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{pmatrix}$, i<=j

commutator relation $\left[A_{ij}, A_{kl}\right] = \delta_{jk}A_{il} - \delta_{li}A_{kj}$,

$\left[A_{ij}, B_{kl}\right] = \delta_{jk}B_{il} - \delta_{li}B_{kj}$, $\left[A_{ij}, C_{kl}\right] = -\delta_{ik}C_{lj} - \delta_{il}C_{kj}$,

$\left[B_{ij}, C_{kl}\right] = \delta_{ik}A_{lj} + \delta_{jk}A_{li} + \delta_{il}A_{kj} + \delta_{jl}A_{ki}$

n simple roots $\alpha_i = (\lambda_i - \lambda_{i+1})$ for i=1,...,n-1 ;

$\alpha_n = 2\lambda_n$

$2n^2$ roots $\pm(\lambda_i \pm \lambda_j)$, $\pm\lambda_j$ $1 \le i < j \le n$

so(2n) even orthogonal

$\{x \in gl(2n), x^T = -x\}$

dim= n(2n-1), rank=n, $H_i = \langle M_{2i-1,2i} \rangle$

generators M_{kl} , $(M_{kl})_{mn} = \delta_{km}\delta_{\ln} - \delta_{kn}\delta_{1m}$,

k,l=1,...2n, k<l

commutator relation

$[M_{mn}, M_{rs}] = \delta_{nr}M_{ms} - \delta_{mr}M_{ns} + \delta_{ms}M_{nr} - \delta_{ns}M_{mr}$

n simple roots $\alpha_i = (\lambda_i - \lambda_{i+1})$, *i=1...n-1* ,

$\alpha_n = (\lambda_{n-1} + \lambda_n)$

2n(n-1) roots $\pm(\lambda_i \pm \lambda_j)$ 1<=i<j<=n

Five exceptional algebras

g_2 (Dimension 14)

f_4 (Dimension 52)

e_6 (Dimension 78)

e_7 (Dimension 133)

e_8 (Dimension 248)

Special linear Lie algebras su(n+1) , Cartan type An

su(n+1)= $\{x \in gl(n+1,C), \bar{x}^T = -x, tr(x) = 0\}$
dim=$(n+1)^2$-1, su(n+1) semi-simple compact,
Cartan subalgebra dim=n
generators Ti , i=1...$(n+1)^2$-1
commutator relation $(T_a)_{jk} = c^a_{jk}$
simple roots $\alpha_i = \lambda_i - \lambda_{i+1}$ for i=1,...,n+1, n(n+1) roots α_i
Killing form is $\kappa(x, y) = 2(n+1)tr(x\, y)$

su(2)=gauge algebra of AK-quantum gravity
generators σ_i Pauli matrices,
commutator relation $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$
dim=3, , Cartan subalgebra dim=1
simple root is $\alpha_1 = \lambda_1$
roots are $\pm\lambda_1$

su(3)=gauge algebra of (strong) color interaction
generators $T_a = i\,\lambda_a/2$, a=1...8,
commutator relation $[T_a, T_b] = c^k_{ab}T_k$
where λ_a are the 8 3x3 Gell-Mann matrices
dim=8 , Cartan subalgebra dim=2
3 simple roots are $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_2 - \lambda_3$, $\alpha_3 = \lambda_1 - \lambda_3$
6 roots are $\pm\alpha_1$, $\pm\alpha_2$, $\pm\alpha_3$
Killing matrix $\kappa_{ij} = -diag_8(\{3,...,3\})$

su(4)=gauge algebra of extended weak interaction
generators $T_a = i\,\lambda_a / 2$, a=1...15,
commutator relation $[T_a, T_b] = c^k_{ab}T_k$
where λ_a are the 15 extended 4x4 Gell-Mann matrices
dim=15 , Cartan subalgebra dim=3
the 3 simple roots are $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_2 - \lambda_3$, $\alpha_3 = \lambda_3 - \lambda_4$,
the 12 roots are $\pm(\lambda_1 - \lambda_2)$, $\pm(\lambda_1 - \lambda_3)$, $\pm(\lambda_1 - \lambda_4)$, $\pm(\lambda_2 - \lambda_3)$,
$\pm(\lambda_2 - \lambda_4)$, $\pm(\lambda_3 - \lambda_4)$
Killing matrix $\kappa_{ij} = -diag_{15}(\{4,...,4\})$

so(1,3)=Lie algebra of the Lorentz group

so(1,3)= $\left\{x \in gl(3,R), x^T\eta = -\eta x\right\}$, η Minkowski $\eta = diag\left(\{-1,1,1,1\}\right)$

so(1,3) is semi-simple non-compact

generators rotations J_i , boosts K_i , i=1,2,3

commutators $\left[J_i, J_j\right] = \varepsilon_{ijk} J_k$, $\left[K_i, K_j\right] = -\varepsilon_{ijk} J_k$, $\left[J_i, K_j\right] = \varepsilon_{ijk} K_k$

dim=6 , Cartan subalgebra dim=2, Cartan type A1

2 simple roots $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_1 + \lambda_2$

4 roots $\pm\lambda_1 \pm \lambda_2$,

Killing matrix is $\kappa_{ij} = diag_6\left(\{-4,-4,-4,4,4,4\}\right)$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Odd orthogonal Lie algebras so(2n+1) , Cartan type Bn

so(2n+1)= $\{x \in gl(2n+1), x^T = -x\}$
dim= n(2n+1), so(2n+1) simple compact,
Cartan subalgebra dim=n
generators M_{kl} , $(M_{kl})_{mn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$, k,l=1,...2n, k<l
commutator relation $[M_{mn}, M_{rs}] = \delta_{nr}M_{ms} - \delta_{mr}M_{ns} + \delta_{ms}M_{nr} - \delta_{ns}M_{mr}$
n simple roots $\alpha_i = (\lambda_i - \lambda_{i+1})$, *i=1...n-1* , $\alpha_n = \lambda_n$
$2n^2$ roots: $\pm\lambda_i \pm \lambda_j$, *1<=i<j<=n* , $\pm\lambda_i$ *1<=i<=n*

so(3) 3x3 skew-symmetric matrices, tangent space 3-dim rotations
dim= 3, simple compact,
Cartan subalgebra dim=1
generators infinitesimal rotations (J_1, J_2, J_3) about x,y,z axes

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

commutator relation $[J_i, J_j] = \varepsilon_{ijk} J_k$
1 simple root λ_1
2 roots $\pm\lambda_1$

so(5)= 5x5 skew-symmetric matrices, tangent space 5-dim rotations
dim= 10, simple compact, so(5) sp(4) $so(5) \cong sp(4)$
Cartan subalgebra dim=2
generators 10 matrices S_{kl} k,l=1,...5, k<l
$(S_{kl})_{mn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$
commutator relation $[S_{mn}, S_{rs}] = \delta_{nr}S_{ms} - \delta_{mr}S_{ns} + \delta_{ms}S_{nr} - \delta_{ns}S_{mr}$
2 simple roots $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_2$
8 roots $\pm\lambda_1$, $\pm\lambda_2$, $\pm\lambda_1 \pm \lambda_2$

Symplectic Lie algebras sp(2n) , Cartan type Cn

sp(2n) 2nx2n matrices of the form

$X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$, with symmetric B, C

dim= n(2n+1), sp(2n) simple non-compact

Cartan subalgebra dim=n

generators $A_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}$ i,j=1...n

$B_{ij} = \begin{pmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{pmatrix}$, i<=j

$C_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij}+E_{ji} & 0 \end{pmatrix}$, i<=j

commutator relation $\left[A_{ij}, A_{kl}\right] = \delta_{jk} A_{il} - \delta_{li} A_{kj}$,

$\left[A_{ij}, B_{kl}\right] = \delta_{jk} B_{il} - \delta_{li} B_{kj}$, $\left[A_{ij}, C_{kl}\right] = -\delta_{ik} C_{lj} - \delta_{il} C_{kj}$,

$\left[B_{ij}, C_{kl}\right] = \delta_{ik} A_{lj} + \delta_{jk} A_{li} + \delta_{il} A_{kj} + \delta_{jl} A_{ki}$

n simple roots $\alpha_i = \left(\lambda_i - \lambda_{i+1}\right)$ for i=1,...,n-1 ; $\alpha_n = 2\lambda_n$

$2n^2$ roots $\pm\left(\lambda_i \pm \lambda_j\right)$, $\pm\lambda_j$ $1 \le i < j \le n$

$\downarrow$

sp(4)= 4x4 block matrices X of the form $X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$, symmetric *B*,*C*

dim= 10, simple non-compact , $J_2 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$

Cartan subalgebra dim=2

4 generators A_{ij} , 3 generators B_{ij} , 3 generators C_{ij}

commutator relation $\left[A_{ij}, A_{kl}\right] = \delta_{jk} A_{il} - \delta_{li} A_{kj}$, $\left[A_{ij}, B_{kl}\right] = \delta_{jk} B_{il} - \delta_{li} B_{kj}$,

$\left[A_{ij}, C_{kl}\right] = -\delta_{ik} C_{lj} - \delta_{il} C_{kj}$, $\left[B_{ij}, C_{kl}\right] = \delta_{ik} A_{lj} + \delta_{jk} A_{li} + \delta_{il} A_{kj} + \delta_{jl} A_{ki}$

2 simple roots $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = 2\lambda_2$

8 roots $\pm\lambda_1 \pm \lambda_2$, $\pm 2\lambda_1$, $\pm 2\lambda_2$

Even orthogonal Lie algebras so(2n) , Cartan type Dn

so(2n)= $\left\{ x \in gl\left(2n\right), x^T = -x \right\}$
dim= n(2n-1), so(2n+1) simple compact (n>=2),
Cartan subalgebra dim=n, $H_i = \left\langle M_{2i-1,2i} \right\rangle$
generators M_{kl} , $\left(M_{kl}\right)_{mn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$, k,l=1,...2n, k<l
commutator relation $\left[M_{mn}, M_{rs}\right] = \delta_{nr}M_{ms} - \delta_{mr}M_{ns} + \delta_{ms}M_{nr} - \delta_{ns}M_{mr}$
n simple roots $\alpha_i = \left(\lambda_i - \lambda_{i+1}\right)$, *i=1...n-1* , $\alpha_n = \left(\lambda_{n-1} + \lambda_n\right)$
2n(n-1) roots $\pm\left(\lambda_i \pm \lambda_j\right)$ 1<=i<j<=n

so(4) 4x4 skew-symmetric matrices $X = \begin{pmatrix} 0 & -b_3 & b_2 & -a_1 \\ b_3 & 0 & -b_1 & -a_2 \\ -b_2 & b_1 & 0 & -a_3 \\ a_1 & a_2 & a_3 & 0 \end{pmatrix}$, tangent space 4-dim rotations

dim= 6, simple compact
Cartan subalgebra dim=2
generators 6 rotations of so(3) A_1, A_2, A_3

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3 boost rotations B_1, B_2, B_3

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

commutator relation $\left[A_i, A_j\right] = \varepsilon_{ijk}A_k$, $\left[B_i, B_j\right] = \varepsilon_{ijk}A_k$, $\left[A_i, B_j\right] = \varepsilon_{ijk}B_k$
2 simple roots $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_1 + \lambda_2$
4 roots $\pm\lambda_1 \pm \lambda_2$

Exceptional Lie algebras

- g_2 (Dimension 14): is the automorphism group of the octonions and the stabilizer of a generic 3-form in 7 dimensions, rank=2
- f_4 (Dimension 52): The Lie algebra of the exceptional group F_4, is the automorphism group of the Albert algebra (a 27-dimensional exceptional Jordan algebra), rank=4

Albert algebra is the space of 3×3 self-adjoint 3x3 matrices over the octonions .

- e_6 (Dimension 78): is the stabilizer of a certain cubic form and appears in the classification of certain projective planes over division algebras, rank=6
- e_7 (Dimension 133): Leaves invariant a symmetric rank-4 tensor in its fundamental 56-dimensional representation, rank=7
- e_8 (Dimension 248): Its root diagram is the E_8 lattice, which models the densest possible sphere packing in 8-dimensional space, rank=8

Structure of Clifford algebras

field K=R, C

Clifford algebra *g* over K

generators $X_1,\ldots,X_n$, anticommutator $\left\{X_i, X_j\right\} = c_{ij} I$

elements $q \in g$,

$$q = q_0 + \sum_{i_1<i_2} q_{i_1 i_2} e_{i_1} e_{i_2} + \sum_{i_1<i_2<i_3} q_{i_1 i_2 i_3} e_{i_1} e_{i_2} e_{i_3} + \ldots + q_{i_1\ldots i_n} e_{i_1} \ldots e_{i_n}$$

g=ring with operations + *

additive commutative group

multiplicative non-commutative half-group

representation: kxk matrices over K

g=vector space over K, basis { $X_1,\ldots,X_n$ }

$g = \langle X_1,\ldots, X_n \rangle$, dim(g)=n

Quaternion field

$$H = \left\{ q = a + bi + c\,j + d\,k \,\middle|\, a,b,c,d \in R \right\}$$

$$i^2 = j^2 = k^2 = -1 \ , \ i\,j = k = -j\,i \ ,$$

$$j\,k = i = -k\,j \ , \ k\,i = j = -ik$$

$$q^{-1} = \left(a + bi + c\,j + d\,k\right)^{-1} = \frac{a - bi - c\,j - d\,k}{a^2 + b^2 + c^2 + d^2}$$

$$q^* = -\frac{1}{2}\left(q + iqi + jq\,j + kqk\right)$$

$$\|q\| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

equivalence $\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\} \to \left\{I, i\sigma_3, i\sigma_2, i\sigma_1\right\}$

Dirac-Clifford spinor algebra

Cl(1,3)(C) generators γ^μ $\left\{\gamma^\mu, \gamma^\mu\right\} = 2\eta_{\mu\nu} I$

Dirac form $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

Weyl form $\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $\gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$, $\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$

General Dirac-Clifford spinor algebra dim=d

General Dirac matrices Γ_a are nxn matrices, $n = \begin{cases} 2^{d/2} & , d = 2m \\ 2^{(d-1)/2} & , d = 2m+1 \end{cases}$

$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} I$ in Cl(p,q)(C)

Weyl basis Euclidean metric d=4 Dirac matrices

$\gamma_E^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $\gamma_E^j = i\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$, $\gamma_E^5 = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

in Cl(d)(C):

for d=2, $\Gamma_a = (i\sigma_1, i\sigma_2)$

for d=3, $\Gamma_a = (i\sigma_1, i\sigma_2, i\sigma_3)$

for d=4, $\Gamma_a^{(4)} = (\gamma_E^1, \gamma_E^2, \gamma_E^3, \gamma_E^0)$

for d=5, $\Gamma_a^{(5)} = (\gamma_E^1, \gamma_E^2, \gamma_E^3, \gamma_E^0)$, $\Gamma_5^{(5)} = \gamma_E^5$,

for d=6, $\Gamma_a^{(6)} = -\sigma_2 \otimes \Gamma_a^{(5)}$ $a = 1,2,3,4,5$; $\Gamma_6^{(6)} = \sigma_1 \otimes I$

for d=7, $\Gamma_a^{(7)} = -\sigma_2 \otimes \Gamma_a^{(5)}$ $a = 1,2,3,4,5$; $\Gamma_6^{(7)} = \sigma_1 \otimes I$, $\Gamma_7^{(7)} = i\sigma_3 \otimes I$

Vector rotations

$x' = \Lambda x$, Lorentz $\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right)$

6 elementary rotations $\omega_{\mu\nu}$,

Kronecker matrices $M_{\mu\nu}$

$(M_{kl})_{mn} = \eta_{km}\delta_{\ln} - \eta_{kn}\delta_{lm}$

$\Lambda = \exp(-i(\vec{\varphi}\,\vec{J} + \vec{v}\,\vec{K}))$

rotation vector $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$,

boost vector $\vec{v} = (v_1, v_2, v_3)$

rotation &boost $(J_i), (K_i)$ Lie algebra

$[K_i, K_j] = -\varepsilon_{ijk} J_k$, $[J_i, K_j] = \varepsilon_{ijk} K_k$

rotation operator $\Phi = (\vec{\varphi}\,\vec{J} + \vec{v}\,\vec{K})$

Spinor rotations

$\Psi' = S\Psi$, $\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$,

$S = exp\left(-\frac{i}{2}\omega_{\mu\nu}\,\Sigma^{\mu\nu} \right)$, spin tensor generators $\Sigma^{\mu\nu}$

$\Sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^{\mu}, \gamma^{\nu} \right] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \tilde{\sigma}^{\mu\nu} \end{pmatrix}$

with rotation vector $\vec{\varphi} = \left(\varphi_1, \varphi_2, \varphi_3 \right)$,

boost vector $\vec{v} = \left(v_1, v_2, v_3 \right)$

$S = \begin{pmatrix} A_L & 0 \\ 0 & A_R \end{pmatrix}$ with the left- and right Weyl-spinor rotations

$A_L = exp\left(-\frac{i}{2}\omega_{\mu\nu}\,\sigma^{\mu\nu} \right) = exp\left(-\frac{i}{2}(\vec{\varphi} - i\,\vec{v}\,)\vec{\sigma} \right)$

$A_R = exp\left(-\frac{i}{2}\omega_{\mu\nu}\,\tilde{\sigma}^{\mu\nu} \right) = exp\left(-\frac{i}{2}(\vec{\varphi} + i\,\vec{v}\,)\vec{\sigma} \right)$

rotation operator $\Phi_s = \frac{1}{2}(\vec{\varphi}\vec{\sigma} \pm i\,\vec{v}\,\vec{\sigma})$

1 Basics

Lie groups are mathematical objects that are simultaneously groups (allowing for operations like multiplication and inversion) and smooth manifolds (allowing for continuous, "smooth" transformations). Lie algebras represent the infinitesimal structure of Lie groups, serving as the "flat" tangent space at the group's identity.

- Lie Groups: Represent continuous global symmetries. Because they are curved manifolds, doing global calculations (like multiplying group elements) can be geometrically complex
- Lie Algebras: Represent infinitesimal symmetries. They form a flat vector space, meaning you can add elements together and scale them.
- The Correspondence: The exponential map connects the two. It translates tangent vectors (from the Lie algebra) into curved trajectories on the Lie group.

Lie Groups

A Lie group G is a set that possesses two compatible structures:

It is a **group**: It is closed under an associative multiplication operation, has an identity element, and every element has an inverse.

It is a **smooth manifold**: It locally resembles n-dimensional Euclidean space R^n, allowing for calculus (differentiation, integration) to be performed on the group.

Examples: The circle group (U(1)), the group of n × n invertible matrices GL(n, R), and the group of 3D rotations SO(3).

Lie Algebras

A Lie algebra g is a vector space equipped with a binary operation called the **Lie bracket**. The Lie bracket, denoted as [.,.] measures the failure of commutativity

The bracket must satisfy three primary rules:

Bilinearity: ([ax + by, z] = a[x,z] + b[y,z])

Antisymmetry: ([x,y] = -[y,x])

Jacobi Identity: ([x, [y,z]] + [y, [z,x]] + [z, [x,y]] = 0)

For matrices, the Lie bracket is the commutator: [A, B] = AB – BA

Structure constants

Structure constants (or structure coefficients) are the numerical values that uniquely define the algebraic multiplication in a Lie algebra. They are the coefficients that appear when the Lie bracket of any two basis vectors is expressed as a linear combination of the basis itself.

For a Lie algebra g equipped with a basis $\{e_i\}$, the Lie bracket $[e_i, e_j]$ of any two basis elements produces a new element within the algebra. Because of bilinearity, you only need to know the products of the basis elements to determine the multiplication for the entire algebra.

The basis vectors (also known as generators) satisfy the commutation relation:

$$\left[e_i, e_j\right] = c_{ij}^k e_k$$

where:

c_{ij}^k are the structure constants , and Einstein summation notation is implied (sum over k).

Structure constants inherently possess specific mathematical properties:

- Anti-symmetry: Since the Lie bracket is anti-symmetric, swapping the first two indices of the structure constants flips the sign:

$$c_{ij}^k = -c_{ji}^k$$

- The Jacobi Identity: The Jacobi identity into a crucial constraint on the structure constants:

$$c_{ij}^m c_{mk}^k + c_{jk}^m c_{mi}^i + c_{ki}^m c_{mj}^j = 0$$

- Killing Form: The structure constants are directly used to compute the Killing form, a symmetric bilinear form that helps classify whether a Lie algebra is semisimple or solvable:

$$\kappa_{ij} = c_{ib}^a c_{ja}^b$$

The properties antisymmetry and Jacobi identity are sufficient in order to define a Lie algebra.

Adjoint representation

The mapping $ad_x\left(y\right) = \left[x, y\right]$ is called the adjoint of $x \in G$ of Lie algebra *G*.

Derivation

A derivation on a Lie algebra G is a linear map D:G→G that satisfies the Leibniz rule for all elements $x, y \in G$

$$D([x,y]) = [Dx,y] + [x,Dy]$$

The collection of all such derivations forms a Lie algebra known as the derivation Lie algebra, denoted *Der(G)* . The key aspects of derivations are

- The Bracket Operation: The Lie bracket on Der(G) is the standard commutator operation, defined for any two derivations $D_1, D_2 \in Der(G)$ as:

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$$

- Inner Derivations: For any element $x \in G$, the adjoint operator $ad_x(y) = [x,y]$ is a derivation. These are called inner derivations and they form an ideal inside *Der(G)* .

Killing form

The Killing form is the bilinear symmetric form $\kappa(x,y) = Tr(ad(x) \circ ad(y))$, with adjoint $ad_x(y) = [x,y]$, for basis vectors $\kappa_{ij} = c^a_{ib} c^b_{ja}$ expressed by the structure constants.

The Killing form characterizes the Lie algebra, and has the properties:

1. Cartan's First Criterion (Detecting Solvability)

A finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is solvable if and only if the Killing form vanishes identically on the commutator subspace:

$\kappa(x,y) = 0$ for all $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$

2. Cartan's Second Criterion (Detecting Semisimplicity)

A finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is semisimple if and only if its Killing form is non-degenerate.

Non-degeneracy means that if

$\kappa(x,y) = 0$ for all $y \in \mathfrak{g}$, then $x = 0$, i..e. $\kappa_{ij} = \kappa(e_i, e_j)$ has full rank $\det(\kappa_{ij}) \neq 0$

3. Nilpotent Lie Algebras

If a Lie algebra $\mathfrak{g}$ is nilpotent, its adjoint operators ad(x) are nilpotent endomorphisms, meaning all their eigenvalues are zero. Consequently, the Killing form is identically zero.

4. Compact Real Forms and Definiteness

For real semisimple Lie algebras, the Killing form provides a characterization based on its signature (the number of strictly positive and negative eigenvalues).

A real semisimple Lie algebra is compact (the Lie algebra of a compact Lie group) if and only if the Killing form is strictly negative definite.

In this case, all eigenvalues of the quadratic matrix $\kappa_{ij} = \kappa(e_i, e_j)$ are strictly negative.

5 Associativity (Adjoint Invariance): The Killing form is invariant under the Lie bracket, meaning that for any $x, y, z \in \mathfrak{g}$, it respects cyclic permutation:

$$\kappa([x,y],z) = \kappa(x,[y,z])$$

6 Orthogonal Ideals: If $\mathfrak{i}$ is an ideal in $\mathfrak{g}$, its orthogonal complement with respect to the Killing form is also an ideal.

$x, y \in g$ are orthogonal resp. $\kappa_{ij} = \kappa(e_i, e_j)$, if their vectors x_b, y_b in the basis b={e_i} satisfy

$$x_b \cdot \kappa(e_i, e_j) \cdot y_b = 0$$

Key Theorems

- **Lie's Third Theorem:** Every finite-dimensional, real Lie algebra is the Lie algebra of some Lie group.
- **The Correspondence:** While two different Lie groups can share the exact same Lie algebra (for example, SU(2) and SO(3)), simply connected Lie groups have a one-to-one correspondence with their Lie algebras.

Applications in physics

- Quantum Mechanics: The angular momentum operators satisfy the commutation relations of the SU(2) Lie algebra.
- Particle Physics: The Standard Model relies heavily on the gauge groups U(1) × SU(2) × SU(3) (where SU(3) relates to quantum chromodynamics).

2 Lie groups and manifolds
A **Lie group** is a mathematical object that is both a smooth manifold and a group, where the group operations of multiplication and inversion are differentiable. This structure allows mathematicians to study continuous symmetries using the tools of calculus and geometry, mapping complex curved spaces into linear structures called Lie algebras.

At its core, a Lie group sits beautifully at the intersection of algebra and geometry. It combines two fundamental mathematical concepts:
The Manifold: A space that locally resembles flat Euclidean space. For instance, a circle looks like a straight line when you zoom in.
The Group: A set of elements coupled with an operation (like rotation or translation) that obeys closure, associativity, has an identity element, and allows for inverses
For the two structures to be compatible, the multiplication map $(x, y) \mapsto x \cdot y$ and the inversion map $x \mapsto x^{-1}$ must be smooth (continuous) functions.
Lie groups can be highly non-linear and curved, making global calculations difficult. The primary breakthrough in Lie theory is the Lie algebra, which essentially acts as a "flat map" or a linear approximation of the curved group.
Tangent Space: The Lie algebra is defined as the tangent space of the Lie group at the identity element.
Exponential Map: The relationship between the Lie algebra and the Lie group is established via the exponential map, which projects tangent vectors (elements of the Lie algebra) directly onto the curved group (the Lie group).
The Lie Bracket: While the group has a multiplication operation, the Lie algebra possesses a special operation called the Lie bracket, which captures the non-commutativity and curvature of the group's operations.
Because Lie algebras are flat (vector spaces), it is often vastly simpler to perform algebraic and geometric computations within them.
Lie algebra $\mathfrak{g}$ is nilpotent if its lower central series terminates in the zero subalgebra. The *lower central* series is the sequence of subalgebras
$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supseteq \ldots$
Lie algebra $\mathfrak{g}$ is *solvable* if its derived series terminates in the zero subalgebra. The *derived Lie algebra* of the Lie algebra $\mathfrak{g}$ is the subalgebra of $\mathfrak{g}$, denoted
$[\mathfrak{g}, \mathfrak{g}]$ that consists of all linear combinations of Lie brackets of pairs of elements of g. The *derived series* is the sequence of subalgebras
$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supseteq \ldots$
Lie algebra $\mathfrak{g}$ is semisimple if it is a direct sum of simple Lie algebras. A simple Lie algebra is a non-abelian Lie algebra without any non-zero proper ideals.
For a finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0, if nonzero, the following conditions are equivalent:
$\mathfrak{g}$ is semisimple;
the Killing form $\kappa(x, y) = tr(ad(x) ad(y))$ is non-degenerate;
$\mathfrak{g}$ has no non-zero abelian ideals;
$\mathfrak{g}$ has no non-zero solvable ideals;
the radical (maximal solvable ideal) of $\mathfrak{g}$ is zero.

2.1 Local groups
[2]
As such, the local theory of Lie groups is completely described (in principle, at least) by the theory of Lie algebras, which leads to a number of useful consequences, such as the following:
- (Local Lie implies Lie) A topological group G is Lie (i.e. it is isomorphic to a Lie group) if and only if it is locally Lie (i.e. the group operations are smooth near the origin).
- (Uniqueness of Lie structure) A topological group has at most one smooth structure on it that makes it Lie.
- (Weak regularity implies strong regularity, I) Lie groups are automatically real analytic. (In fact one only needs a local $C^{1,1}$ regularity on the group structure to obtain real analyticity.)

• (Weak regularity implies strong regularity, II) A continuous homomorphism from one Lie group to another is automatically smooth (and real analytic).
The connection between Lie groups and Lie algebras also highlights the role of *one-parameter subgroups* of a topological group, which will play a central role in the solution of Hilbert's fifth problem.
We note that there is also a very important *algebraic* structure theory of Lie groups and Lie algebras, in which the Lie algebra is split into solvable and semisimple components, with the latter being decomposed further into simple components, which can then be completely classified using Dynkin diagrams.

Definition 1 (Local group) A *local topological group* $G = (G, \Omega, \Lambda, 1, \cdot, ()^{-1})$, or *local group* for short, is a topological space G equipped with an identity element $1 \in G$, a partially defined but continuous multiplication operation $\cdot : \Omega \to G$ for some domain $\Omega \subset G \times G$, and a partially defined but continuous inversion operation $()^{-1} : \Lambda \to G$, where $\Lambda \subset G$, obeying the following axioms:

- (Local closure) Ω is an open neighbourhood of $G \times \{1\} \cup \{1\} \times G$, and Λ is an open neighbourhood of 1.
- (Local associativity) If $g, h, k \in G$ are such that $(g \cdot h) \cdot k$ and $g \cdot (h \cdot k)$ are both well-defined in G, then they are equal. (Note however that it may be possible for one of these products to be defined but not the other.)
- (Identity) For all $g \in G$, $g \cdot 1 = 1 \cdot g = g$.
- (Local inverse) If $g \in G$ and g^{-1} is well-defined in G, then $g \cdot g^{-1} = g^{-1} \cdot g = 1$. (In particular this, together with the other axioms, forces $1^{-1} = 1$.)

Definition 2 (Restriction) If G is a local group, and U is an open neighbourhood of the identity in G, then we define the *restriction* $G\lfloor_U$ of G to U to be the topological space U with domains $\Omega\lfloor_U := \{(g,h) \in \Omega : g, h, g \cdot h \in U\}$ and $\Lambda\lfloor_U := \{g \in \Lambda : g, g^{-1} \in U\}$, and with the group operations $\cdot, ()^{-1}$ being the restriction of the group operations of G to $\Omega\lfloor_U$, $\Lambda\lfloor_U$ respectively. If U is symmetric (in the sense that g^{-1} is well-defined and lies in U for all $g \in U$), then this restriction $G\lfloor_U$ will also be symmetric. If G is a global or local Lie group, then $G\lfloor_U$ will also be a local Lie group.
Thus, for instance, one can take the Euclidean space $\mathbf{R}^d$, and restrict it to a ball B centred at the origin, to obtain an additive local group $\mathbf{R}^d\lfloor_B$. In this group, two elements x, y in B have a well-defined sum $x + y$ only when their sum in $\mathbf{R}^d$ stays inside B. Intuitively, this local group behaves like the global group $\mathbf{R}^d$ as long as one is close enough to the identity element 0, but as one gets closer to the boundary of B, the group structure begins to break down.

Example 1 Let G be a global or local Lie group of some dimension d, and let $\phi : U \to V$ be a smooth coordinate chart from a neighbourhood U of the identity 1 in G to a neighbourhood V of the origin 0 in $\mathbf{R}^d$, such that ϕ maps 1 to 0. Then we can define a local group $\phi_* G\lfloor_U$ which is the set V (viewed as a smooth submanifold of $\mathbf{R}^d$) with the local group identity 0, the local group multiplication law $*$ defined by the formula

$$x * y := \phi(\phi^{-1}(x) \cdot \phi^{-1}(y))$$

defined whenever $\phi^{-1}(x), \phi^{-1}(y), \phi^{-1}(x) \cdot \phi^{-1}(y)$ are well-defined and lie in U, and the local group inversion law $()^{*-1}$ defined by the formula

$$x^{*-1} := \phi(\phi^{-1}(x)^{-1})$$

defined whenever $\phi^{-1}(x), \phi^{-1}(x)^{-1}$ are well-defined and lie in U. One easily verifies that $\phi_* G\lfloor_U$ is a local Lie group.

Definition 3 (Homomorphism) A *continuous homomorphism* $\phi : G \to H$ between two local groups G, H is a continuous map from G to H with the following properties:

- ϕ maps the identity 1_G of G to the identity 1_H of H: $\phi(1_G) = 1_H$.
- If $g \in G$ is such that g^{-1} is well-defined in G, then $\phi(g)^{-1}$ is well-defined in H and is equal to $\phi(g^{-1})$.

- If $g, h \in G$ are such that $g \cdot h$ is well-defined in G, then $\phi(g) \cdot \phi(h)$ is well-defined and equal to $\phi(g \cdot h)$.

A *smooth homomorphism* $\phi : G \to H$ between two local Lie groups G, H is a continuous homomorphism that is also smooth (differentiable).

2.2 Some differential geometry

[2]

Definition 4 (Tangent space) Let M be a smooth d-dimensional manifold. At every point x of this manifold, we can define the tangent space $T_x M$ of M at x. Formally, this tangent space can be defined as the space of all continuously differentiable curves $\gamma : I \to M$ defined on an open interval I containing 0 with $\gamma(0) = x$, modulo the relation that two curves γ_1, γ_2 are considered equivalent if they have the same derivative at 0, in the sense that

$$\frac{d}{dt}\phi(\gamma_1(t))|_{t=0} = \frac{d}{dt}\phi(\gamma_2(t))|_{t=0}$$

where $\phi : U \to V$ is a coordinate chart of G defined in a neighbourhood of x; it is easy to see from the chain rule that this equivalence is independent of the actual choice of ϕ. Using such a coordinate chart, one can identify the tangent space $T_x M$ with the Euclidean space $\mathbf{R}^d$, by identifying γ with $\frac{d}{dt}\phi(\gamma(t))|_{t=0}$. One easily verifies that this gives $T_x M$ the structure of a d-dimensional vector space, in a manner which is independent of the choice of coordinate chart ϕ. Elements of $T_x M$ are called *tangent vectors* of M at x. If $\gamma : I \to M$ is a continuously differentiable curve with $\gamma(0) = x$, the equivalence class of γ in $T_x M$ will be denoted $\gamma'(0)$.

The space $TM := \bigcup_{x \in M} (\{x\} \times T_x M)$ of pairs (x, v), where x is a point in M and v is a tangent vector of M at x, is called the tangent bundle.

If $\Phi : M \to N$ is a smooth map between two manifolds, we define the derivative map $D\Phi : TM \to TN$ to be the map defined by setting

$$D\Phi((x, \gamma'(0))) := (\Phi(x), (\Phi \circ \gamma)'(0))$$

for all continously differentiable curves $\gamma : I \to M$ with $\gamma(0) = x$ for some $x \in M$. We also write $(\Phi(x), D\Phi(x)(v))$ for $D\Phi(x, v)$, so that for each $x \in M$, $D\Phi(x)$ is a map from $T_x M$ to $T_{\Phi(x)} N$. One can easily verify that this latter map is linear. We observe the *chain rule*

$$D(\Psi \circ \Phi) = (D\Psi) \circ (D\Phi) \qquad (1)$$

for any smooth maps $\Phi : M \to N$, $\Psi : N \to O$. (Indeed, one can view the tangent operator T and the derivative operator D together as a single covariant functor from the category of smooth manifolds to itself, although we will not need to use this perspective here.)

Observe that if V is an open subset of $\mathbf{R}^d$, then TV may be identified with $V \times \mathbf{R}^d$. In particular, every coordinate chart $\phi : U \to V$ of M gives rise to a coordinate chart $D\phi : TU \to V \times \mathbf{R}^d$ of TM, which gives TM the structure of a smooth $2d$-dimensional manifold.

Definition 5 (Vector fields) A smooth vector field on M is a smooth map $X : M \to TM$ which is a right inverse for the projection map $\pi : TM \to M$, thus (by slight abuse of notation) X maps x to $(x, X(x))$ for some $X(x) \in T_x M$. The space of all smooth vector fields is denoted $\Gamma(TM)$. It is clearly a real vector space. In fact, it is a $C^\infty(M)$-module: given a smooth vector field $X \in \Gamma(TM)$ and a smooth function $f \in C^\infty(M)$ (i.e. a smooth map $f : M \to \mathbf{R}$), one can define the product fX in the obvious manner: $fX(x) := f(x)X(x)$, and one easily verifies the module axioms.

Given a smooth function $f \in C^\infty(M)$ and a smooth vector field $X \in \Gamma(TM)$, we define the directional derivative $\nabla_X f \in C^\infty(M)$ of f along X by the formula

$$\nabla_X f(x) := \frac{d}{dt} f(\gamma(t))|_{t=0}$$

whenever $\gamma : I \to M$ is a continuously differentiable function with $\gamma(0) = x$ and $\gamma'(0) = X(x)$; one easily verifies that $\nabla_X f$ is well-defined and is an element of $C^\infty(M)$.

There is a fundamental link between smooth vector fields and derivations of $C^\infty(M)$:

Lemma 2 (Correspondence between smooth vector fields and derivations) Let M be a smooth manifold.

- If $X \in \Gamma(TM)$ is a smooth vector field, Then $\nabla_X : C^\infty(M) \to C^\infty(M)$ is a derivation on the (real) algebra $C^\infty(M)$, i.e. a (real) linear map that obeys the Leibniz rule

$$\nabla_X(fg) = f\nabla_X g + (\nabla_X f)g \qquad (2)$$

  for all $f, g \in C^\infty(M)$.

- Conversely, if $d : C^\infty(M) \to C^\infty(M)$ is a derivation on $C^\infty(M)$, Then there exists a unique smooth vector field X such that $d = \nabla_X$.

Lemma 3 (Commutator of derivations is a derivation) Let $d_1, d_2 : A \to A$ be two derivations on an algebra A. The commutator $[d_1, d_2] := d_1 \circ d_2 - d_2 \circ d_1$ is also a derivation on A

From the preceding, we can define the Lie bracket $[X, Y]$ of two vector fields $X, Y \in \Gamma(TM)$ by the formula $\nabla_{[X,Y]} := [\nabla_X, \nabla_Y]$.

This gives the space $\Gamma(TM)$ of smooth vector fields the structure of an (infinite-dimensional) Lie algebra

Definition 6 (Lie algebra) A (real) Lie algebra is a real vector space V (possibly infinite dimensional), together with a bilinear map $[,] : V \times V \to V$ which is anti-symmetric (thus $[X, Y] = -[Y, X]$ for all $X, Y \in V$, or equivalently $[X, X] = 0$ for all $X \in V$) and obeys the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

for all $X, Y, Z \in V$.

Lemma 4 If M is a smooth manifold, Then $\Gamma(TM)$ (equipped with the Lie bracket) is a Lie algebra.

3 Lie algebras

Lie Algebras

A Lie algebra g is a vector space equipped with a binary operation called the **Lie bracket**. The Lie bracket, denoted as [.,.] measures the failure of commutativity

The bracket must satisfy three primary rules:

Bilinearity: ([ax + by, z] = a[x,z] + b[y,z])

Antisymmetry: ([x,y] = -[y,x])

Jacobi Identity: ([x, [y,z]] + [y, [z,x]] + [z, [x,y]] = 0)

For matrices, the Lie bracket is the commutator: [A, B] = AB – BA

Structure constants

Structure constants (or structure coefficients) are the numerical values that uniquely define the algebraic multiplication in a Lie algebra. They are the coefficients that appear when the Lie bracket of any two basis vectors is expressed as a linear combination of the basis itself.

For a Lie algebra g equipped with a basis $\{e_i\}$, the Lie bracket $[e_i, e_j]$ of any two basis elements produces a new element within the algebra. Because of bilinearity, you only need to know the products of the basis elements to determine the multiplication for the entire algebra.

The basis vectors (also known as generators) satisfy the commutation relation:

$$\left[e_i, e_j\right] = c_{ij}^k e_k$$

where:

c_{ij}^k are the structure constants , and Einstein summation notation is implied (sum over k).

Structure constants inherently possess specific mathematical properties:

- Anti-symmetry: Since the Lie bracket is anti-symmetric, swapping the first two indices of the structure constants flips the sign:

$$c_{ij}^k = -c_{ji}^k$$

- The Jacobi Identity: The Jacobi identity into a crucial constraint on the structure constants:

$$c_{ij}^m c_{mk}^k + c_{jk}^m c_{mi}^i + c_{ki}^m c_{mj}^j = 0$$

- Killing Form: The structure constants are directly used to compute the Killing form, a symmetric bilinear form that helps classify whether a Lie algebra is semisimple or solvable:

$$\kappa_{ij} = c_{ib}^a c_{ja}^b$$

The properties antisymmetry and Jacobi identity are sufficient in order to define a Lie algebra.

Adjoint representation

The mapping $ad_x(y) = [x, y]$ is called the adjoint of $x \in G$ of Lie algebra *G*.

Derivation

A derivation on a Lie algebra G is a linear map D:G→G that satisfies the Leibniz rule for all elements $x, y \in G$

$$D([x, y]) = [Dx, y] + [x, Dy]$$

The collection of all such derivations forms a Lie algebra known as the derivation Lie algebra, denoted *Der(G)* . The key aspects of derivations are

- The Bracket Operation: The Lie bracket on Der(G) is the standard commutator operation, defined for any two derivations $D_1, D_2 \in Der(G)$ as:

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$$

- Inner Derivations: For any element $x \in G$, the adjoint operator $ad_x(y) = [x, y]$ is a derivation. These are called inner derivations and they form an ideal inside *Der(G)* .

Killing form

The Killing form is the bilinear symmetric form $\kappa(x, y) = Tr(ad(x) \circ ad(y))$, with adjoint $ad_x(y) = [x, y]$, for basis vectors $\kappa_{ij} = c_{ib}^a c_{ja}^b$ expressed by the structure constants.

The Killing form characterizes the Lie algebra, and has the properties:

1. Cartan's First Criterion (Detecting Solvability)
A finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is solvable if and only if the Killing form vanishes identically on the commutator subspace:
$\kappa(x, y) = 0$ for all $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$
2. Cartan's Second Criterion (Detecting Semisimplicity)
A finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is semisimple if and only if its Killing form is non-degenerate.
Non-degeneracy means that if
$\kappa(x, y) = 0$ for all $y \in \mathfrak{g}$, then $x = 0$, i..e. $\kappa_{ij} = \kappa(e_i, e_j)$ has full rank $\det(\kappa_{ij}) \neq 0$
3. Nilpotent Lie Algebras
If a Lie algebra $\mathfrak{g}$ is nilpotent, its adjoint operators ad(x) are nilpotent endomorphisms, meaning all their eigenvalues are zero. Consequently, the Killing form is identically zero.
4. Compact Real Forms and Definiteness
For real semisimple Lie algebras, the Killing form provides a characterization based on its signature (the number of strictly positive and negative eigenvalues).
A real semisimple Lie algebra is compact (the Lie algebra of a compact Lie group) if and only if the Killing form is strictly negative definite.
In this case, all eigenvalues of the quadratic matrix $\kappa_{ij} = \kappa(e_i, e_j)$ are strictly negative.
5 Associativity (Adjoint Invariance): The Killing form is invariant under the Lie bracket, meaning that for any $x, y, z \in \mathfrak{g}$, it respects cyclic permutation:
$\kappa([x, y], z) = \kappa(x, [y, z])$
6 Orthogonal Ideals: If $\mathfrak{i}$ is an ideal in $\mathfrak{g}$, its orthogonal complement with respect to the Killing form is also an ideal.
$x, y \in g$ are orthogonal resp. $\kappa_{ij} = \kappa(e_i, e_j)$, if their vectors x_b, y_b in the basis b={e_i} satisfy
$x_b \cdot \kappa(e_i, e_j) \cdot y_b = 0$

3.1 The Lie algebra of a Lie group
[2]
Let G be a (global) Lie group. By definition, G is then a smooth manifolds, so we can thus define the tangent bundle TG and smooth vector fields $X \in \Gamma(TG)$ as in the preceding section. In particular, we can define the tangent space $T_1 G$ of G at the identity element 1.
If $g \in G$, then the left multiplication operation $\rho_g^{left} : x \mapsto gx$ is, by definition of a Lie group, a smooth map from G to G. This creates a derivative map $D\rho_g^{left} : TG \to TG$ from the tangent bundle TG to itself. We say that a vector field $X \in \Gamma(TG)$ is *left-invariant* if one has $(\rho_g^{left})_* X = X$ for all $g \in G$, or equivalently if $(D\rho_g^{left}) \circ X = X \circ \rho_g^{left}$ for all $g \in G$.
From the above, we can identify the tangent space $T_1 G$ with the left-invariant vector fields on TG, and the Lie bracket structure on the latter then induces a Lie bracket (which we also call $[,]$) on $T_1 G$. The vector space $T_1 G$ together with this Lie bracket is then a (finite-dimensional) Lie algebra, which we call the *Lie algebra* of the Lie group G, and we write as $\mathfrak{g}$.
Lemma 7 Let $\phi : G \to H$ be a smooth homomorphism between (global) Lie groups. Then the derivative map $D\phi(1_G)$ at the identity element 1_G is then a Lie algebra homomorphism from the Lie algebra $\mathfrak{g}$ of G to the Lie algebra $\mathfrak{h}$ of H (thus this map is linear and preserves the Lie bracket).

3.2 The exponential map
The exponential map $x \mapsto \exp(x)$ on the reals $\mathbf{R}$ (or its extension to the complex numbers $\mathbf{C}$) is of course fundamental to modern analysis. It can be defined in a variety of ways, such as the following:
(i) $\exp : \mathbf{R} \to \mathbf{R}$ is the differentiable map obeying the ODE $\frac{d}{dx}\exp(x) = \exp(x)$ and the initial condition $\exp(0) = 1$.

(ii) $\exp : \mathbf{R} \to \mathbf{R}$ is the differentiable map obeying the homomorphism property $\exp(x+y) = \exp(x)\exp(y)$ and the initial condition $\frac{d}{dx}\exp(x)|_{x=0} = 1$.
(iii) $\exp : \mathbf{R} \to \mathbf{R}$ is the limit of the functions $x \mapsto (1+\frac{x}{n})^n$ as $n \to \infty$.
(iv) $\exp : \mathbf{R} \to \mathbf{R}$ is the limit of the infinite series $x \mapsto \sum_{n=0}^\infty \frac{x^n}{n!}$.

Definition 7 ($C^{1,1}$ local group) A $C^{1,1}$ *local group* is a local group V that is an open neighbourhood of the origin 0 in a Euclidean space $\mathbf{R}^d$, with group identity 0, and whose group operation $*$ obeys the estimate

$$x * y = x + y + O(|x||y|) \qquad (4)$$

for all sufficiently small x, y, where the implied constant in the $O()$ notation can depend on V but is uniform in x, y.

Remark 6 In real analysis, a (locally) $C^{1,1}$ function is a function $f : U \to \mathbf{R}^m$ on a domain $U \subset \mathbf{R}^n$ which is continuously differentiable (i.e. in the regularity class C^1), and whose first derivatives ∇f are (locally) Lipschitz (i.e. in the regularity class $C^{0,1}$) the $C^{1,1}$ regularity class is slightly weaker (i.e. larger) than the class C^2 of twice continuously differentiable functions, but much stronger than the class C^1 of singly continuously differentiable functions.
The reason for the terminology $C^{1,1}$ in the above definition is that $C^{1,1}$ regularity is essentially the minimal regularity for which one has the Taylor expansion
$f(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + O(|x-x_0|^2)$
for any x_0 in the domain of f, and any x sufficiently close to x_0; note that the asymptotic (4) is of this form.

We can now define the *exponential map* $\exp : V' \to V$ on this $C^{1,1}$ local group by defining
$\exp(x) := \lim_{n\to\infty} (\frac{1}{2^n}x)^{*2^n} \qquad (6)$
for any x in a sufficiently small neighbourhood V' of the origin in V.
Lemma 10 Let V be a local $C^{1,1}$ group.
(i) Then if V' is a sufficiently small neighbourhood of the origin in V, then the limit in (6) exists for all $x \in V'$. (*Hint:* use the previous exercise to estimate the distance between $(\frac{1}{2^n}x)^{*2^n}$ and $(\frac{1}{2^{n+1}}x)^{*2^{n+1}}$.) Establish the additional estimate
$\exp(x) = x + O(|x|^2). \qquad (7)$
(ii) Then if $\gamma : I \to G$ is a smooth curve with $\gamma(0) = 1$, and $\gamma'(0)$ is sufficiently small, then
$\exp(\gamma'(0)) = \lim_{n\to\infty} \gamma(1/2^n)^{*2^n}$.
(iii) Then for all sufficiently small x, y, one has the bilipschitz property
$|(\exp(x) - \exp(y)) - (x-y)| \leq \frac{1}{2}|x-y|$.
We conclude in particular that for V' sufficiently small, $\exp$ is a homeomorphism between V' and an open neighbourhood $\exp(V')$ of the origin.
Then
$\exp(sx) * \exp(tx) = \exp((s+t)x) \qquad (8)$
for $s, t \in \mathbf{R}$ and $x \in \mathbf{R}^d$ with sx, tx sufficiently small. (*Hint:* first handle the case when $s, t \in \mathbf{Z}[\frac{1}{2}]$ are dyadic numbers.)
(iv) Then for any sufficiently small $x, y \in \mathbf{R}^d$, one has
$\exp(x+y) = \lim_{n\to\infty} (\exp(x/2^n) * \exp(y/2^n))^{*2^n}. \qquad (9)$
We conclude the stronger estimate
$\exp(x+y) = \lim_{n\to\infty} (\exp(x/n) * \exp(y/n))^{*n}. \qquad (10)$
(v) Then that for any sufficiently small $x, y \in \mathbf{R}^d$, one has
$\exp(x+y) = \exp(x) * \exp(y) + O(|x||y|)$.

Let us say that a $C^{1,1}$ local group is *radially homogeneous* if one has
$sx * tx = (s+t)x \qquad (11)$
whenever $s, t \in \mathbf{R}$ and $x \in \mathbf{R}^d$ are such that sx, tx are sufficiently small. (In particular, this implies that $x^{*-1} = -x$ for sufficiently small x.) From the above exercise, we see that any $C^{1,1}$ local group V can be made

into a radially homogeneous $C^{1,1}$ local group V' by first restricting to an open neighbourhood $\exp(V')$ of the identity, and then applying the logarithmic homeomorphism $\exp^{-1}$. Thus:
Corollary 8 Every $C^{1,1}$ local group has a neighbourhood of the identity which is isomorphic (as a topological group) to a radially homogeneous $C^{1,1}$ local group.

Now we study the exponential map on global Lie groups. If G is a global Lie group, and $\mathfrak{g}$ is its Lie algebra, we define the exponential map $\exp : \mathfrak{g} \to G$ on a global Lie group G by setting
$$\exp(\gamma'(0)) := \lim_{n\to\infty} \gamma(1/2^n)^{2^n}$$
whenever $\gamma : I \to G$ is a smooth curve with $\gamma(0) = 1$.

Lemma 11 Let G be a global Lie group.
(i) Then the exponential map is well-defined. (*Hint:* First handle the case when $\gamma'(0)$ is small, using the previous exercise, then bootstrap to larger values of $\gamma'(0)$.)
(ii) Then for all $x, y \in \mathfrak{g}$ and $s, t \in \mathbf{R}$, one has
$$\exp(sx)\exp(tx) = \exp((s+t)x) \qquad (12)$$
and
$$\exp(x+y) = \lim_{n\to\infty} (\exp(x/n)\exp(y/n))^n. \qquad (13)$$
(iii) Then the exponential map is continuous.
(iv) Then for each $x \in \mathfrak{g}$, the function $t \mapsto \exp(tx)$ is the unique homomorphism from $\mathbf{R}$ to G that is differentiable at $t = 0$ with derivative equal to x.

Proposition 9 (Lie's first theorem) Let G be a Lie group. Then the exponential map is smooth. Furthermore, there is an open neighbourhood U of the origin in $\mathfrak{g}$ and an open neighbourhood V of the identity in G such that the exponential map $\exp$ is a diffeomorphism from U to V.

Lemma 12 (Classification of one-parameter subgroups) Let G be a Lie group. For any $X \in \mathfrak{g}$, Then the map $t \mapsto \exp(tX)$ is a one-parameter subgroup. Conversely, if $\phi : \mathbf{R} \to G$ is a one-parameter subgroup, there exists a unique $X \in \mathfrak{g}$ such that $\phi(t) = \exp(tX)$ for all $t \in \mathbf{R}$

Proposition 10 (Weak regularity implies strong regularity) Let G, H be global Lie groups, and let $\Phi : G \to H$ be a continuous homomorphism. Then Φ is smooth.
Corollary 11 (Uniqueness of Lie structure) Any (global) topological group can be made into a Lie group in at most one manner. More precisely, given a topological group G, there is at most one smooth structure one can place on G that makes the group operations smooth.
Lemma 13 Let G be a connected (global) Lie group, let H be another (global) Lie group, and let $\Phi : G \to H$ be a continuous homomorphism (which is thus smooth by Proposition 10). Then Φ is uniquely determined by the derivative map $D\Phi(1) : \mathfrak{g} \to \mathfrak{h}$. In other words, if $\Phi' : G \to H$ is another continuous homomorphism with $D\Phi(1) = D\Phi'(1)$, then $\Phi = \Phi'$. (*Hint:* first prove this in a small neighbourhood of the origin. What group does this neighbourhood generate?) What happens if G is not connected?
Lemma 14 (Weak regularity implies strong regularity, local version) Let G, H be local Lie groups, and let $\Phi : G \to H$ be a continuous homomorphism. Then Φ is smooth in a neighbourhood of the identity in G.
Lemma 15 (Local Lie implies Lie) Let G be a global topological group. Suppose that there is an open neighbourhood U of the identity such that the local group $G\lfloor_U$ can be given the structure of a local Lie group. Then G can be given the structure of a global Lie group.

4 Fundamental theorems

The Fundamental Theorems of Lie Theory—often called Lie's Theorems—form the backbone of the relationship between Lie groups (continuous symmetries) and Lie algebras (their infinitesimal counterparts). They establish a direct dictionary between the calculus of smooth manifolds and linear algebra.

Lie's Three Theorems

The core of Lie theory revolves around three foundational theorems that connect a Lie group G with its Lie algebra $\mathfrak{g}$ (typically represented as the tangent space at the group's identity element).

Lie's First Theorem: This states that every finite-dimensional (local) differentiable manifold or group can be embedded into a local Lie group. It establishes that the tangent space at any point fundamentally characterizes the infinitesimal transformations of the group.

Lie's Second Theorem: This theorem establishes a direct link between the structure of a Lie group and its algebra. It states that two elements (x,y) correspond to the commutator product, meaning the structure of the algebra is entirely determined by its structure constants.

Lie's Third Theorem: This is the most profound result. It states that every finite-dimensional, real Lie algebra is the Lie algebra of some unique, connected, and simply connected Lie group.

The Homomorphisms and Subgroups Theorems

The Homomorphisms Theorem: If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, and the domain group G is simply connected, there exists a unique global Lie group homomorphism $\Phi : G \to H$ whose derivative at the identity maps to φ.

The Subgroups–Subalgebras Theorem: For any Lie group G and any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there is a unique connected Lie subgroup $H \subset G$ that has $\mathfrak{h}$ as its Lie algebra.

Engel's Theorem

This theorem applies to nilpotent Lie algebras. It states that a finite-dimensional Lie algebra $\mathfrak{g}$ is nilpotent if and only if every element in the adjoint representation is a nilpotent endomorphism. Consequently, there exists a common basis in which all matrices representing elements of $\mathfrak{g}$ are strictly upper triangular.

Lie's Theorem solvable algebras

This applies to solvable Lie algebras. It states that if $\mathfrak{g}$ is a solvable Lie algebra over an algebraically closed field of characteristic zero (like C), and $\rho : \mathfrak{g} \to End(V)$ is a finite-dimensional representation (on a matrix space V), then all elements of $\mathfrak{g}$ can be represented by upper triangular matrices simultaneously.

Cartan's Criterion

This provides a powerful test for the semisimplicity of a Lie algebra. It states that a finite-dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form (a symmetric bilinear form defined by the trace of the adjoint representations) is non-degenerate.

Levi's Decomposition Theorem

This is a cornerstone for the structure theory of all finite-dimensional Lie algebras. It asserts that any finite-dimensional Lie algebra $\mathfrak{g}$ can be decomposed as a semi-direct sum of its radical (the unique maximal solvable ideal) and a semisimple subalgebra (known as the Levi factor).

4.1 The Baker-Campbell-Hausdorff formula

[2]

Theorem 12 (Baker-Campbell-Hausdorff formula, qualitative version) Let $V \subset \mathbf{R}^d$ be a radially homogeneous $C^{1,1}$ local group. Then the group operation $*$ is real analytic near the origin. In particular, after restricting V to a sufficiently small neighbourhood of the origin, one obtains a local Lie group.

Explanation: In geometric group theory and the study of Hilbert's Fifth Problem, a radially homogeneous $C^{1,1}$ local group refers to a local topological group equipped with a differentiable structure where the multiplication and inversion operations are locally Lipschitz continuously differentiable $C^{1,1}$.

Remark 7 In the case where V comes from viewing a general linear group $GL_n(\mathbf{C})$ in local exponential coordinates, the group operation $*$ is given by $x * y = \log(\exp(x)\exp(y))$ for sufficiently small $x, y \in M_n(\mathbf{C})$. Thus, a corollary of Theorem 12 is that this map is real analytic.

Lemma 16 (Lipschitz bounds) If $x, y, z \in V$ are sufficiently small, establish the bounds

$$x * y = x + y + O(|x + y||y|) \qquad (15)$$

$x*y = x+y+O(|x+y||x|)$ (16)
$x*y = x*z+O(|y-z|)$ (17)
and
$y*x = z*x+O(|y-z|).$ (18)
Now we exploit the radial homogeneity to describe the conjugation operation $y \mapsto x*y*(-x)$ as a linear map:
Lemma 13 (Adjoint representation) For all x sufficiently close to the origin, there exists a linear transformation $\mathrm{Ad}_x : \mathbf{R}^d \to \mathbf{R}^d$ such that $x*y*(-x) = \mathrm{Ad}_x(y)$ for all y sufficiently close to the origin.
Remark 8 Using the matrix example from Remark 7, we are asserting here that
$\exp(x)\exp(y)\exp(-x) = \exp(\mathrm{Ad}_x(y))$
for some linear transform $\mathrm{Ad}_x(y)$ of y, and all sufficiently small x, y. Indeed, using the basic matrix identity $\exp(AxA^{-1}) = A\exp(x)A^{-1}$ for invertible A (coming from the fact that the conjugation map $x \mapsto AxA^{-1}$ is a continuous ring homomorphism) we see that we may take $\mathrm{Ad}(x) = \exp(x)y\exp(-x)$ here.

$$x*(y+z)*(-x) = n(x*\frac{1}{n}y*(-x)+x*\frac{1}{n}z*(-x)+O(\frac{1}{n^2}))$$
$$= x*y*(-x)+x*z*(-x)+O(\frac{1}{n});$$

Remark 8a
From (4) we see that
$\|\mathrm{Ad}_x - I\|_{op} = O(|x|)$
for x sufficiently small. Also from the associativity property we see that
$\mathrm{Ad}_{x*y} = \mathrm{Ad}_x\mathrm{Ad}_y$ (19)
for all x, y sufficiently small. Combining these two properties (and using (15)) we conclude in particular that
$\|\mathrm{Ad}_x - \mathrm{Ad}_y\|_{op} = O(|x-y|)$ (20)
for x, y sufficiently small. Thus we see that Ad is a (locally) continuous linear representation. In particular, $t \mapsto \mathrm{Ad}_{tx}$ is a (locally) continuous homomorphism into a linear group, and so (by Proposition 1 of Notes 0) we have the *Hadamard lemma*
$\mathrm{Ad}_x = \exp(\mathrm{ad}_x)$
for all sufficiently small x, where $\mathrm{ad}_x : \mathbf{R}^d \to \mathbf{R}^d$ is the linear transformation
$\mathrm{ad}_x = \frac{d}{dt}\mathrm{Ad}_{tx}|_{t=0}.$
From te above, we see that
$\mathrm{Ad}_{tx}\mathrm{Ad}_{ty} = \mathrm{Ad}_{t(x+y)} + O(|t|^2)$
for x, y, t sufficiently small, and so by the product rule we have
$\mathrm{ad}_{x+y} = \mathrm{ad}_x + \mathrm{ad}_y.$
Also we clearly have $\mathrm{ad}_{tx} = t\mathrm{ad}_x$ for x, t small. Thus we see that ad_x is linear in x, and so we have
$\mathrm{ad}_x y = [x, y]$ (21)
for some bilinear form $[,] : \mathbf{R}^d \to \mathbf{R}^d$.
One can show that this bilinear form in fact defines a Lie bracket (i.e. it is anti-symmetric and obeys the Jacobi identity), but for now, all we need is that it is manifestly real analytic (since all bilinear forms are polynomial and thus analytic). In particular ad_x and Ad_x depend analytically on x.

Lemma 14 For x, y sufficiently small, we have
$x*y = x+F(\mathrm{Ad}_x)y+O(|y|^2)$
where
$F(z) := \frac{z\log z}{z-1}.$

Corollary 15 (Baker-Campbell-Hausdorff-Dynkin formula) For x, y sufficiently small, one has
$x*y = x+\int_0^1 F(\mathrm{Ad}_x\mathrm{Ad}_{ty})y\ dt.$

Lemma 17 From the Taylor-type expansion
$$F(z) = 1 - \frac{1/z-1}{2} + \frac{(1/z-1)^2}{3} - \frac{(1/z-1)^3}{4} + \dots$$
we obtain the explicit expansion
$$x * y = x + \sum_{n=0}^{\infty} \frac{(-1)^m}{n+1}$$
where $m := n + r_1 + \ldots + r_n + s_1 + \ldots + s_n + 1$, and show that the series is absolutely convergent for x, y small enough.
Inverting this , we obtain the alternate expansion
$$x * y = y + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

Lemma 18 Let V be a radially homogeneous $C^{1,1}$ local group. By Theorem 12, an open neighbourhood of the origin in V has the structure of a local Lie group, and thus is associated to a Lie algebra. Then this Lie algebra is isomorphic to $\mathbf{R}^d$ and the Lie bracket $[,]$ is given by the standard commutator.
Note that this establishes *a posteriori* the fact that the bracket $[,]$ is anti-symmetric and obeys the Jacobi identity.

4.2. Consequences of the Baker-Campbell-Hausdorff formula, Lie theorems
[2]
Theorem (Lie groups are analytic) Let G be a global Lie group. Then G is a real analytic manifold (i.e. one can find an atlas of smooth coordinate charts whose transition maps are all real analytic), and that the group operations are also real analytic (i.e. they are real analytic when viewed in the above-mentioned coordinate charts).
Furthermore, any continuous homomorphism between Lie groups is also real analytic.
Theorem (Lie's second theorem) Let G, H be global Lie groups, and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then there exists an open neighbourhood U of the identity in G and a homomorphism $\Phi : U \to H$ from the local Lie group $G\mid_U$ to H such that $D\Phi(1) = \phi$. If G is connected and simply connected, Then one can take U to be all of G.
Theorem (Lie's third theorem) Ado's theorem asserts that every finite-dimensional Lie algebra is isomorphic to a subalgebra of $\mathfrak{gl}_n(\mathbf{R})$ for some n. This (somewhat difficult) theorem and its proof is discussed in this previous blog post.
Assuming Ado's theorem as a "black box", we conclude the following claims:
(i) (Lie's third theorem, local version) Every finite-dimensional Lie algebra is isomorphic to the Lie algebra of some local Lie group.
(ii) Every local or global Lie group has a neighbourhood of the identity that is isomorphic to a local linear Lie group (i.e. a local Lie group contained in $GL_n(\mathbf{R})$ or $GL_n(\mathbf{C})$ for some n).
(iii) (Lie's third theorem, global version) Every finite-dimensional Lie algebra $\mathfrak{g}$ is isomorphic to the Lie algebra of some global Lie group. (Hint: from (i) and (ii), one may identify $\mathfrak{g}$ with the Lie algebra of a local linear Lie group. Now consider the space of all smooth curves in the ambient linear group that are everywhere "tangent" to this local linear Lie group modulo "homotopy", and use this to build the global Lie group.)
(iv) (Lie's third theorem, simply connected version) Every finite-dimensional Lie algebra $\mathfrak{g}$ is isomorphic to the Lie algebra of some global connected, simply connected Lie group. Furthermore, this Lie group is unique up to isomorphism.
(v) Every local Lie group G has a neighbourhood of the identity that is isomorphic to a neighbourhood of the identity of a global connected, simply connected Lie group. Furthermore, this Lie group is unique up to isomorphism.

Remark 11 Lie's three theorems can be interpreted as establishing an equivalence between three different categories: the category of finite-dimensional Lie algebras; the category of local Lie groups (or more precisely, the category of local Lie group germs, formed by identifying local Lie groups that are identical near the origin); and the category of global connected, simply connected Lie groups.

Lemma 16 (Criterion for Lie structure) Let G be a topological group. Then G is Lie if and only if there is a neighbourhood of the identity in G which is isomorphic (as a topological group) to a $C^{1,1}$ local group.
Remark 12 Informally, Lemma 16 asserts that $C^{1,1}$ regularity can automatically be upgraded to smooth (C^∞) or even real analytic (C^ω) regularity for topological groups.
Lemma 22 Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $X, Y \in \mathfrak{g}$, then
$\exp([X,Y]) = \lim_{n \to \infty} (\exp(X/n)\exp(Y/n)$
$\exp(-X/n)\exp(-Y/n))^{n^2}$.

5 Roots, weights, Cartan subalgebra, Dynkin diagrams

Root system

[1]

Let E be a finite-dimensional Euclidean vector space, with the standard Euclidean inner product denoted by (·,·). A root system Φ in E is a finite set of non-zero vectors (called roots) that satisfy the following conditions:

1 The roots span E.

2 The only scalar multiples of a root α∈Φ that belong to Φ are α itself and −α.

3 For every root α∈Φ, the set Φ is closed under reflection through the hyperplane perpendicular to α.

4 (Integrality) If α and β are roots in Φ, then the projection of β onto the line through α is an integer or half-integer multiple of α.

Equivalent ways of writing conditions 3 and 4, respectively, are as follows:

3 For any two roots α,β∈Φ, the set Φ contains the projection

$$\sigma_\alpha(\beta)=\beta-2\alpha\frac{(\alpha,\beta)}{(\alpha,\alpha)}$$

4 For any two roots α,β∈Φ , the number (double projection β onto α)

$$\langle\beta,\alpha\rangle=\alpha\frac{(\alpha,\beta)}{(\alpha,\alpha)}$$

is an integer.

The **rank** of a root system Φ is the dimension of *E*.

Two root systems (E1, Φ1) and (E2, Φ2) are called isomorphic if there is an invertible linear transformation E1 → E2 which sends Φ1 to Φ2 such that for each pair of roots, the projection ⟨x,y⟩ is preserved

The group of isometries of E generated by reflections through hyperplanes associated to the roots of Φ is called the Weyl group of Φ. As it acts faithfully on the finite set Φ, the Weyl group is always finite.

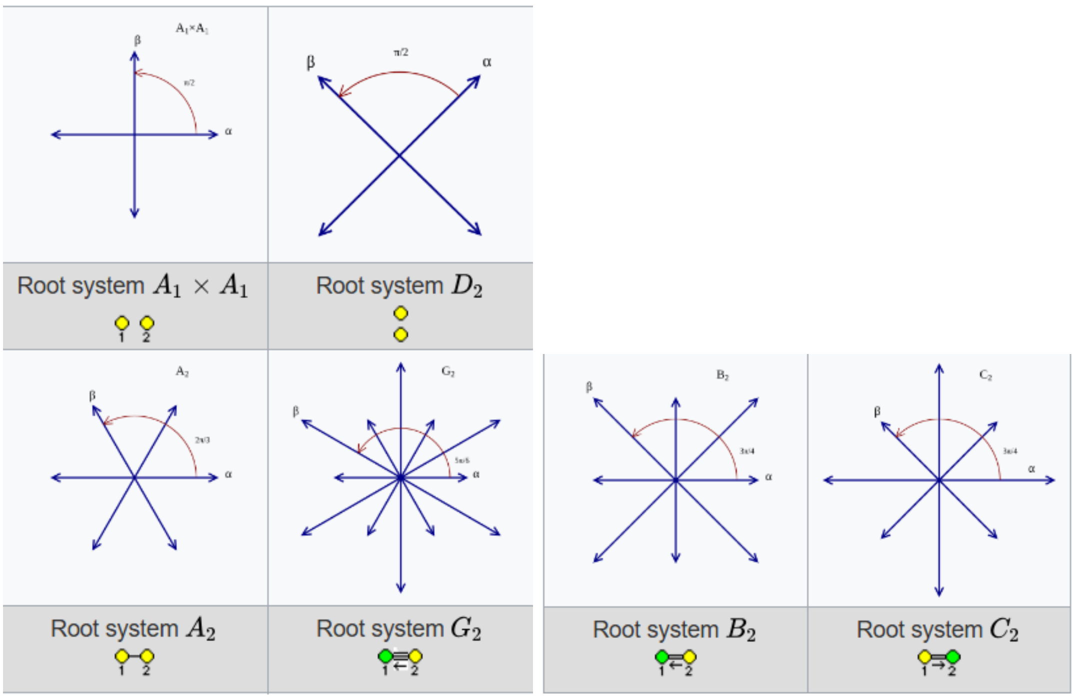
Root system rank 2

In rank 2 there are four possibilities, corresponding to $\sigma_\alpha(\beta)=\beta+n\alpha$, where n=0,1,2,3 .https://en.wikipedia.org/wiki/Root_system - cite_note-8

The root system is not determined by the lattice that it generates: A1×A1 and B2 both generate a square lattice while A2 and G2 both generate a hexagonal lattice.

Whenever Φ is a root system in E, and S is a subspace of E spanned by Ψ = Φ ∩ S, then Ψ is a root system in S. Thus, the exhaustive list of four root systems of rank 2 shows the geometric possibilities for any two roots chosen from a root system of arbitrary rank.

In particular, two such roots must meet at an angle of 0, 30, 45, 60, 90, 120, 135, 150, or 180 degrees.



Rank 2 root systems

Lie Algebras roots and weights
[3]
Lie Algebra Root
The roots of a semisimple Lie algebra g are the Lie algebra weights occurring in its adjoint representation. The set of roots form the root system, and are completely determined by g. It is possible to choose a set of Lie algebra positive roots, every root α is either positive or -α is positive. The Lie algebra simple roots are the positive roots which cannot be written as a sum of positive roots.
The simple roots can be considered as a linearly independent finite subset of Euclidean space, and they generate the root lattice.
For example, in the special Lie algebra $sl_2(C)$ of two by two matrices with zero matrix trace, has a basis given by the matrices

$$H=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad X=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad Y=\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

The adjoint representation is given by the brackets

$$ad(H)(X)=[H,X]=2X$$
$$ad(H)(Y)=[H,Y]=-2Y$$

so there are two roots of $sl_2(C)$ given by α(H)=2 and - α(H)=-2. The Lie algebraic rank of $sl_2(C)$ is one, and it has one positive root α(H)=2.
Adjoint representation
In mathematics, the adjoint representation (or adjoint action) of a Lie group G is a way of representing the elements of the group as linear transformations of the group's Lie algebra, considered as a vector space.
For example, if G =GL(n,R), the Lie group of real n-by-n invertible matrices, then the adjoint representation is the group homomorphism that sends an invertible nxn matrix g to an endomorphism of the vector space of all linear transformations of R^n defined by:

$$x\mapsto g\,x\,g^{-1}$$

Roots in ei basis
For a Lie algebra with dimension *d=2n* or *d=2n+1* (e.g. orthogonal Lie algebras so(2n) , so(2n+1), sp(2n)) we use an orthonormal basis of extended Kronecker matrices $i,j=1...n$

$$E^1_{jk}=E_{jk}-E_{k+n,j+n}\ ,\ E^2_{jk}=E_{j,k+n}-E_{k,j+n}\ ,\ E^3_{jk}=E_{j+n,k}-E_{k+n,j}\ ,\ E^3_{jk}=E_{j+n,k}-E_{k+n,j}\ ,$$

where the ordinary Kronecker matrices are $E_{ij}=\left(\delta_{ik}\delta_{jl}\right)_{kl}$.

For su(n+1), we use the basis $E^1_{jk}=E_{jk}$.

Cartan subalgebra is spanned by the diagonal operators $h=\langle E_{ii}\rangle$,

the remaining subspace is spanned by the non-diagonal operators $h_\perp=\langle E^1_{ij}\rangle\ \ i\neq j$

We make the ansatz for the Cartan subalgebra h : $H=\sum_{i=1}^{n}\lambda_i E_{ii}$ (n=dim(h)

we obtain the commutator relations

$$[H,E^1_{jk}]=(\lambda_j-\lambda_k)E^1_{jk}\ \ i\neq j$$
$$[H,E^2_{jk}]=(\lambda_j+\lambda_k)E^2_{jk}\ \ i\leq j$$
$$[H,E^3_{jk}]=-(\lambda_j+\lambda_k)E^3_{jk}\ \ i\leq j$$

For the classical Lie algebras we obtain simple root vectors e_i in the form

$$e_1=E^1_{12}\ ,\ e_2=E^1_{23}\ ,\ ...,\ e_n=f\left(E_{ij}\right)$$

with the commutators

$$r_1=[H,e_1]=(\lambda_1-\lambda_2)e_1$$
$$r_2=[H,e_2]=(\lambda_1-\lambda_2)e_2$$

...

$r_n = [H, e_n] = (c_{n-1}\lambda_{n-1} + c_n\lambda_n) e_n$
Here, the final term differs
for su(n+1) $r_n = (\lambda_n - \lambda_{n+1}) E_{n,n+1}$
for so(2n) $r_n = (\lambda_{n-1} + \lambda_n)(E_{2n-1,n} - E_{2n,n-1})$
for so(2n+1) $r_n = \lambda_n (E_{n,2n+1} - E_{2n+1,2n})$
for sp(2n) $r_n = 2\lambda_n (E_{n,2n} - E_{2n,n})$

Roots in Weyl-Cartan representation

We start for the Lie algebra *g* with the generator set $g = \langle H_1, ..., H_k, Y_1, ..., Y_{n-k} \rangle$,
where the Cartan algebra *h* is spanned by the first *k* generators $h = \langle H_1, ..., H_k \rangle$.
The form ladder operators $L_{1\pm} = Y_1 \pm Y_2, ..., L_{(n-k-1)/2\pm} = Y_{n-k-1} \pm Y_{n-k}$ and let them span the remaining subspace
$h_\perp = \langle L_{1\pm}, ..., L_{(n-k-1)/2\pm} \rangle$
The mapping $ad(H_i)(X) = [H_i, X]$ $X \in h_\perp$ maps the subspace $h_\perp$ into itself, it is a *(n-k)x(n-k)* matrix $ad(H_i)$ on the vectors of $h_\perp$. The *k* common eigenvectors of these matrices, which belong to the eigenvalues $(\lambda_{p1}(H_1), ..., \lambda_{pk}(H_k))$, *p=1,..k*
are the *k* simple roots of the Lie algebra *g*.

Lie Algebra Weight

Consider a collection of diagonal matrices $H_1,...,H_k$, which span a subspace h. Then the ith eigenvalue, i.e., the ith entry along the diagonal, is a linear functional on *h*, and is called a weight.
The general setting for weights occurs in a Lie algebra representation of a semisimple Lie algebra, in which case the Cartan subalgebra *h* is Abelian and can be put into diagonal form.
For example, consider the standard representation of the special linear Lie algebra sl3(C) on C^3.

Then $H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

span the Cartan subalgebra *h*.
There are three eigenvectors (standard orthonormal basis) e_i with $e_i[j] = \delta_{ij}$
There are three weights,
$\alpha = (1, -2, 1)$,
which are the eigenvalues of $H_1 + H_2$,
corresponding to the decomposition of
$C^3 = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \langle e_3 \rangle$
into its eigenspaces.

Note that $\alpha_1+\alpha_2+\alpha_3=0$, because the matrices have zero matrix trace. The eigenvectors e_1, e_2, e_3 are called weight vectors, and the corresponding eigenspaces are called weight spaces.
In the important special case of the adjoint representation of a semisimple Lie algebra, the weights are called Lie algebra roots and the weight space is called the root space. The roots generate a discrete lattice, called the root lattice, in the dual vector space *h**. The set of all possible weights forms a weight lattice, which contains the root lattice. The Lie Algebra representations of g can be classified using the weight lattice.
Cartan matrix
A Cartan matrix $A=(a_{ij})$ is an n×n integer matrix that uniquely encodes the structure and root system of a semisimple Lie algebra. It captures the angular and length relationships between the simple roots.
For the simple roots (α_i, α_j) the matrix entries are defined as:

$$a_{ij}=\frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$$

- Diagonal entries are always $a_{ij}=2$.
- Off-diagonal entries are non-positive integers $a_{ij}\le 0 \quad i\ne j$ and represent how roots reflect into one another.
- The product $a_{ij}\, a_{ji}$ can only equal 0,1,2,3 .

The Cartan matrices of the types An, Bn, Cn, Dn are
- A_n(su(n+1))

$$\begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 \\ -1 & 2 & -1 & ... & 0 & 0 \\ 0 & -1 & -2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

- B_n(so(2n+1))

$$\begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 \\ -1 & 2 & -1 & ... & 0 & 0 \\ 0 & -1 & -2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

- C_n(sp(2n))

$$\begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 \\ -1 & 2 & -1 & ... & 0 & 0 \\ 0 & -1 & -2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{pmatrix}$$

- D_n(so(2n))

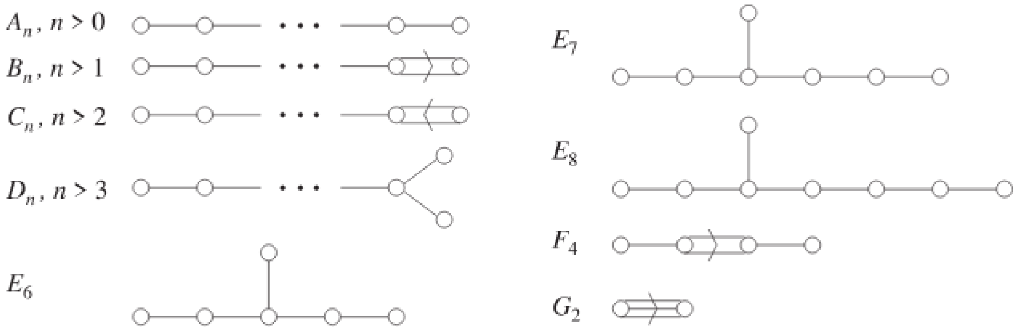
$$\begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 & 0 \\ -1 & 2 & -1 & ... & 0 & 0 & 0 \\ 0 & -1 & -2 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 2 & -1 & -1 \\ 0 & 0 & 0 & ... & -1 & 2 & 0 \\ 0 & 0 & 0 & ... & -1 & 0 & 2 \end{pmatrix}$$

Five exceptional algebras:
- E6: A (6 times 6) matrix characterized by a "Y" shape on its Dynkin diagram with a central branch.
- E7: A (7 times 7) matrix corresponding to a 7-vertex diagram.
- E8: An (8 times 8) matrix associated with the largest simple exceptional Lie group.
- F4: A (4 times 4) matrix corresponding to the 24-dimensional exceptional Lie algebra.
- G2: The smallest exceptional Lie algebra (14-dimensional), whose Cartan matrix is $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

Simple roots and weights
In general, for a simple Lie algebra g, the relation between the simple roots α_i and the fundamental weights ω_j reads
α_i = j A_{ij} ω_j ,
where A is the Cartan matrix of g

Cartan Subalgebra
Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over some field *K*.
A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a Cartan subalgebra if it is nilpotent and equal to its normalizer, which is the set of those elements $x \in \mathfrak{g}$ such that $[x, \mathfrak{h}] \subset \mathfrak{h}$.
Lie algebra $\mathfrak{g}$ is nilpotent if its lower central series terminates in the zero subalgebra. The lower central series is the sequence of subalgebras
$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supseteq \dots$$
Let $\mathfrak{g}$ be the Lie algebra of all endomorphisms *f* of vector space k^n (for some natural number n), with
$$[f, g] = f \circ g - g \circ f$$
Then the set of all endomorphisms *f* of k^n of the form
$$f(x_1, \dots, x_n) = (\lambda_1 x_1, \dots, \lambda_n x_n)$$
$$f\ (x_1,\ \dots,\ x_n) = (\lambda_1\ x_1,\ \dots,\ \lambda_n\ x_n)$$
is a Cartan subalgebra of $\mathfrak{g}$.
Properties of Cartan subalgebra
1. If *K* is infinite, then $\mathfrak{g}$ has Cartan subalgebras.
2. If the characteristic of *K* is equal to 0, then all Cartan subalgebras of $\mathfrak{g}$ have the same dimension.
3. If *K* is algebraically closed and its characteristic is equal to 0, then, given two Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ of $\mathfrak{g}$, there is an automorphism *f* of $\mathfrak{g}$ such that $f(\mathfrak{h}) = \mathfrak{h}'$ f(h)=h'.
4. If $\mathfrak{g}$ is semisimple and *K* is an infinite field whose characteristic is equal to 0, then all Cartan subalgebras of $\mathfrak{g}$ are Abelian.
5. Over field C, a Cartan subalgebra is precisely equivalent to a maximal commutative (or abelian) subalgebra.
6. In general, if a maximal commutative subalgebra contains non-semisimple (nilpotent) elements, it fails to be a Cartan subalgebra.
7. Every Cartan subalgebra of a Lie algebra $\mathfrak{g}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$. However, a maximal nilpotent subalgebra of $\mathfrak{g}$ doesn't have to be a Cartan subalgebra.
Semisimple Lie Algebra
A Lie algebra over a field of characteristic zero is called semisimple if its Killing form is nondegenerate. The following properties can be proved equivalent for a finite-dimensional algebra L over a field of characteristic 0:
1. L is semisimple.
2. L has no nonzero Abelian ideal.
3. L has zero ideal radical (the radical is the biggest solvable ideal).
4. Every representation of L is fully reducible, i.e., is a sum of irreducible representations.
5. L is a (finite) direct product of simple Lie algebras (Lie algebra is called simple if it is not Abelian and has no nonzero ideal smaller than L).
The radical of an ideal $\mathfrak{a}$ in a ring *R* is the ideal $r(\mathfrak{a}) = \{x; \exists n > 0 : x^n \in \mathfrak{a}\}$

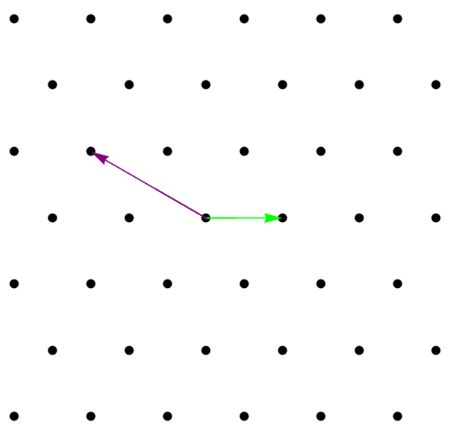
Dynkin Diagram



Every semisimple Lie algebra $\mathfrak{g}$ is classified by its Dynkin diagram. A Dynkin diagram is a graph with a few different kinds of possible edges. The connected components of the graph correspond to the irreducible subalgebras of $\mathfrak{g}$. So a simple Lie algebra's Dynkin diagram has only one component.
In fact, there are only certain possibilities for each component, corresponding to the classification of semi-simple Lie algebras.
The roots of a complex Lie algebra form a lattice of rank *k* in a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, where *k* is the Lie algebra rank of $\mathfrak{g}$. Hence, the root lattice can be considered a lattice in R^k.
A vertex, or node, in the Dynkin diagram is drawn for each Lie algebra simple root, which corresponds to a generator of the root lattice. Between two nodes α and β, an edge is drawn if the simple roots are not perpendicular.
One line is drawn if the angle θ between them is $2\pi/3$, two lines if the angle is $3\pi/4$, and three lines are drawn if the angle is $5\pi/6$. There are no other possible angles between Lie algebra simple roots.
Alternatively, the number of lines *N* between the simple roots α and β is given by

$$N = A_{\alpha\beta} A_{\beta\alpha} = \frac{2\langle\alpha,\beta\rangle}{|\alpha|^2}\frac{2\langle\beta,\alpha\rangle}{|\beta|^2} = \cos^2\theta$$

where $A_{\alpha\beta}$ is an entry in the Cartan matrix. In a Dynkin diagram, an arrow is drawn from the longer root to the shorter root (when the angle θ is $3\pi/4$ or $5\pi/6$).



The picture above shows the two simple roots for G_2, at an angle of $5\pi/6$, in the root lattice. Therefore, the Dynkin diagram for G_2 has two nodes, with three lines between them.
Here are some properties of admissible Dynkin diagrams.
Recover a simple Lie algebra from its Dynkin diagram
The simplest way to recover a simple Lie algebra from its Dynkin diagram is to first reconstruct its Cartan matrix A_{ij}. The ith node and jth node are connected by $A_{ij}A_{ji}$ lines. Since $A_{ij} = 0$ iff $A_{ji} = 0$, and otherwise $A_{ij} \in \{-3,-2,-1\}$, it is easy to find A_{ij} and A_{ji}, up to order, from their product.
The arrow in the diagram indicates which is larger. For example, if node 1 and node 2 have two lines between them, from node 1 to node 2, then $A_{12} = -1$ and $A_{21} = -2$.

Weyl Group

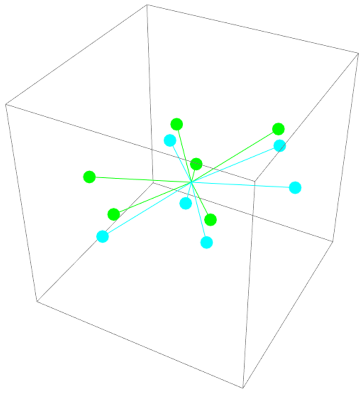
Let L be a finite-dimensional split semisimple Lie algebra over a field K of field characteristic 0, H a splitting Cartan subalgebra, and Λ a weight of H in a representation of L. Then

$$\Lambda' = \Lambda S_\alpha = \lambda - \frac{2(\Lambda,\alpha)}{(\alpha,\alpha)}(\alpha)$$

$$\Lambda' = \Lambda S_\alpha = \lambda - \frac{2\,(\Lambda,\,\alpha)}{(\alpha,\,\alpha)}\,(\alpha)$$

is also a weight. Furthermore, the reflections S_α with root α, generate a group of linear transformations in H_0^* called the Weyl group W of L relative to H, where H^* is the algebraic conjugate space of H and H_0^* is the Q-space spanned by the roots.

The Weyl group acts on the roots of a semisimple Lie algebra, and it is a finite group.



Weyl Group acting on the roots of the Cartan matrix of the infinite family of semisimple lie algebras A_3

Calculation of the simple roots of a simple Lie algebra

To calculate the simple roots of a simple Lie algebra, you find a linearly independent subset of the algebra's roots that cannot be written as the sum of other positive roots. These simple roots act as a basis for the root system, meaning every other root can be expressed as a linear combination of these roots with non-negative integer coefficients.

The number of simple roots will always equal the rank of the Lie algebra (the dimension of the Cartan subalgebra).

For standard classical Lie algebras (e.g. A_i , B_i , C_i , D_i the simple roots are commonly defined using the standard orthonormal basis vectors e_i with factors λ_i

- An =su(n+1) The simple roots are $\alpha_i = \lambda_i - \lambda_{i+1}$ for $i=1,\ldots,n$
- Bn =so(2n+1) The simple roots are $\alpha_i = \lambda_i - \lambda_{i+1}$ for $i=1,\ldots,n-1$; $\alpha_n = \lambda_n$
- Cn =sp(2n) The simple roots are $\alpha_i = \lambda_i - \lambda_{i+1}$ for $i=1,\ldots,n-1$; $\alpha_n = 2\lambda_n$
- Dn =so(2n) The simple roots are $\alpha_i = \lambda_i - \lambda_{i+1}$ for $i=1,\ldots,n-1$; $\alpha_n = \lambda_{n-1} + \lambda_n$

6 Classical and exceptional Lie algebras

Four classical Lie algebras

The classical Lie algebras are finite-dimensional Lie algebras over a field which can be classified into four types An , Bn , Cn and Dn , where gl(n) is the general linear Lie algebra and I_n = n×n identity matrix

An =su(n+1)=sl(n+1,C) $=\left\{x\in gl\left(n+1,C\right),\bar{x}^T=-x,tr\left(x\right)=0\right\}$ the special linear complex Lie algebra

Bn:=so*(2n+1)*= $\left\{x\in gl\left(2n+1\right),x^T=-x\right\}$, the odd orthogonal Lie algebra;

Cn:=*sp(2n)*= $\left\{x\in gl\left(2n\right),x^TJ_n=-J_nx,J_n=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}\right\}$, the symplectic Lie algebra;

Dn:=so*(2n)*= $\left\{x\in gl\left(2n\right),x^T=-x\right\}$ the even orthogonal Lie algebra.

Except for the low-dimensional cases D_1=so(2) and D_2=so(4) , the classical Lie algebras are simple .

Five Exceptional Lie Algebras

- g_2 (Dimension 14): The smallest of the exceptional Lie algebras, it is the automorphism group of the octonions and the stabilizer of a generic 3-form in 7 dimensions.
- f_4 (Dimension 52): The Lie algebra of the exceptional group F_4, it is related to the symmetries of the Albert algebra (a 27-dimensional exceptional Jordan algebra).
- e_6 (Dimension 78): Can be understood through the stabilizer of a certain cubic form and appears in the classification of certain projective planes over division algebras.
- e_7 (Dimension 133): Leaves invariant a symmetric rank-4 tensor in its fundamental 56-dimensional representation.
- e_8 (Dimension 248): The largest of the exceptional Lie algebras. It is heavily associated with the E_8 lattice, which models the densest possible sphere packing in 8-dimensional space.

6.1 General examples

- Four-dim nilpotent Lie-algebra

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Every 4-dimensional nilpotent Lie algebra over a field K of characteristic zero is either abelian or isomorphic to the direct sum $n_3(K)\oplus K$, where $n_3(K)$ denotes the 3-dimensional Heisenberg Lie algebra, or to the filiform nilpotent Lie algebra $f_4(K)$ with adapted basis $(e_1,\dots e_4)$ and brackets determined by $[e_1,e_i]=e_{i+1}$ for i=2,3. This is well known and not difficult to show. Obviously, both algebras contain an abelian ideal of dimension 3. For example, $\langle e_2,e_3,e_4\rangle$ is an abelian ideal in $f_4(K)$.

- **Lie4(ε) algebra**

The 4 generators T_i are $\left(T_j\right)_{ik}=\varepsilon_{ijk}$.

Commutator relations are

$\left[T_k,T_m\right]=\varepsilon_{kmi}T_i$

where ε_{kmi} is the 3-dimensional Levi-Civitta (antisymmetric) 4x4x4 tensor with indices k,m,i=1,...,4 .

The structure constants $c^i_{km}=\varepsilon_{kmi}$ satisfy

antisymmetry $c^i_{km}=-c^i_{mk}$

and Jacobi-identity $c^m_{ij}c^k_{mk}+c^m_{jk}c^i_{mi}+c^m_{ki}c^j_{mj}=0$

so they define a Lie algebra [1].

However, the four generators T_i are linearly dependent, so the Lie algebra is 3-dimensional and has to be reformulated.

The 4x4 Killing matrix is $\begin{pmatrix}-6 & -2 & 2 & -2\\ -2 & -6 & -2 & 2\\ 2 & -2 & -6 & -2\\ -2 & 2 & -2 & -6\end{pmatrix}$ [6]

has eigenvalues (-8,-8,-8,0),

so the reduced 3x3 Killing matrix has eigenvalues (-8,-8,-8)

- **3-dimensional Lie3(ε) algebra**

Lie3(ε) is a reformulation of the Lie4(ε) algebra, which results from the replacement of the generator T_4 by a linear combination of (T_1,T_2,T_3) .
Commutator relations are
$[T_1,T_2]=T_1-T_2+2T_3$, $[T_1,T_3]=T_1-2T_2+T_3$, $[T_2,T_3]=2T_1-T_2+T_3$
with generic formula $[T_a,T_b]=\sum_i(1+\varepsilon_{abi})T_i$

The 3-dimensional Lie algebra defined by these commutator relations is isomorphic to so(3) over reals R or sl(2,C) over complex nmbers C.

The Killing matrix is $\begin{pmatrix} -6 & -2 & 2 \\ -2 & -6 & -2 \\ 2 & -2 & -6 \end{pmatrix}$ and has eigenvalues (-8,-8,-2).

Characterization of Lie3(ε)

The derived algebra [g,g] in orthonormal basis e_i is spanned by the vectors
$e_1-e_2+2e_3$, $e_1-2e_2+e_3$, $2e_1-e_2+e_3$

The corresponding coefficient matrix is $M=\begin{pmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$, det(M)=4, so dim([g,g])=3 .
Killing form is negative-definite, and by Cartan's criterion, a semisimple real Lie algebra is compact if and only if its Killing form is negative-definite.
Cartan subalgebra has dim=1, the simple root is λ_1, the 2 roots are $\pm\lambda_1$
The unique 3-dimensional compact simple real Lie algebra is so(3), isomorphic to su(2).

- **Heisenberg algebra**

The Lie algebra $\mathfrak{h}$ of the Heisenberg group H (over the real numbers) is known as the Heisenberg algebra.
It may be represented using the space of 3×3 matrices of the form
$$\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$
with a,b,c∈R.
The following three generators form a basis for $\mathfrak{h}$:
$$x=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ p=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ z=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
they are all nilpotent $p^2=y^2=z^2=0$
These basis elements satisfy the commutation relations
$[x,p]=z$, $[x,z]=0$, $[p,z]=0$
Cartan subalgebra is generated by Z, so rank(H)=1.
The Heisenberg algebra H is the Lie algebra of the conjugated variables in quantum mechanics, i.e. it has the Heisenberg commutation relations
$[x,p]=i\hbar I$, $[x,i\hbar I]=0$, $[p,i\hbar I]=0$
where x,p are the position resp. the momentum operator, and $\hbar$ is the Planck constant, $z=i\hbar I$ is the Cartan generator (i.e. a multiple of the identity operator).
The Heisenberg group generated by the 3 generators has the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

with a,b,c∈R.

The Heisenberg group H has the special property that the exponential map is a bijective (one-to-one) map from the Lie algebra $\mathfrak{h}$ to the Heisenberg group H

$$\exp\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a & c+ab/2 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

• Heisenberg algebra in R^{n+2}

Heisenberg algebra with dimension $2n+1$ has the form of (n+2)x(n+2) matrices

$$\begin{pmatrix} 0 & a & c \\ 0_n & 0_{nxn} & b \\ 0 & 0_n & 0 \end{pmatrix}$$ with n-vectors a, b, and scalar c ∈R .

With the canonical basis $(e_1, e_2, ..., e_n)$ in R^n , we obtain for the generators

$$x_i = \begin{pmatrix} 0 & e_i & 0 \\ 0_n & 0_{nxn} & 0_n \\ 0 & 0_n & 0 \end{pmatrix} \;,\; p_i = \begin{pmatrix} 0 & 0_n & 0 \\ 0_n & 0_{nxn} & e_i \\ 0 & 0_n & 0 \end{pmatrix} \;,\; z = \begin{pmatrix} 0 & 0_n & 1 \\ 0_n & 0_{nxn} & 0_n \\ 0 & 0_n & 0 \end{pmatrix}$$

with the commutators

$$\left[x_i, p_j\right] = \delta_{ij} z \;,\; \left[p_j, z\right] = 0 \;,\; \left[x_i, z\right] = 0$$

Here, p_i are the generalized momentums, and x_i are the generalized positions, z is the Cartan operator corresponding to $i\hbar I$ in quantum mechanics.

• Heisenberg algebra in quantum mechanics

The generalized momentums p_i are derived from the Lagrangian L and have the properties

$$p_i = \frac{\partial L}{\partial\left(\partial x_i / \partial t\right)}$$, Euler-Lagrange equation: $$\frac{\partial}{\partial t} p_i = \frac{\partial L}{\partial x_i}$$

The Heisenberg algebra yields for them the commutator relations

$$\left[x_i, p_j\right] = i\hbar\delta_{ij}$$

and the uncertainty relations $\sigma(x_i)\sigma(p_j) \geq \frac{\hbar}{2}\delta_{ij}$ where $\sigma(x) = \sqrt{\left\langle (x-\bar{x})^2 \right\rangle}$ is the standard deviation.

6.2 Special linear su-Lie-algebras su(n+1), Cartan type An
[4]
The special unitary Lie algebra su(n+1) consists of (n+1)x(n+1) skew-Hermitian matrices with a trace of zero. It is the compact real form of the complex special linear Lie algebra sl(n+1,C). Its Cartan structure corresponds to the classical An.
su(n+1) is compact and semi-simple.
su(n+1) has the properties:
-dim=$(n+1)^2$-1
-rank=n , dimension of the Cartan subalgebra
-the n(n+1) roots are all permutations of (1, -1, 0, ..., 0)
The simple roots are $\alpha_i=e_i - e_{i+1}$ for i=1,...,n+1
The generators are traceless skew-symmetric matrices Ti , i=1...$(n+1)^2$-1
with commutator relations $\left[T_a,T_b\right]=c_{ab}^k T_k$
In the $((n+1)^2 - 1)$-dimensional adjoint representation, the generators are represented by $((n+1)^2 - 1) \times ((n+1)^2 - 1)$ matrices, whose elements are defined by the structure constants themselves.
$\left(T_a\right)_{jk} = c_{jk}^a$
The Killing form is $\kappa\left(x,y\right)=2\left(n+1\right)tr\left(x\,y\right)$
Roots in ei-basis
We use as basis the matrices $E_{ij}=\left(\delta_{ik}\delta_{jl}\right)_{kl}$.
We make the ansatz for the Cartan subalgebra $H=\sum_{i=1}^{n+1}\lambda_i E_{ii}$, $\sum_{i=1}^{n+1}\lambda_i=0$,
and for the root vectors $e_i=E_{i,i+1}$, *i=1...n*
We obtain the commutator relations
$\left[H,e_i\right]=\left(\lambda_i-\lambda_{i+1}\right)e_i$
and the simple roots $\alpha_i=\left(\lambda_i-\lambda_{i+1}\right)$, *i=1...n*

- **su(2) algebra**

A Lie algebra with structure constants proportional to the totally antisymmetric tensor ε_{abc} is the su(2) algebra. Its generators satisfy $\left[T_a,T_b\right]=2i\varepsilon_{abc}T_c$, where ε_{abc} is the Levi-Civita symbol. This algebra describes the rotational symmetry of 3D space and the spin of quantum particles.
The generators T_a are the Pauli matrices σ_i
$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$
The rank is 1, meaning the maximal abelian subalgebra (Cartan subalgebra h) is 1-dimensional.
Cartan generator: It is typically chosen as the diagonal Pauli matrix:
$H=\sigma_z$
The remaining dimensions of the 3-dimensional algebra form the root spaces. They are spanned by the ladder (raising and lowering) operators
raising operator $E=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $E=\sigma_x+i\,\sigma_y$
lowering operator $F=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $F=\sigma_x-i\,\sigma_y$
The action of H on E, F is
$[H,E]=2E$, $[H,F]=-2F$
The roots in Weyl-Cartan representation are $\alpha_i=\left\{+2,-2\right\}$, normalized simple root $\alpha=1$ (one-dimensional)
In orthonormal basis we have : the simple root is $\alpha_1=\lambda_1$, the 2 roots are $\pm\lambda_1$

- **su(3) algebra**

[4]

The su(3) Lie algebra is the real vector space of 3 x 3 skew-Hermitian, traceless matrices. It is an 8-dimensional Lie algebra of rank 2. Its generators are $T_a = i\,\lambda_a/2$, a=1...8, where λ_a are the eight traceless and Hermitian Gell-Mann matrices. The fundamental representations are used heavily in particle physics, such as the Eightfold Way and Quantum Chromodynamics.

The commutator relations are $[T_a, T_b] = c_{ab}^{k} T_k$

The 8 Gell-Mann matrices λ_a are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The structure constants are [6]

$$\begin{pmatrix}
\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\-1\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\-\frac{1}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\\frac{1}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\-\frac{1}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\\frac{1}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}-1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\-\frac{1}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\-\frac{1}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\\frac{1}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\\frac{1}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\-1\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\-\frac{1}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\\frac{1}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\\frac{1}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\-\frac{1}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\0\\0\\0\\0\\\frac{1}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\\frac{1}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\\frac{1}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\-\frac{1}{2}\\0\\0\\0\\0\\-\frac{\sqrt{3}}{2}\end{pmatrix} & \begin{pmatrix}0\\-\frac{1}{2}\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}-\frac{1}{2}\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\\frac{\sqrt{3}}{2}\\0\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\0\\0\\0\\-\frac{1}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\\frac{1}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\-\frac{1}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\\frac{1}{2}\\0\\0\\0\\0\\\frac{\sqrt{3}}{2}\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}\frac{1}{2}\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\-\frac{1}{2}\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\-\frac{\sqrt{3}}{2}\\0\\0\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\0\\0\\\frac{1}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\-\frac{1}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\-\frac{1}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\\frac{1}{2}\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}-\frac{1}{2}\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\\frac{1}{2}\\0\\0\\0\\0\\-\frac{\sqrt{3}}{2}\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\\frac{\sqrt{3}}{2}\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\0\\-\frac{1}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\-\frac{1}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\\frac{1}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}\frac{1}{2}\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\\frac{1}{2}\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\-\frac{1}{2}\\0\\0\\0\\0\\\frac{\sqrt{3}}{2}\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\-\frac{\sqrt{3}}{2}\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\-\frac{\sqrt{3}}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\\frac{\sqrt{3}}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\-\frac{\sqrt{3}}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\\frac{\sqrt{3}}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}
\end{pmatrix}$$

The root system for the su(3) Lie algebra is of type A2.

It consists of 6 non-zero roots spanning a two-dimensional Cartan subalgebra.
Geometrically, these roots point to the vertices of a regular hexagon, with two simple roots α1 and α2 subtending an angle of π/3 .
- Rank: n-1 = 2
- Dimension: n^2-1 = 8 generators (2 Cartan diagonal generators, 6 non-diagonal root generators)

Cartan matrix $A_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ is $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Roots in Cartan-Weyl basis

Cartan-Weyl basis with integer coefficients is (2 Cartan operators $\hat{I}_3, \hat{Y}$, and 2x3 ladder operators $\hat{I}_\pm$, $\hat{U}_\pm$, $\hat{V}_\pm$)

$\hat{F}_i = \frac{1}{2}\lambda_i$, $\begin{matrix} \hat{V}_\pm = \hat{F}_4 \pm i\hat{F}_5 \\ \hat{U}_\pm = \hat{F}_6 \pm i\hat{F}_7 \end{matrix}$, $\hat{I}_\pm = \hat{F}_1 \pm i\hat{F}_2$

$\hat{Y} = 2\hat{F}_8$, $\hat{I}_3 = 2\hat{F}_3$

has the commutator action of the two Cartan operators $\hat{Y}$, $\hat{I}_3$

$\left[\hat{Y},\hat{I}_\pm\right] = 0$, $\left[\hat{Y},\hat{U}_\pm\right] = \pm\sqrt{3}\hat{U}_\pm$, $\left[\hat{Y},\hat{V}_\pm\right] = \pm\sqrt{3}\hat{V}_\pm$

$\left[\hat{I}_3,\hat{I}_\pm\right] = \pm 2\hat{I}_\pm$, $\left[\hat{I}_3,\hat{U}_\pm\right] = \mp\hat{U}_\pm$, $\left[\hat{I}_3,\hat{V}_\pm\right] = \pm\hat{V}_\pm$

2 action vectors of Cartan operators on ladder operators (3-vectors) [6]

raw simple roots (3-vectors) are , $\beta_1 = (2,-1,1)$ $\beta_2 = \sqrt{3}(0,1,1)$ [6]

from this we obtain 2 simple roots

$\alpha_1 = (2,0)$, $\alpha_2 = \left(-1,\sqrt{3}\right)$

Positive Roots: α1, α2, (α1+ α2)
Negative Roots: -α1, -α2, -(α1+ α2)

Roots representation in e_i-basis

the 2 simple roots are $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_2 - \lambda_3$

the 6 roots are $\pm(\lambda_1 - \lambda_2)$, $\pm(\lambda_1 - \lambda_3)$, $\pm(\lambda_2 - \lambda_3)$

The Killing matrix [6] $\kappa_{ij} = c^a_{ib} c^b_{ja}$ of su(3) is $\kappa_{ij} = -diag_8\left(\{3,...,3\}\right)$

- **su(4) algebra**

[4]
The su(4) Lie algebra is the mathematical framework for the special unitary group SU(4).
It is a 15-dimensional real Lie algebra consisting of 4 × 4 traceless, skew-Hermitian matrices.
In physics, it is most commonly represented using traceless Hermitian matrices. It is isomorphic to the orthogonal algebra so(6) .
The dimension is n=15 .
The generators are $T_a = i\,\lambda_a/2$, a=1...15, where λ_a a=1...15 are extended 4 x 4 Gell-Mann matrices .
Exponentiating these generators yields the continuous group transformations of SU(4) .
The commutator relations are $\left[T_a, T_b\right] = c^k_{ab} T_k$.

su(4) is a compact, semi-simple Lie algebra with properties:
- Rank 3: There are 3 simultaneously diagonalizable matrices in the Cartan Subalgebra , denoted H_1, H_2, H_3.
- 12 Roots: The remaining 12 generators act as raising and lowering operators.
- Dynkin Diagram: Its roots map geometrically into R^3, structurally identical to the A_3 Dynkin diagram, which consists of three interconnected nodes.

The λ_a a=1...15 are extended 4 x 4 Gell-Mann matrices
for su(2) subgroup a=1...3

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

remaining Gell-Mann matrices a=4...8

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

remaining index-4 matrices a=9...15

$$\lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\lambda_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

The first 3 structure constant matrices are [6]

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\begin{pmatrix}
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0 \\
0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0 \\
0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\begin{pmatrix}
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

Cartan matrix $A_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ is $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Root system Weyl-Cartan representation

We obtain from the 3x2x6 commutator relations with 2x6 ladder operators [6]:
3 action vectors of Cartan operators on ladder operators (6-vectors)

$\beta_1 = (2,1,-1,1,-1,0)$,

$\beta_2 = \sqrt{3}(0,1,1,1/3,1/3,-2/3)$,

$\beta_3 = \left(\sqrt{1/6}+\sqrt{3/2}\right)(0,0,0,1,1,1)$

we obtain 3 simple root vectors α1 α2 α3 in Weyl-Cartan representation

$\alpha_i = \left((2,0,0),(1,1/\sqrt{3},0),(1,1/\sqrt{3},\gamma)\right)$, $\gamma = \left(\sqrt{1/6}+\sqrt{3/2}\right)$

All 12 roots are $\pm\{\alpha_1,\alpha_2,\alpha_3,\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_1+\alpha_2+\alpha_3\}$

Representation in ei-basis

In orthonormal basis $\{e_1,e_2,e_3,e_4\}$ we obtain:

the 3 simple roots are $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_2 - \lambda_3$, $\alpha_3 = \lambda_3 - \lambda_4$,

the 12 roots are $\pm(\lambda_1-\lambda_2)$, $\pm(\lambda_1-\lambda_3)$, $\pm(\lambda_1-\lambda_4)$, $\pm(\lambda_2-\lambda_3)$, $\pm(\lambda_2-\lambda_4)$, $\pm(\lambda_3-\lambda_4)$

The Killing matrix [6] $\kappa_{ij} = c_{ib}^{a} c_{ja}^{b}$ of su(4) is $\kappa_{ij} = -diag_{15}(\{4,...,4\})$

- **so(1,3) Lie algebra**

The Lie algebra of the Lorentz group O(1,3) is a 6-dimensional, non-compact, and semi-simple real Lie algebra denoted as so(1,3). It consists of infinitesimal Lorentz transformations, which are generated by three spatial rotations Ji and three velocity boosts Ki .
It is semi-simple and non-compact.
Complex $so(1,3)_C$ is isomorphic to direct sum of sl(2,C) , $so(1,3)_C \cong sl(2,C) \oplus sl(2,C)$
The Killing form is indefinite rather than negative-definite, it defines a hyperbolic space, which is not bounded, therefore so(1,3) is non-compact.
The matrices X of so(1,3) satisfy the condition $X^T\eta + \eta X = 0$

With the matrices M_{kl} , $(M_{kl})_{mn} = \eta_{km}\delta_{ln} - \eta_{kn}\delta_{lm}$, k,l=1,...4, k<l ,

and the matrices $\bar{M}_{kl}$, $(\bar{M}_{kl})_{mn} = \eta_{km}\delta_{ln} + \eta_{kn}\delta_{lm}$, k,l=1,...4, k<l ,

the generators are [8]

$J_i = -(M_{34}, M_{42}, M_{23})$, $K_i = (\bar{M}_{12}, \bar{M}_{13}, \bar{M}_{14})$

$$J_1 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}$$

with the commutators $[J_i, J_j] = \varepsilon_{ijk} J_k$, $[K_i, K_j] = -\varepsilon_{ijk} J_k$, $[J_i, K_j] = \varepsilon_{ijk} K_k$

Ki-commutator differs in sign from the corresponding relation in the so(4) algebra.

Cartan subalgebra $h = \langle J_1, K_1 \rangle$ is 2-dimensional (so(1,3) has rank 2), type A1

In Weyl-Cartan representation, the simple roots are $\alpha_1 = (i,1)$, $\alpha_2 = (1,i)$ [6].

In the ei-basis the simple roots are $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_1 + \lambda_2$

The 4 roots are $\pm\lambda_1 \pm \lambda_2$,

Killing matrix is $\kappa_{ij} = diag_6(\{-4,-4,-4,4,4,4\})$

6.3 Odd orthogonal so-Lie-algebras so(2n+1) , Cartan type Bn
[4]
The odd-dimensional orthogonal Lie algebras so(2n+1) compact simple Lie algebras of Cartan type Bn. They correspond to the Lie algebras of the Special Orthogonal groups SO(2n+1) and consist of skew-symmetric (2n+1) × (2n+1) matrices, the dimension of so(2n+1) is n(2n+1) .
The rank of the algebra is n, which the dimension of the Cartan subalgebra.
Generators are M_{kl} , $(M_{kl})_{mn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$, k,l=1,...2n+1, k<l
with commutators $[M_{mn}, M_{rs}] = \delta_{nr}M_{ms} - \delta_{mr}M_{ns} + \delta_{ms}M_{nr} - \delta_{ns}M_{mr}$
Cartan subalgebra is spanned by $H_i = M_{2i-1,1}$ i=1,...,n
Roots in ei-basis
with index 0=2n+1;
We use the Kronecker matrices $E_{ij} = (\delta_{ik}\delta_{jl})_{kl}$, and we use as basis the extended Kronecker matrices
$E^1_{jk} = E_{jk} - E_{k+n,j+n}$, $E^2_{jk} = E_{j,k+n} - E_{k,j+n}$, $E^3_{jk} = E_{j+n,k} - E_{k+n,j}$, $E^3_{jk} = E_{j+n,k} - E_{k+n,j}$,
$E^4_j = E_{0j} - E_{j+n,0}$, $E^5_j = E_{j0} - E_{0,j+n}$
We make the ansatz for the Cartan subalgebra $H = \sum_{i=1}^{n} \lambda_i E_{ii}$,
we obtain the commutator relations
$[H, E^1_{jk}] = (\lambda_j - \lambda_k)E^1_{jk} \quad i \neq j$
$[H, E^2_{jk}] = (\lambda_j + \lambda_k)E^2_{jk} \quad i \leq j$
$[H, E^3_{jk}] = -(\lambda_j + \lambda_k)E^3_{jk} \quad i \leq j$
$[H, E^4_j] = -\lambda_j E^4_j$
$[H, E^5_j] = \lambda_j E^5_j$
and for the root vectors $e_i = E^1_{i,i+1}$, *i=1...n-1* , $e_n = E^5_n$
The simple root vectors are then
$(\lambda_1 - \lambda_2)E^1_{12}$
$(\lambda_2 - \lambda_3)E^1_{23}$
...
$\lambda_n E^5_n$
with n simple roots $\alpha_i = (\lambda_i - \lambda_{i+1})$, *i=1...n-1* , $\alpha_n = \lambda_n$
$2n^2$ roots: $\pm\lambda_i \pm \lambda_j$, *1<=i<j<=n* , $\pm\lambda_i$ *1<=i<=n*

- **so(5) Lie algebra**

The so(5) Lie algebra is the real, 10-dimensional Lie algebra associated with the 5-dimensional special orthogonal group SO(5), which describes rotations in 5-dimensional Euclidean space. It is a classical simple Lie algebra of rank 2 and is mathematically isomorphic to the symplectic Lie algebra sp(4).
The standard representation is by 5x5 real, skew-symmetric matrices.
It is of Cartan type B2, its Cartan subalgebra has dimension 2, i.e. it has rank=2.
The generator basis for so(5) consists of the 10 matrices S_{kl} k,l=1,...5, k<l
$(M_{kl})_{mn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$
$M_{12}, M_{13}, M_{14}, M_{15}, M_{23}, M_{24}, M_{25}, M_{34}, M_{35}, M_{45}$
The commutator relations are $[M_{mn}, M_{rs}] = \delta_{nr}M_{ms} - \delta_{mr}M_{ns} + \delta_{ms}M_{nr} - \delta_{ns}M_{mr}$
In the adjoint representation of a Lie algebra, ad(Sij) is a linear operator $\mathrm{ad}(S_i)(X) = [S_i, X]$. .
Roots in Weyl-Cartan representation
Cartan subalgebra is $h = \langle M_{12}, M_{34} \rangle$

the remaining space is spanned by the 8 ladder operators $h_{\perp} = \langle M_{13} \pm M_{23}, M_{14} \pm M_{24}, M_{15} \pm M_{25}, M_{35} \pm M_{45} \rangle$

In the basis $h_{\perp}$, we have the eigenvalues and eigenvectors [6]:

for H_1

```
{i, i, i, -i, -i, -i, 0, 0}

{{0, 0, 0, 0, -i, 1, 0, 0}, {0, 0, -i, 1, 0, 0, 0, 0}, {-i, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, i, 1, 0, 0},
 {0, 0, i, 1, 0, 0, 0, 0}, {i, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 1, 0}}
```

for H_2

```
{i, i, i, -i, -i, -i, 0, 0}

{{0, 0, 0, 0, 0, 0, -i, 1}, {0, i, 0, 1, 0, 0, 0, 0}, {i, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, i, 1},
 {0, -i, 0, 1, 0, 0, 0, 0}, {-i, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}}
```

We obtain common eigenvectors for H_1 and H_2 for the eigenvalue doublets (= simple roots)
$(\lambda(H_1), \lambda(H_2))$ = $(0,i)$, $(-i,0)$, so the simple roots are $\alpha_1 = (0,i)$, $\alpha_2 = (-i,0)$

and all 8 roots are $\pm\alpha_1$, $\pm\alpha_2$, $\pm\alpha_1 \pm \alpha_2$

Roots in ei basis

In orthonormal basis e_i , the 2 simple roots are $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_2$

The root system consists of 8 roots $\pm\lambda_1$, $\pm\lambda_2$, $\pm\lambda_1 \pm \lambda_2$

- **so(3) Lie algebra , isomorphic to su(2)**

The so(3) Lie algebra is the tangent space at the identity of the SO(3) rotation group. It consists of all 3x3 real, skew-symmetric matrices.

The algebra is spanned by three standard basis matrices, which correspond to infinitesimal rotations (J_1, J_2, J_3) about the x,y,z axes

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \; J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \; J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with commutator relations of angular momentum $\left[J_i, J_j \right] = \varepsilon_{ijk} J_k$

Cartan subalgebra is generated by J3, it has dim=1,

the simple root is λ_1, the 2 roots are $\pm\lambda_1$

6.4 Symplectic Lie algebras sp(2n) , Cartan type Cn
[4]
The symplectic Lie algebra sp(2n) is a simple non-compact Lie algebra corresponding to the Cartan classification type Cn. It consists of the infinitesimal symmetries of a non-degenerate, skew-symmetric bilinear form (the symplectic form), with dim=n(2n+1).
sp(2n) is realized as the set of 2n x 2n block matrices X that satisfy the condition
$X^T J_n = -J_n X$
where J is the canonical symplectic matrix:
$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$
Matrices are of the form
$X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$, with symmetric nxn matrices B, C .
Generators are of three types [7]
where the basis elements are:
n^2 generators A_{ij}
$A_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}$ i,j=1...n
n(n+1)/2 generators for symmetric B_{ij}
$B_{ij} = \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix}$, i<=j
n(n+1)/2 generators for symmetric C_{ij}
$C_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{pmatrix}$, i<=j
Here, the E_{ij} are nxn ij-Kronecker matrices $\left(E_{ij}\right)_{kl} = \delta_{ik}\delta_{jl}$
The generators are $A_{ij} = E^1_{ij}\ \ i, j = 1...n$, $B_{(ij)} = E^2_{ij}\ \ 1 \le i \le j \le n$, $C_{(ij)} = E^3_{ij}\ \ 1 \le i \le j \le n$
with extended Kronecker matrices
$E^1_{ij} = E_{jk} - E_{k+n, j+n}$, $E^2_{ij} = E_{j,k+n} - E_{k,j+n}$, $E^3_{ij} = E_{j+n,k} - E_{k+n,j}$
The commutator relations are:
$\left[A_{ij}, A_{kl}\right] = \delta_{jk} A_{il} - \delta_{li} A_{kj}$,
$\left[A_{ij}, B_{kl}\right] = \delta_{jk} B_{il} + \delta_{lj} B_{ik}$, $\left[A_{ij}, C_{kl}\right] = -\delta_{il} C_{jk} - \delta_{ik} C_{jl}$, $\left[B_{ij}, C_{kl}\right] = \delta_{jk} A_{il} + \delta_{il} A_{jk} + \delta_{lj} A_{ik} + \delta_{ik} A_{jl}$
$\left[B_{ij}, A_{kl}\right] = -\left(\delta_{li} B_{kj} + \delta_{lj} B_{ki}\right)$, $\left[C_{ij}, A_{kl}\right] = \delta_{kj} C_{li} + \delta_{ki} C_{lj}$, $\left[C_{ij}, B_{kl}\right] = -\left(\delta_{li} A_{kj} + \delta_{kj} A_{li} + \delta_{jl} A_{ki} + \delta_{ki} A_{lj}\right)$
Cartan subalgebra is spanned by $h = \left\langle A_{11}, ..., A_{kk} \right\rangle$
Roots in ei-basis
With the ansatz for Cartan elements $H = \sum_{j=1}^{k} \lambda_j A_{jj}$
we obtain the commutator relations
$\left[H, A_{jk}\right] = \left(\lambda_j - \lambda_k\right) A_{jk}\ \ i \ne j$
$\left[H, B_{jk}\right] = \left(\lambda_j + \lambda_k\right) B_{jk}\ \ i \le j$
$\left[H, C_{jk}\right] = -\left(\lambda_j + \lambda_k\right) C_{jk}\ \ i \le j$
The simple root vectors are then
$\left(\lambda_1 - \lambda_2\right) A_{12}$
$\left(\lambda_2 - \lambda_3\right) A_{23}$
...

$2\lambda_n B_{nn}$

The root system of type Cn has $2n^2$ roots,,
in ei basis they are:

n simple roots are $\alpha_i = (\lambda_i - \lambda_{i+1})$ for i=1,...,n-1 ; $\alpha_n = 2\lambda_n$

$2n^2$ roots are : $\pm(\lambda_i \pm \lambda_j)$, $\pm\lambda_j \quad 1 \le i < j \le n$

- **sp(4) symplectic Lie algebra**

The sp(4) Lie algebra is the 10-dimensional, rank-2 simple Lie algebra associated with the 4 × 4 symplectic group.

sp(2n) is realized as the set of 4x4 block matrices X that satisfy the condition

$X^T J_2 = -J_2 X$

where J_2 is the canonical symplectic matrix:

$$J_2 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

Matrices are of the form

$X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$, with symmetric nxn matrices B, C .

4 generators A_{ij} , 3 generators B_{ij} , 3 generators C_{ij}

$$A_{ij} = \begin{pmatrix} A_{2,ij} & 0 \\ 0 & -A_{2,ij}{}^T \end{pmatrix} ,\ B_{ij} = \begin{pmatrix} 0 & B_{2,ij} \\ 0 & 0 \end{pmatrix} ,\ C_{ij} = \begin{pmatrix} 0 & 0 \\ C_{2,ij} & 0 \end{pmatrix}$$

generators 4 $A_{ij} = E^1_{ij}$, 3 $B_{(ij)} = E^2_{ij}$, 3 $C_{(ij)} = E^3_{ij}$

with extended Kronecker matrices

$$E^1_{jk} = E_{jk} - E_{k+n,j+n} \ ,\ E^2_{jk} = E_{j,k+n} - E_{k,j+n} \ ,\ E^3_{jk} = E_{j+n,k} - E_{k+n,j} \ ,\ E^3_{jk} = E_{j+n,k} - E_{k+n,j}$$

The commutator relations are:

$$\left[A_{ij}, A_{kl}\right] = \delta_{jk} A_{il} - \delta_{li} A_{kj} \ ,\ \left[A_{ij}, B_{kl}\right] = \delta_{jk} B_{il} + \delta_{jl} B_{ik} \ ,\ \left[A_{ij}, C_{kl}\right] = -\delta_{ik} C_{lj} - \delta_{ik} C_{jl} \ ,$$

$$\left[B_{ij}, C_{kl}\right] = \delta_{ik} A_{lj} + \delta_{jk} A_{li} + \delta_{il} A_{kj} + \delta_{jl} A_{ki}$$

The 10 generators are

$A_{11}, A_{22}, A_{12}, A_{21}, B_{11}, B_{12}, B_{22}, C_{11}, C_{12}, C_{22}$

Killing matrix is $\begin{pmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 24 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 & 0 \end{pmatrix}$ [6]

with eigenvalues $\{-24, -24, 24, 24, -12, -12, 12, 12, 12, 12\}$

Cartan algebra $h = \langle A_{11}, A_{22} \rangle$ has dim=2, so sp(4) has rank=2.

Roots in the Weyl-Cartan representation

Cartan subalgebra $h = \langle H_1, H_2 \rangle$ is spanned by $H_1 = A_{11}$, $H_2 = A_{22}$

the remaining space is spanned by ladder operators

$$h_\perp = \langle A_{12} + A_{21}, A_{12} - A_{21}, B_{11} + B_{12}, B_{11} - B_{12}, B_{22} + C_{11}, B_{22} - C_{11}, C_{12} + C_{22}, C_{12} - C_{22} \rangle$$

In the basis $h_\perp$, we have the eigenvalues and eigenvectors of ad[H_i] [6]:

for H_1

$\{-\sqrt{14}, \sqrt{14}, 2, 2, 0, 0, 0, 0\}$

$$\left\{\left\{0, 0, 0, 0, 0, 0, -\sqrt{\frac{2}{7}}, 1\right\}, \left\{0, 0, 0, 0, 0, 0, \sqrt{\frac{2}{7}}, 1\right\}, \{0, 0, 0, 0, 0, 1, 0, 0\}, \{0, 0, 0, 0, 1, 0, 0, 0\}, \{0, 0, 0, 1, 0, 0, 0, 0\}, \{0, 0, 1, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0, 0\}, \{1, 0, 0, 0, 0, 0, 0, 0\}\right\}$$

for H_2

$$\{-2, -2, -2, 2, 1, 0, 0, 0\}$$

$$\{\{0, 0, 0, 0, 0, 0, 0, 1\}, \{0, 0, 0, 0, -1, -1, 3, 0\}, \{0, 0, -1, 1, 0, 0, 0, 0\}, \{0, 0, 1, 1, 0, 0, 0, 0\}, \{0, 0, 0, 0, 1, 1, 0, 0\}, \{0, 0, 0, 0, 1, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0, 0\}, \{1, 0, 0, 0, 0, 0, 0, 0\}\}$$

We obtain common eigenvectors for H_1 and H_2 for the simple eigenvalue doublets

$\left(\lambda(H_1), \lambda(H_2)\right) = (0,2)$, $(2,0)$

Roots in ei basis

In orthonormal basis the simple roots are

$\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = 2\lambda_2$,

8 roots are $\pm\lambda_1 \pm \lambda_2$, $\pm 2\lambda_1$, $\pm 2\lambda_2$

6.5 Even orthogonal so-Lie-algebras so(2n) , Cartan type Dn

[4]

The even-dimensional orthogonal Lie algebra so(2n) corresponds to the Cartan type Dn. It consists of all 2n × 2n skew-symmetric matrices. These algebras are simple for n ≥ 4 and represent infinitesimal rotations and reflections that preserve an even-dimensional non-degenerate symmetric bilinear form.

so(2n) has dim=n(2n-1), it is a simple compact (n>=2) Lie algebra.

Cartan subalgebra h , dimension *n*, consists of block-diagonal matrices with 2 × 2 skew-symmetric blocks along the diagonal.

Generators are M_{kl} , $(M_{kl})_{mn} = \delta_{km}\delta_{\ln} - \delta_{kn}\delta_{1m}$, k,l=1,...2n, k<l

with commutators $[M_{mn}, M_{rs}] = \delta_{nr}M_{ms} - \delta_{mr}M_{ns} + \delta_{ms}M_{nr} - \delta_{ns}M_{mr}$

Cartan subalgebra is spanned by $H_i = M_{2i-1,2i}$ i=1,...,n

Cartan type Dn has 2n(n-1) roots.

Roots in ei-basis

We use as basis the matrices $E_{ij} = \left(\delta_{ik}\delta_{jl}\right)_{kl}$, extended to

$$E^1_{jk} = E_{jk} - E_{k+n,j+n} \ , \ E^2_{jk} = E_{j,k+n} - E_{k,j+n} \ , \ E^3_{jk} = E_{j+n,k} - E_{k+n,j} \ , \ E^3_{jk} = E_{j+n,k} - E_{k+n,j} \ ,$$

We make the ansatz for the Cartan subalgebra $H = \sum_{i=1}^{n} \lambda_i E_{ii}$,

we obtain the commutator relations

$$\left[H, E^1_{jk}\right] = \left(\lambda_j - \lambda_k\right) E^1_{jk} \quad i \neq j$$

$$\left[H, E^2_{jk}\right] = \left(\lambda_j + \lambda_k\right) E^2_{jk} \quad i \leq j$$

$$\left[H, E^3_{jk}\right] = -\left(\lambda_j + \lambda_k\right) E^3_{jk} \quad i \leq j$$

The simple root vectors are then

$\left(\lambda_1 - \lambda_2\right) E^1_{12}$

$\left(\lambda_2 - \lambda_3\right) E^1_{23}$

...

$\left(\lambda_{n-1} + \lambda_n\right) E^2_{n-1,n}$

with n simple roots $\alpha_i = \left(\lambda_i - \lambda_{i+1}\right)$, *i=1...n-1* , $\alpha_n = \left(\lambda_{n-1} + \lambda_n\right)$

the 2n(n-1) roots are $\pm\left(\lambda_i \pm \lambda_j\right)$ 1<=i<j<=n

- **so(4) Lie algebra**

The so(4) Lie algebra is the tangent space of the SO(4) group (4D rotations) and consists of 4 × 4 real, skew-symmetric matrices. It has a dimension of 6 and is uniquely characterized by its isomorphism to the direct sum of two su(2) (or equivalently so(3)) Lie algebras:

$$so(4) = so(3) \oplus so(3)$$

so(4) is a semi-simple and compact Lie algebra.

Elements of so(4) are skew-symmetric and have the generic form

$$X = \begin{pmatrix} 0 & -b_3 & b_2 & -a_1 \\ b_3 & 0 & -b_1 & -a_2 \\ -b_2 & b_1 & 0 & -a_3 \\ a_1 & a_2 & a_3 & 0 \end{pmatrix}$$

The six generators are the 3-dimensional rotations of so(3) A_1, A_2, A_3

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \ A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \ A_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the 3 boost rotations B1, B2, B3 (in physics: velocity transformations, including the last (time) coordinate)

$$B_1=\begin{pmatrix}0&0&0&-1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix},\ B_2=\begin{pmatrix}0&0&0&0\\0&0&0&-1\\0&0&0&0\\0&1&0&0\end{pmatrix},\ B_3=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}$$

with the commutator relations

$$\left[A_i,A_j\right]=\varepsilon_{ijk}A_k\ ,\ \left[B_i,B_j\right]=\varepsilon_{ijk}A_k\ ,\ \left[A_i,B_j\right]=\varepsilon_{ijk}B_k$$

Roots in the Weyl-Cartan representation

Cartan subalgebra $h=\langle H_1,H_2\rangle$ is spanned by $H_1=A_3+B_3$, $H_2=A_3-B_3$

the remaining space is $h_\perp=\langle A_1,A_2,B_1,B_2\rangle$,

In the basis $h_\perp$, we have the eigenvalues and eigenvectors [6]:

for H_1

```
{2 i, -2 i, 0, 0}
{{-i, 1, -i, 1}, {i, 1, i, 1}, {0, -1, 0, 1}, {-1, 0, 1, 0}}
```

for H_2

```
{2 i, -2 i, 0, 0}
{{i, -1, -i, 1}, {-i, -1, i, 1}, {0, 1, 0, 1}, {1, 0, 1, 0}}
```

We obtain common eigenvectors for H_1 and H_2 for the simple eigenvalue doublets

$$\left(\lambda(H_1),\lambda(H_2)\right)=(0,2i),\ (2i,0)$$

Equivalent ladder generators and commutators

An equivalent ladder generator set is

$$X_i=\left(A_i+B_i\right)/2\ ,\ Y_i=\left(A_i-B_i\right)/2$$

with the commutator relations

$$\left[X_i,X_j\right]=\varepsilon_{ijk}X_k\ ,\ \left[Y_i,Y_j\right]=\varepsilon_{ijk}Y_k\ ,\ \left[X_i,Y_j\right]=0$$

which are the basis for the direct sum $so(4)=so(3)\oplus so(3)$

with the structure constants $c_{ab}^d=\begin{cases}\varepsilon_{ijk,} & a=i,b=j,d=k\\ \varepsilon_{ijk,} & a=i+3,b=j+3,d=k+3\\ 0, & else\end{cases}$

The Killing form is $\kappa_{ij}=c_{ib}^a c_{ja}^b$ and becomes here $\kappa_{ij}=-2\delta_{ij}$

Cartan subalgebra is 2-dimensional (so(4) has rank 2) and is spanned by

$$H_1=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\ H_2=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$$

Roots representation in e_i-basis

In orthonormal basis $\{e_1,e_2\}$

the simple roots are $\alpha_1=\lambda_1-\lambda_2$, $\alpha_2=\lambda_1+\lambda_2$

all 4 roots are $\pm\lambda_1\pm\lambda_2$.

- **so(2) Lie algebra**

The **so(2)** Lie algebra is the tangent space at the identity for the Lie group SO(2), which represents 2D rotations. It is a 1-dimensional, abelian (commutative) Lie algebra. The elements of **so(2)** are 2 × 2 real, skew-symmetric matrices.

The generator is $J=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$.

6.6 sl(n,C) algebras
sl(n,C) is Lie algebra is the set of all complex n × n matrices with trace of zero. It is a simple Lie algebra of dimension n^2 - 1.
Cartan algebra (dimension n-1) consists of traceless diagonal matrices.
The root system is A_{n-1} , with n(n-1) roots.

- **sl(2,C) Lie algebra**

The Lie algebra sl(2,C)is the set of all 2x2 matrices with complex entries and a trace of zero.
generators $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
commutator relations $[H, X] = 2X$, $[H,Y] = -2Y$, $[X,Y] = H$
Cartan subalgebra is generated by *H*, has dim=1,
in Weyl-Cartan representation the roots are {2,-2},
in orthonormal basis the roots are $\pm\lambda_1$.

6.7 Exceptional Lie algebras

• g_2 (Dimension 14): is the automorphism group of the octonions and the stabilizer of a generic 3-form in 7 dimensions, rank=2

Cartan matrix $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$, rank=2

2 simple roots $e_1-e_2 = (1,-1,0)$, $-e_1+2e_2-e_3 = (-1,2,-1)$

12 roots

(1,−1,0), (−1,1,0)	(2,−1,−1), (−2,1,1)
(1,0,−1), (−1,0,1)	(1,−2,1), (−1,2,−1)
(0,1,−1), (0,−1,1)	(1,1,−2), (−1,−1,2)

• f_4 (Dimension 52): The Lie algebra of the exceptional group F_4, is the automorphism group of the Albert algebra (a 27-dimensional exceptional Jordan algebra), rank=4

Albert algebra is the space of 3×3 self-adjoint 3x3 matrices over the octonions .

f4 is the automorphism group of Albert algebra.

f4 is the group of automorphisms of the following set of 3 polynomials in 27 variables

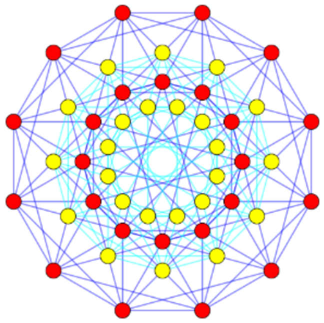
$$C_1 = x + y + z$$
$$C_2 = x^2 + y^2 + z^2 + 2X\overline{X} + 2Y\overline{Y} + 2Z\overline{Z}$$
$$C_3 = xyz - xX\overline{X} - yY\overline{Y} - zZ\overline{Z} + XYZ + \overline{XYZ}$$

where x,y,z are reals and X,Y,Z are octonions.

Cartan matrix $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, rank=4

4 simple roots $\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

8 roots , red and yelllow vertices in the diagram



• e_6 (Dimension 78): is the stabilizer of a certain cubic form and appears in the classification of certain projective planes over division algebras, rank=6

The minimal representation of e6 is the Albert algebra.

The fundamental group of the adjoint form of e6 ad(e6) is the cyclic group Z3 .

Cartan matrix is $\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$

The rank=6, simple roots are

$(0,0,0;0,0,0;0,1,-1)$

$(0,0,0;0,0,0;1,-1,0)$

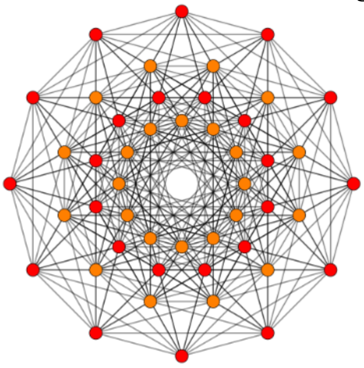
$(0,0,0;0,1,-1;0,0,0)$

$(0,0,0;1,-1,0;0,0,0)$

$(0,1,-1;0,0,0;0,0,0)$

$\left(\frac{1}{3},-\frac{2}{3},\frac{1}{3};-\frac{2}{3},\frac{1}{3},\frac{1}{3};-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)$

72 roots , vertices in the diagram



• e_7 (Dimension 133): Leaves invariant a symmetric rank-4 tensor in its fundamental 56-dimensional representation, rank=7

The rank=7, simple roots are

$(0,-1,1,0,0,0,0,0)$

$(0,0,-1,1,0,0,0,0)$

$(0,0,0,-1,1,0,0,0)$

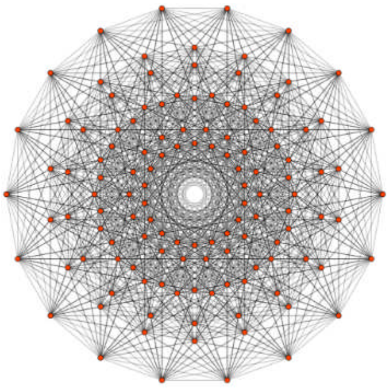
$(0,0,0,0,-1,1,0,0)$

$(0,0,0,0,0,-1,1,0)$

$(0,0,0,0,0,0,-1,1)$

$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$

126 roots , vertices in the diagram



• e_8 (Dimension 248): Its root diagram is the E_8 lattice, which models the densest possible sphere packing in 8-dimensional space, rank=8

e8 has rank=8

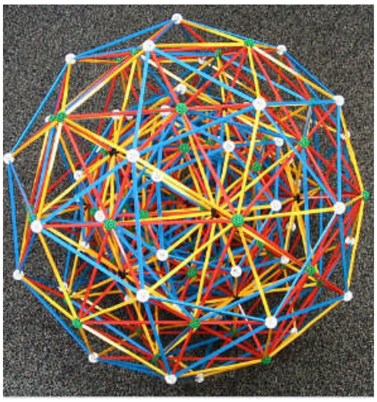
Cartan matrix is

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

simple roots are

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

240 roots are the vertices of the 3-dim diagram



7 Characterization of low-dimensional Lie algebras

7.1 Four-dimensional algebras

Four-dimensional Lie algebras are entirely classified. The classification generally breaks down by the algebraic properties of their derived algebras: solvable (or nilpotent), semisimple, or decomposable into direct sums of lower-dimensional algebras. Because they are the tangent spaces to four-dimensional Lie groups, they are deeply important in general relativity, differential geometry, and theoretical physics.
The most widely used classification of real 4D Lie algebras is due to G.M. Mubarakzyanov. They are typically divided into indecomposable and decomposable algebras.

Indecomposable Algebras

These cannot be written as a direct sum of smaller Lie algebras:
Nilpotent: There is only one unique 4-dimensional indecomposable nilpotent Lie algebra, defined by the non-zero commutation relations:
$[e_2, e_3] = e_1$, $[e_2, e_4] = e_3$
Solvable: These have non-abelian derived algebras and play a fundamental role in geometry. They are classified into several families (e.g. A4,1 through A4,12) depending on the eigenvalues of the adjoint representation matrix. Notable examples are the oscillator algebra and the Euclidean algebra of the plane e(2)
Semisimple: The only simple 4-dimensional real Lie algebra is $su(2) \oplus su(1)$,which is the Lie algebra for the group U(2).

Decomposable algebras

These are direct sums of 2D or 3D Lie algebras

- $R \oplus aff\,(R)$ sum a 1D abelian and 2D non-abelian affine algebra
- $so(3) \oplus R$ or $su(2) \oplus R$ rotation group algebra and a central 1D charge
- $sl(2, R) \oplus R$ (2+1)-dimensional spacetime in GR

Complex four-dim algebras

When working over the complex numbers C, the picture is simpler, and the classification reveals fascinating geometric properties. The moduli space of 4D complex Lie algebras forms an orbifold determined by the natural action of the symmetric group on a 2D complex projective space.

7.2 Classification of low-dimensional real Lie algebras

[5]
Let $\mathfrak{g}_n$ gn be n-dimensional Lie algebra over the field of real numbers with generators $e_1, e_2, ..., e_n$, $n \leq 4$

Two-dimensional

- $2\mathfrak{g}_1$, abelian $\mathbb{R}^2$;
- $\mathfrak{g}_{2.1}$, solvable $\mathfrak{aff}(1) = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \,:\, a, b \in \mathbb{R} \right\}$,

$$[e_1, e_2] = e_1.$$

Three-dimensional

- $3\mathfrak{g}_1$, abelian, Bianchi I;
- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$, decomposable solvable, Bianchi III;
- $\mathfrak{g}_{3.1}$, Heisenberg–Weyl algebra, nilpotent, Bianchi II,
  $$[e_2, e_3] = e_1;$$
- $\mathfrak{g}_{3.2}$, solvable, Bianchi IV,
  $$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2;$$
- $\mathfrak{g}_{3.3}$, solvable, Bianchi V,
  $$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_2;$$
- $\mathfrak{g}_{3.4}$, solvable, Bianchi VI, Poincaré algebra $\mathfrak{p}(1,1)$ when $\alpha = -1$,
  $$[e_1, e_3] = e_1, \quad [e_2, e_3] = \alpha e_2, \quad -1 \leq \alpha < 1, \quad \alpha \neq 0;$$
- $\mathfrak{g}_{3.5}$, solvable, Bianchi VII,
  $$[e_1, e_3] = \beta e_1 - e_2, \quad [e_2, e_3] = e_1 + \beta e_2, \quad \beta \geq 0;$$
- $\mathfrak{g}_{3.6}$, simple, Bianchi VIII, $\mathfrak{sl}(2, \mathbb{R})$,
  $$[e_1, e_2] = e_1, \quad [e_2, e_3] = e_3, \quad [e_1, e_3] = 2e_2;$$
- $\mathfrak{g}_{3.7}$, simple, Bianchi IX, $\mathfrak{so}(3)$,
  $$[e_2, e_3] = e_1, \quad [e_3, e_1] = e_2, \quad [e_1, e_2] = e_3.$$

Algebra $\mathfrak{g}_{3.3}$ can be considered as an extreme case of $\mathfrak{g}_{3.5}$, when $\beta \to \infty$, forming contraction of Lie algebra.

Over the field $\mathbb{C}$ algebras $\mathfrak{g}_{3.5}$, $\mathfrak{g}_{3.7}$ are isomorphic to $\mathfrak{g}_{3.4}$ and $\mathfrak{g}_{3.6}$, respectively.

Four-dimensional

- $4\mathfrak{g}_1$, abelian;
- $\mathfrak{g}_{2.1} \oplus 2\mathfrak{g}_1$, decomposable solvable,

  $$[e_1, e_2] = e_1;$$
- $2\mathfrak{g}_{2.1}$, decomposable solvable,

  $$[e_1, e_2] = e_1 \quad [e_3, e_4] = e_3;$$
- $\mathfrak{g}_{3.1} \oplus \mathfrak{g}_1$, decomposable nilpotent,

  $$[e_2, e_3] = e_1;$$
- $\mathfrak{g}_{3.2} \oplus \mathfrak{g}_1$, decomposable solvable,

  $$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2;$$
- $\mathfrak{g}_{3.3} \oplus \mathfrak{g}_1$, decomposable solvable,

  $$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_2;$$
- $\mathfrak{g}_{3.4} \oplus \mathfrak{g}_1$, decomposable solvable,

  $$[e_1, e_3] = e_1, \quad [e_2, e_3] = \alpha e_2, \quad -1 \leq \alpha < 1, \quad \alpha \neq 0;$$
- $\mathfrak{g}_{3.5} \oplus \mathfrak{g}_1$, decomposable solvable,

  $$[e_1, e_3] = \beta e_1 - e_2 \quad [e_2, e_3] = e_1 + \beta e_2, \quad \beta \geq 0;$$
- $\mathfrak{g}_{3.6} \oplus \mathfrak{g}_1$, unsolvable,

  $$[e_1, e_2] = e_1, \quad [e_2, e_3] = e_3, \quad [e_1, e_3] = 2e_2;$$
- $\mathfrak{g}_{3.7} \oplus \mathfrak{g}_1$, unsolvable,

  $$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2;$$
- $\mathfrak{g}_{4.1}$, indecomposable nilpotent,

  $$[e_2, e_4] = e_1, \quad [e_3, e_4] = e_2;$$
- $\mathfrak{g}_{4.2}$, indecomposable solvable,

  $$[e_1, e_4] = \beta e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3, \quad \beta \neq 0;$$
- $\mathfrak{g}_{4.3}$, indecomposable solvable,

  $$[e_1, e_4] = e_1, \quad [e_3, e_4] = e_2;$$

- $\mathfrak{g}_{4.4}$, indecomposable solvable,

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = e_1 + e_2, \quad [e_3, e_4] = e_2 + e_3;$$

- $\mathfrak{g}_{4.5}$, indecomposable solvable,

$$[e_1, e_4] = \alpha e_1, \quad [e_2, e_4] = \beta e_2, \quad [e_3, e_4] = \gamma e_3, \quad \alpha\beta\gamma \neq 0;$$

- $\mathfrak{g}_{4.6}$, indecomposable solvable,

$$[e_1, e_4] = \alpha e_1, \quad [e_2, e_4] = \beta e_2 - e_3, \quad [e_3, e_4] = e_2 + \beta e_3, \quad \alpha > 0;$$

- $\mathfrak{g}_{4.7}$, indecomposable solvable,

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = 2e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3;$$

- $\mathfrak{g}_{4.8}$, indecomposable solvable,

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = (1+\beta)e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = \beta e_3, \quad -1 \leq \beta \leq 1;$$

- $\mathfrak{g}_{4.9}$, indecomposable solvable,

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = 2\alpha e_1, \quad [e_2, e_4] = \alpha e_2 - e_3, \quad [e_3, e_4] = e_2 + \alpha e_3, \quad \alpha \geq 0;$$

- $\mathfrak{g}_{4.10}$, indecomposable solvable,

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_2, \quad [e_1, e_4] = -e_2, \quad [e_2, e_4] = e_1.$$

Algebra $\mathfrak{g}_{4.3}$ can be considered as an extreme case of $\mathfrak{g}_{4.2}$, when $\beta \to 0$, forming contraction of Lie algebra.

8 Spin representation and Lie algebras

In quantum mechanics, spin is mathematically described through the representations of the su(2) Lie algebra.

- spin-1 state resides in a 3-dimensional representation

Lie algebra is so(3) (isomorphic su(2)), 3x3 representation with J-matrices

$$J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

commutators $\left[J_k, J_l\right] = i\,\varepsilon_{klm} J_m$

- spin-1/2 particle resides in a 2-dimensional representation,

Lie algebra is su(2) , representation is by 2x2 trace 0, skew-symmetric matrices, generated by Pauli matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with the usual commutator relations $\left[\sigma_k, \sigma_l\right] = i\,\varepsilon_{klm}\sigma_m$

- spin-2 particle corresponds to a 5-dimensional irreducible representation (quintuplet).

Lie algebra is su(2) , the irreducible representation of su(2) corresponding to spin 2 is 5-dimensional., with J-matrices as generators.

$$J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{3/2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{3/2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & -i\sqrt{3/2} & 0 & 0 \\ 0 & i\sqrt{3/2} & 0 & -i\sqrt{3/2} & 0 \\ 0 & 0 & i\sqrt{3/2} & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

Their commutation relations are given by:

$$\left[J_k, J_l\right] = i\,\varepsilon_{klm} J_m$$

and normalization

$$J_1^{\,2} + J_2^{\,2} + J_3^{\,2} = 6I$$

The wavefunction for spin-2-particles is a 2x2 symmetric matrix, with 5 parameters, and two polarizations, in total 10=5x2 degrees-of-freedom.

The wavefunction has therefore the form

$$h = \begin{pmatrix} d_1 & h_{12} & h_{13} & h_{14} \\ h_{12} & d_2 & h_{23} & h_{24} \\ h_{13} & h_{23} & d_3 & h_{34} \\ h_{14} & h_{24} & h_{34} & d_4 \end{pmatrix}$$

There are two 5-component spin vectors s_1 and s_2 , for polarization 1 and polarization 2.

$$s_1 = \left(d_1, d_2, h_{14}, d_3, d_4\right)$$

$$s_2 = \left(h_{12}, h_{13}, h_{23}, h_{24}, h_{34}\right)$$
The 5x5 J-matrices act on the spin vectors via matrix multiplication, and act on the wave function accordingly.

9 Clifford algebras
A Clifford algebra [9] is an associative algebra generated by a vector space with a quadratic form, generalizing complex numbers and quaternions, and widely used in geometry and physics.
Definition and Construction
A Clifford algebra is constructed from a vector space V over a field K equipped with a quadratic form Q. Formally, it is defined as the quotient of the tensor algebra T(V) by the ideal generated by elements of the form $v\otimes v-Q(v)1$ for all $v\in V$.
This ensures that the square of each vector in the algebra corresponds to its quadratic form value. The algebra is associative but generally non-commutative, and its dimension is $2^{\dim V}$.
Generators and Relations
The algebra is generated by the elements of V, called generators, with the defining relation:
$v^2 = Q(v)$ for all $v\in V$
or equivalently, using the symmetric bilinear form B associated with Q:
$vw + wv = 2B(v,w)1$ for all $v,w\in V$. When the quadratic form is zero, the Clifford algebra reduces to the *exterior algebra*.
Applications
Clifford algebras have broad applications in both mathematics and physics:
Geometry: They provide a framework for geometric algebra, allowing the representation of rotations, reflections, and orthogonal transformations ..
Spinors and Quantum Physics:
Clifford algebras define spinor spaces and the Spin group, which are essential in describing fermions like electrons and in constructing spinor bundles on manifolds.
Representation Theory: They are used to study Lie algebras, root systems, and weights, providing a natural setting for constructing spinor modules.
Combinatorics and Algebra:
Clifford algebras generalize complex numbers, quaternions, and octonions, and are useful in defining hypercomplex numbers and various algebraic operations .
Structure and Properties
Grading: Clifford algebras are often $\mathbb{Z}2$-graded, separating even and odd elements.
Tensor Products: The Clifford algebra of a direct sum of vector spaces can be expressed as a tensor product of the Clifford algebras of the summands .
Normed Division Algebras: Real Clifford algebras are closely related to normed division algebras, connecting them to quaternions and octonions .
Clifford algebras thus provide a unifying algebraic framework that extends classical number systems and underpins many modern mathematical and physical theories.
Abstract definition
A Clifford algebra is a unital (with 1) associative algebra that contains and is generated by a vector space V over a field K, where V is equipped with a bilinear form $\langle u,v\rangle$: $V\times V \to K$. , which becomes a quadratic form $Q(v)$ $V \to K$ for $v\in V$
The Clifford algebra Cl(V, Q) is the unital associative algebra generated by V , defined by multiplication $u\cdot v$ with the anticommutator condition (Clifford product)
$\{u,v\} = u\cdot v + v\cdot u = 2\langle u,v\rangle 1$ for $u,v\in V$
here, $Q(v)=\langle v,v\rangle$ and $\langle u,v\rangle = \frac{1}{2}\left(Q(u+v)-Q(u)-Q(v)\right)$
If Q(v)=0, Cl(V, Q) becomes the external algebra $\wedge V$.
Usually, the bilinear form is the inner product $\langle u,v\rangle = u_1v_1 + ... + u_nv_n$
A graded Clifford algebra $\mathrm{Cl}_{p,q}(V, Q)$ has the quadratic form $\langle v,v\rangle = v_1{}^2 + ... + v_p{}^2 - v_{p+1}{}^2 - ... - v_{p+q}{}^2$ with p positive and q negative terms.
The canonical Clifford construction
The Clifford algebra Cl(V, Q) can be formulated in a standard orthogonal basis $(e_1, e_2, ..., e_n)$ with the relations $e_i\cdot e_j = -e_j\cdot e_i$, $e_i\cdot e_i = 1$, and the quadratic form $\langle v,v\rangle = v_1{}^2 + ... + v_p{}^2 - v_{p+1}{}^2 - ... - v_{p+q}{}^2$.

The general element of the Clifford algebra is given by

$$q = q_0 + \sum_{i_1<i_2} q_{i_1 i_2} e_{i_1} e_{i_2} + \sum_{i_1<i_2<i_3} q_{i_1 i_2 i_3} e_{i_1} e_{i_2} e_{i_3} + ... + q_{i_1 ... i_n} e_{i_1} ... e_{i_n}$$

The basis $(e_1, e_2, ..., e_n)$ are the *n* generators of the Clifford algebra, with dimension dim(Cl(*V*, *Q*))=*n* .
The linear combination of the even degree elements of Cl(V, Q) defines the even subalgebra $Cl^{[0]}$(V, Q) with the general element

$$q = q_0 + \sum_{i_1<i_2} q_{i_1 i_2} e_{i_1} e_{i_2} + \sum_{i_1<i_2<i_3<i_4} q_{i_1 i_2 i_3 i_4} e_{i_1} e_{i_2} e_{i_3} e_{i_4} + ...$$

The quaternion algebra
The even subalgebra $Cl_{3,0}$[0](R, $\langle v, v \rangle$) with the generic form

$$q = q_0 + q_{12} e_1 e_2 + q_{23} e_2 e_3 + q_{31} e_3 e_1$$

is Hamilton's real quaternion algebra *H* with quaternions (i, j, k) and the relations

$$i^2 = \alpha \ , \ j^2 = \beta \ , \ i\,j = k \ , \ j\,i = -k \ , \ k^2 = i\,j\,i\,j = -ii\,j\,j = -\alpha\beta$$

where ($\alpha = \beta = -1$) for Hamilton’s quaternions *H*
and (α =-1, β= +1) for split-quaternions, isomorphic to the ring of the 2×2 real matrices .
Hamilton’s quaternions have the properties

$$i^2 = j^2 = k^2 = -1 \ , \ i\,j = k = -j\,i \ , \ j\,k = i = -k\,j \ , \ k\,i = j = -ik$$

H has the generic form $q = a + bi + c\,j + d\,k$

H is actually a field, with the inverse $q^{-1} = (a + bi + c\,j + d\,k)^{-1} = \dfrac{a - bi - c\,j - d\,k}{a^2 + b^2 + c^2 + d^2}$

The conjugated element is $q^* = -\dfrac{1}{2}(q + iqi + jq\,j + kqk)$

te norm is $\|q\| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}$

According to the Frobenius theorem, there are only two division-free algebras over the field of real numbers R : complex numbers *C* and Hamilton’s quaternions *H* .
A 2x2 matrix representation of *H* is by Pauli matrices {1, i, j, k} $\rightarrow \{I, i\sigma_3, i\sigma_2, i\sigma_1\}$.

The dual quaternion algebra
With *V*=R^4 and the degenerate bilinear form $d(v, w) = v_1 w_1 + v_2 w_2 + v_3 w_3$

and the Clifford product $vw + wv = -2d(v, w)$, we obtain the Clifford algebra Cl_4(**R**, *d*) with 16 components and the basis relations

$$e_1^{\ 2} = e_2^{\ 2} = e_3^{\ 2} = -1 \ , \ e_4^{\ 2} = 0 \ , \ e_m e_n = -e_n e_m \ \ m \neq n$$

The linear combination of the even degree elements defines the even subalgebra $Cl^{[0]}$(R4, d) with the general element

$$h = h_0 + h_1 e_2 e_3 + h_2 e_3 e_1 + h_3 e_1 e_2 + h_4 e_4 e_1 + h_5 e_4 e_2 + h_6 e_4 e_3 + h_7 e_1 e_2 e_3 e_4$$

The basis elements can be identified with the quaternion basis elements *i, j, k* and the dual unit *ε* asi=e2e3,j=e3e1,k=e1e2,ε=e1e2e3e4.

$$i = e_2 e_3 \ , \ j = e_3 e_1 \ , \ k = e_1 e_2 \ , \ \varepsilon = e_1 e_2 e_3 e_4$$

This is an isomorphism with the dual quaternion algebra.
Dual quaternions H_d are an 8-dimensional real algebra isomorphic to the tensor product of the quaternions and the dual numbers with the generic form $A + \varepsilon B$, where A, B are quaternions, and ε is the dual unit, which commutes with all quaternions and is nilpotent: ε^2=0 .

Super algebra
Clifford algebras are superalgebras, i.e. the reflection α: v ↦ −v preserves the quadratic form Q and an algebra automorphism.α:Cl(V,Q)→Cl(V,Q).
We can decompose Cl(V, Q) into direct sum of even and odd subspaces

$$Cl(V, Q) = Cl^{[0]}(V, Q) \oplus Cl^{[1]}(V, Q)$$

where $Cl^{[i]}(V, Q) = \left\{ x \in Cl(V, Q) \middle| \alpha(x) = (-1)^i x \right\}$

Transposition

A transpose of an element reverses the order in product terms

$$q = q_0 + \sum_{i_1<i_2} q_{i_1 i_2} e_{i_1} e_{i_2} + \sum_{i_1<i_2<i_3} q_{i_1 i_2 i_3} e_{i_1} e_{i_2} e_{i_3} + \ldots$$

$$q^t = q_0 + \sum_{i_1<i_2} q_{i_1 i_2} e_{i_2} e_{i_1} + \sum_{i_1<i_2<i_3} q_{i_1 i_2 i_3} e_{i_3} e_{i_2} e_{i_1} + \ldots$$

The Clifford scalar product is then

$\langle x, y\rangle = \langle x^{\mathrm{t}} y\rangle_0$, where $\langle x^{\mathrm{t}} y\rangle_0$ is the scalar part q_0 of the product $x^t y$

Pin and Spin-group

$Pin_{p,q} = \left\{ v_1 \ldots v_{2r} \,\middle|\, \|v_i\| = \pm 1 \right\}$ consists of products of invertible unit vectors , where the spinor norm is $\|v\| = \left(v^t v\right)_0$,

$Pin_{p,q}$ is the full rotation group O(p,q) ,

$Spin_{p,q}$ is the proper (without reflections) rotation group SO(p,q) , it consists of even-number-products of unit vectors

$$Spin_{p,q} = \left\{ v_1 \ldots v_{2r} \,\middle|\, \|v_i\| = \pm 1 \right\} = \{ q \in Cl(V) \,|\, q^t q = I, \|q\| = 1 \}$$

10 Dirac matrices, spinors and Lorentz Lie algebra

Dirac Clifford algebra

Dirac algebra is the algebra with the basis of Dirac matrices matrices γ_0, ..., γ_3 with the anticommutator $\left\{\gamma^{\mu},\gamma^{\mu}\right\}=2\eta_{\mu\nu}I$ in (1,3) space, Minkowski signature (+,---)

This is the complexification $Cl_{1,3}(R)_C$ of the Clifford algebra $Cl_{1,3}(R)$, which is isomorphic to the 4x4 complex matrices $M_4(C)$.

The Dirac matrices (Dirac basis) are explicitly given by the Pauli matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

with the relations $\left[\sigma_j,\sigma_k\right]=2i\varepsilon_{jkl}\sigma_l$, $\left\{\sigma_j,\sigma_k\right\}=2\delta_{jk}I$

in the form

$$\gamma^0=\begin{pmatrix}I&0\\0&-I\end{pmatrix},\ \gamma^j=\begin{pmatrix}0&\sigma_j\\-\sigma_j&0\end{pmatrix},\ \gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}0&I\\I&0\end{pmatrix}$$

$$\gamma^0=\sigma_3\otimes I_2\ ,\ \gamma^j=i\sigma_2\otimes\sigma_j\ ,\ \gamma^5=\sigma_1\otimes I_2$$

$$\gamma^0=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\ \gamma^1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix},\ \gamma^2=i\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix},\ \gamma^3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix},$$

$$\gamma^5=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$$

In chiral Weyl basis they are

$$\gamma^0=\begin{pmatrix}0&I\\I&0\end{pmatrix},\ \gamma^j=\begin{pmatrix}0&\sigma^j\\-\sigma^j&0\end{pmatrix},\ \gamma^5=\begin{pmatrix}-I&0\\0&I\end{pmatrix}$$

The corresponding spin Lie algebra so(1,3) is generated by the spin generators $\Sigma^{\mu\nu}=\frac{i}{4}\left[\gamma^{\mu},\gamma^{\nu}\right]$, with the Lie bracket

$$\left[\Sigma^{\mu\nu},\Sigma^{\rho\tau}\right]=-i\left(\eta^{\tau\mu}\Sigma^{\rho\nu}+\eta^{\nu\tau}\Sigma^{\mu\rho}-\eta^{\rho\mu}\Sigma^{\tau\nu}-\eta^{\nu\rho}\Sigma^{\mu\tau}\right)$$

so(1,3) Lie algebra

The Lie algebra of the Lorentz group O(1,3) is a 6-dimensional, non-compact, and semi-simple real Lie algebra denoted as so(1,3). It consists of infinitesimal Lorentz transformations, which are generated by three spatial rotations Ji and three velocity boosts Ki .

It is semi-simple and non-compact.

Complex $so(1,3)_C$ is isomorphic to direct sum of sl(2,C) , $so(1,3)_C\cong sl(2,C)\oplus sl(2,C)$

The Killing form is indefinite rather than negative-definite, it defines a hyperbolic space, which is not bounded, therefore so(1,3) is non-compact.

With the matrices M_{kl} , $\left(M_{kl}\right)_{mn}=\eta_{km}\delta_{\ln}-\eta_{kn}\delta_{1m}$, k,l=1,...4, k<l ,

and the matrices $\bar{M}_{kl}$, $\left(\bar{M}_{kl}\right)_{mn}=\eta_{km}\delta_{\ln}+\eta_{kn}\delta_{1m}$, k,l=1,...4, k<l ,

the generators are [8] $J_k=\frac{1}{2}\varepsilon_{ijk}M_{ij}$, $K_j=M_{0j}$

$$J_i=-\left(M_{34},M_{42},M_{23}\right)\ ,\ K_i=\left(\bar{M}_{12},\bar{M}_{13},\bar{M}_{14}\right)$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

with the commutators , $\left[K_i, K_j\right] = -\varepsilon_{ijk} J_k$, $\left[J_i, K_j\right] = \varepsilon_{ijk} K_k$

Lorentz group SO(1,3)

The rotations Λ [8] of the Lorentz groups $x' = \Lambda x$ can be written in the Kronecker matrices with 6 elementary rotations $\omega_{\mu\nu}$ and Kronecker matrices $M_{\mu\nu}$

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right)$$

or in the rotation and boost generators $\left(J_i\right),\left(K_i\right)$ with rotation vector $\vec{\varphi} = \left(\varphi_1, \varphi_2, \varphi_3\right)$ and boost vector $\vec{v} = \left(v_1, v_2, v_3\right)$ we obtain with

$\omega_{0i} = v_i = -\omega_{i0}$, $\omega_{ij} = \varepsilon_{ijk}\varphi_k$

in the form $\Lambda = \exp(-i\,(\vec{\varphi}\,\vec{J} + \vec{v}\,\vec{K}))$

rotation operator $\Phi = (\vec{\varphi}\,\vec{J} + \vec{v}\,\vec{K})$

Spinor group Spin(1,3)

The corresponding spin rotations of the Dirac spinor $\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$, $\Psi' = S\Psi$

can be formulated with the spin generators $\Sigma^{\mu\nu}$, $\Sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \tilde{\sigma}^{\mu\nu} \end{pmatrix}$

in the form $S = exp\left(-\frac{i}{2}\omega_{\mu\nu}\,\Sigma^{\mu\nu}\right)$

or in the rotation-boost generators

$$S = \begin{pmatrix} A_L & 0 \\ 0 & A_R \end{pmatrix} = \begin{pmatrix} exp\left(-\frac{i}{2}(\vec{\varphi} - i\vec{v}\,)\vec{\sigma}\right) & 0 \\ 0 & exp\left(-\frac{i}{2}(\vec{\varphi} + i\vec{v}\,)\vec{\sigma}\right) \end{pmatrix},$$

with Pauli spin-tensor $\sigma^{\mu\nu} = \frac{i}{4}[\sigma^\mu, \tilde{\sigma}^\nu]$, $\tilde{\sigma}^{\mu\nu} = \frac{i}{4}[\tilde{\sigma}^\mu, \sigma^\nu]$, $\sigma^\mu = \frac{1}{2}(1, \vec{\sigma})$, $\tilde{\sigma}^\mu = \frac{1}{2}(1, -\vec{\sigma})$

and Pauli 3-vector $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$

$S = \begin{pmatrix} A_L & 0 \\ 0 & A_R \end{pmatrix}$ with the left- and right Weyl-spinor rotations

$A_L = exp\left(-\frac{i}{2}\omega_{\mu\nu}\,\sigma^{\mu\nu}\right) = exp\left(-\frac{i}{2}(\,\vec{\varphi} - i\,\vec{v}\,)\vec{\sigma}\right)$, left-handed Weyl-spinor

$A_R = exp\left(-\frac{i}{2}\omega_{\mu\nu}\,\tilde{\sigma}^{\mu\nu}\right) = exp\left(-\frac{i}{2}(\,\vec{\varphi} + i\,\vec{v}\,)\vec{\sigma}\right)$, right-handed Weyl-spinor

rotation operator $\Phi_s = \frac{1}{2}(\vec{\varphi}\vec{\sigma} \pm i\vec{v}\vec{\sigma})$

General Dirac matrices

General Dirac matrices Γ_a [10] generate the Dirac group G in dim=d . metrics (p,q) in the form

$x = i^n \Gamma_{a_1} \Gamma_{a_2} ... \Gamma_{a_d}$, $x \in G$, in canonical order $a_1 < a_2 < ... < a_d$

with the anti-commutator $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} I$ for Cl(p,q)(C)

In SO(4) with the Euclidean metric $\delta_{\mu\nu}$, the Dirac matrices become

$\gamma_E^0 = \gamma^0$, $\gamma_E^j = i\gamma^j$, $\gamma_E^5 = -i\gamma^5$

$$\gamma_E^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} , \gamma_E^j = i\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} , \gamma_E^5 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

compactly $\gamma_E^0 = \sigma_3 \otimes I_2$, $\gamma_E^j = -\sigma_2 \otimes \sigma_j$, $\gamma_E^5 = -i\sigma_1 \otimes I_2$

with the anticommutator $\{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta^{\mu\nu} I_4$

In chiral Weyl basis the Euclidean Dirac matrices are

$$\gamma_E^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} , \gamma_E^j = i\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} , \gamma_E^5 = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

compactly $\gamma_E^0 = \sigma_1 \otimes I_2$, $\gamma_E^j = -\sigma_2 \otimes \sigma_j$, $\gamma_E^5 = i\sigma_3 \otimes I_2$

In SO(2,2) (bi-Lorentzian) they are

$\gamma_{(2,2)}^0 = \gamma^0$, $\gamma_{(2,2)}^1 = i\gamma^1$ (time-like)

$\gamma_{(2,2)}^2 = \gamma^2$, $\gamma_{(2,2)}^3 = \gamma^3$ (space-like)

General Dirac matrices Γ_a are nxn matrices, where $n = \begin{cases} 2^{d/2} & , d = 2m \\ 2^{(d-1)/2} & , d = 2m+1 \end{cases}$ for even and odd dimension

In Euclidean space we have in Cl(d)(C) the Dirac generators

for d=2, $\Gamma_a = (i\sigma_1, i\sigma_2)$

for d=3, $\Gamma_a = (i\sigma_1, i\sigma_2, i\sigma_3)$

for d=4, $\Gamma_a^{(4)} = (\gamma_E^1, \gamma_E^2, \gamma_E^3, \gamma_E^0)$, $\gamma_E^0 = \sigma_1 \otimes I_2$, $\gamma_E^j = -\sigma_2 \otimes \sigma_j$, $\gamma_E^5 = i\sigma_3 \otimes I_2$,

for d=5, $\Gamma_a^{(5)} = (\gamma_E^1, \gamma_E^2, \gamma_E^3, \gamma_E^0)$, $\Gamma_5^{(5)} = \gamma_E^5$,

for d=6, $\Gamma_a^{(6)} = -\sigma_2 \otimes \Gamma_a^{(5)}$ $a = 1,2,3,4,5$; $\Gamma_6^{(6)} = \sigma_1 \otimes I$

for d=7, $\Gamma_a^{(7)} = -\sigma_2 \otimes \Gamma_a^{(5)}$ $a = 1,2,3,4,5$; $\Gamma_6^{(7)} = \sigma_1 \otimes I$, $\Gamma_7^{(7)} = i\sigma_3 \otimes I$

This scheme is continued accordingly for higher dimensions.

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