

Functional analysis with p-adic, real, complex and quaternionic numbers

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Abstract

Real and complex analysis is the most important part of the mathematical fundament of physics.

Bosonic vectors (field-carriers with integer spin, e.g. electromagnetic 4-vector) are described by complex analysis, mass-carrying spinors (spin-1/2 particles like leptons and quarks) are described by Pauli-matrices, which obey quaternion-based arithmetic and analysis.

P-adic analysis based on p-adic numbers yields an alternative, which is similar in many respects, but has a completely different topology.

P-adic analysis is applicable for the description of Planck-scale physics.

This paper is an introduction to functional analysis with p-adic, real, complex and quaternionic numbers.

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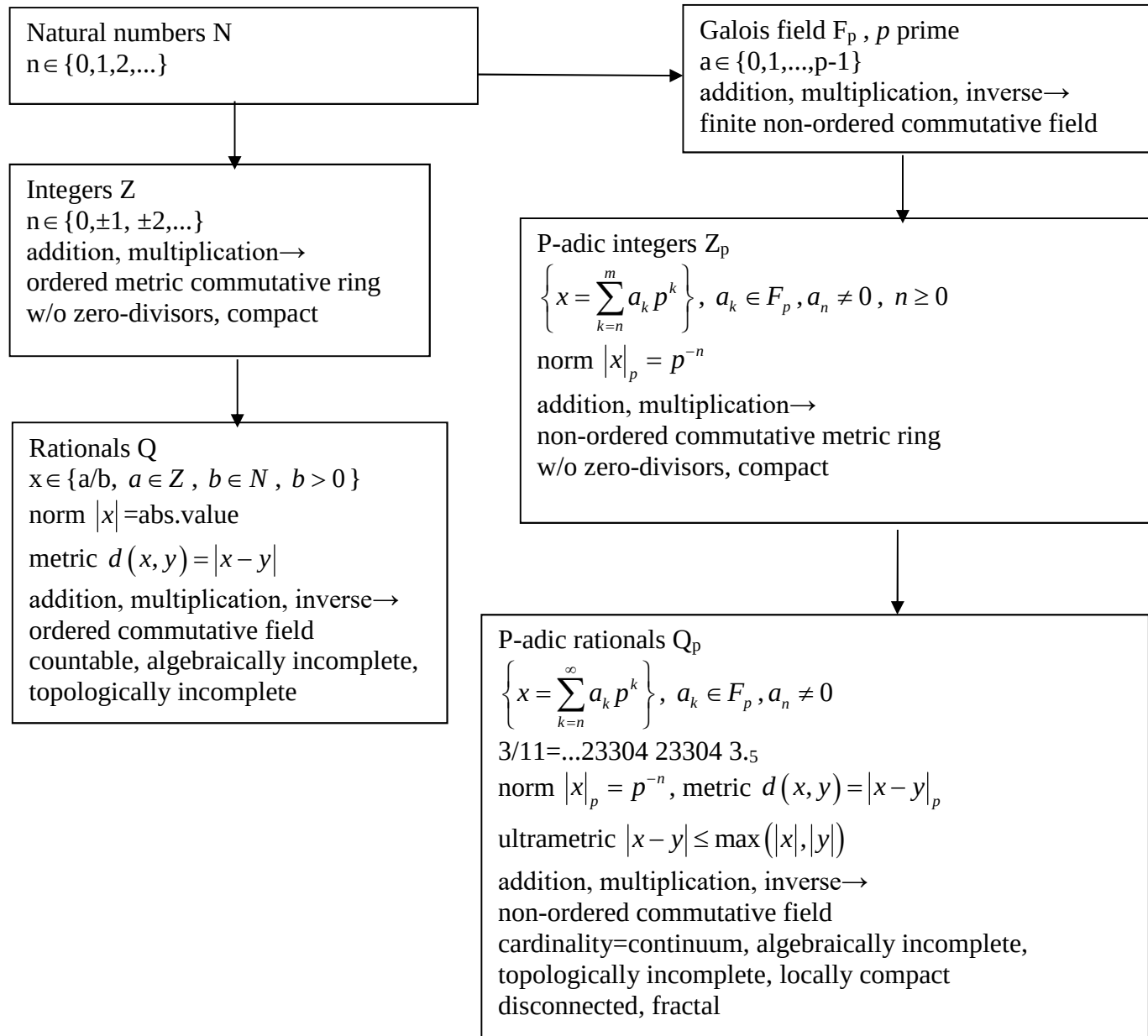
3 Complex analysis

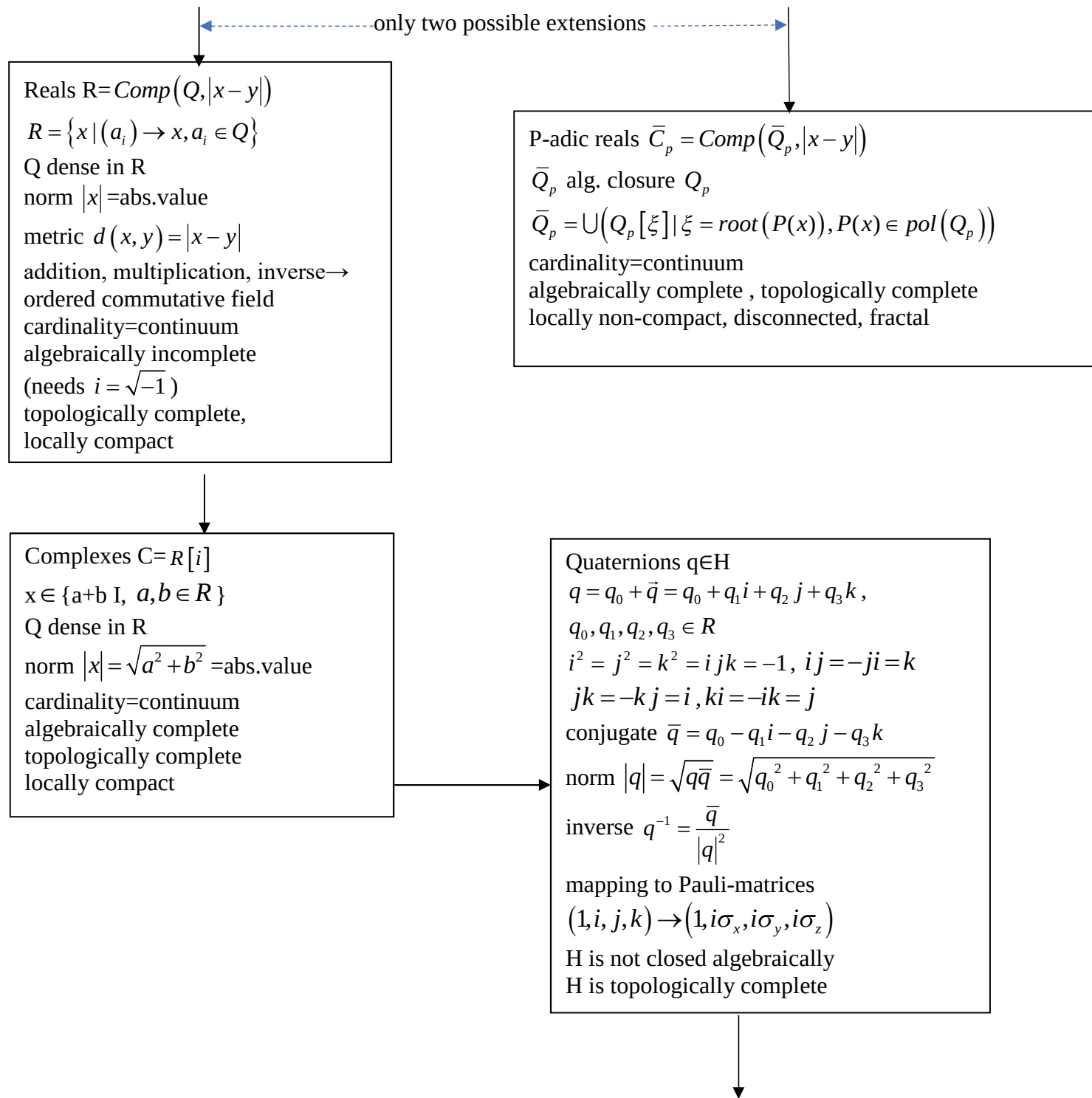
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Numbers





Biquaternions $q \in H_C$

$$q = q_0 + \vec{q} = q_0 + q_1 i + q_2 j + q_3 k, q_0, q_1, q_2, q_3 \in C$$

$$\text{conjugate } \bar{q} = q_0 - q_1 i - q_2 j - q_3 k$$

$$q\bar{q} = \bar{q}q \in C$$

$$\text{semi-norm } |q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

$$\text{inverse } q^{-1} = \{\bar{q}(q\bar{q})^{-1} \text{ if } (q\bar{q}) \neq 0$$

$$q = z + w e_2 = (q_0 + q_1 e_1) + (q_2 + q_3 e_1) e_2$$

$$Q = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}$$

complex 4-vector

$$x = (x_0, x_1, x_2, x_3)$$

vector rotation $x' = \Lambda x$

$$\Lambda = \exp\left(-\frac{i}{2} \omega_{\mu\nu} M^{\mu\nu}\right)$$

$$= \exp(-i(\vec{\varphi} \vec{J} + \vec{v} \vec{K}))$$

field bosons $S=1, m=0$

e.g. photon A^μ

spinor=biquaternion

$$q = q_0 + \vec{q} = q_0 + q_1 i + q_2 j + q_3 k$$

$$Q = \begin{pmatrix} q_0 + q_1 e_1 & q_2 + q_3 e_1 \\ -(q_2^* + q_3^* e_1) & (q_0^* + q_1^* e_1) \end{pmatrix}$$

spinor rotation $Q' = SQ$

$$S = \begin{pmatrix} \exp\left(\frac{i}{2}(\vec{\varphi} - i\vec{v})\vec{\sigma}\right) & 0 \\ 0 & \exp\left(\frac{i}{2}(\vec{\varphi} + i\vec{v})\vec{\sigma}\right) \end{pmatrix}$$

particles=Dirac-bispinors $S=1/2, m \neq 0$

e.g. electron Ψ

Polynomials and power series in reals $\mathbb{R}$, p -adics $\mathbb{Q}_p$

sums in $\mathbb{R}$: Cauchy convergence

$$\left(\sum_{i=0}^{\infty} a_i \mid a_i \in \mathbb{R} \right) \rightarrow a \text{ iff}$$

$$\sum_{i=m}^n |a_i| < \varepsilon, \forall m, n$$



polynomials in $\mathbb{Q}$: $P(x) \in \mathbb{Q}[x]$

algebraically solvable in $\mathbb{C}$

zeros: Newton-Raphson iteration

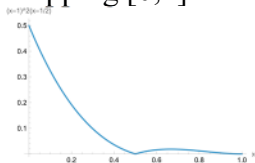
$$x_{n+1} = x_n - f(x_n) / f'(x_n)$$

$$\text{error } \varepsilon_n = |\alpha - x_n|, \xi_n \in [x_n, \alpha]$$

$$\text{converges } |\varepsilon_{n+1}| = \varepsilon_n^2 \frac{|f''(\xi_n)|}{2|f'(\xi_n)|}$$

$$\text{example } P(x) = (x-1)^2(x-1/2)$$

mapping $[0,1] \rightarrow \mathbb{R}$ compact, connected



sums in $\mathbb{Q}_p$: coefficients $\rightarrow 0$

$$\left(\sum_{i=0}^{\infty} a_i \mid a_i \in \mathbb{Q}_p \right) \rightarrow a \text{ iff}$$

$$|a_i|_p \rightarrow 0$$



polynomials in $\mathbb{Q}_p$: $P(x) \in \mathbb{Q}_p[x]$

algebraically solvable in $\mathbb{C}_p$

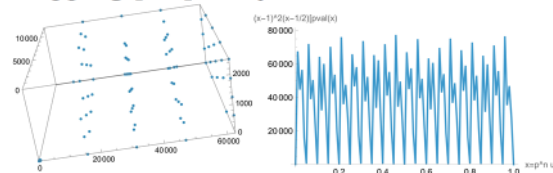
zeros: Hensel iteration

$$\hat{x}(x) = x - P(x) / P'(x)$$

converges if order $v(P'(x)) < n/2$

$$\text{example } P(x) = (x-1)^2(x-1/2)$$

mapping $[0,1] \rightarrow \mathbb{Q}_p$ fractal, disconnected





polynomial approximation in R
Taylor series

$$f(x) = \sum_{i=0}^k \frac{f^{(i)}(a)}{i!} (x-a)^i + h_k(x)$$

$$h_k(x) = \frac{f^{(k+1)}(\xi)}{(k+1)!} (x-a)^{k+1}$$

Weierstraß approximation

f continuous $f: [a, b] \rightarrow \mathbb{R}$

$$p_n(x) = \sum_{i=0}^n f\left(a + \frac{i}{n}(b-a)\right) B_{i,n}^{a,b}(x)$$

$$p_n(x) \rightarrow f(x) \text{ uniform}$$

$$B_{i,n}^{a,b}(x) \text{ Bernstein polynomials}$$



polynomial approximation in $\mathbb{Q}_p$
Mahler series

$$f(x) = \sum_{k \geq 0} \nabla^k f(0) \binom{x}{k} \text{ uniform conv.}$$

$$\binom{x}{i} = \frac{x(x-1)\dots(x-i+1)}{i!} \text{ binomial polynomials}$$

forward difference operator $\nabla f(x) = f(x+1) - f(x)$

van-der-Putt approximation

$f: \mathbb{Z}_p \rightarrow \bar{\mathbb{Q}}_p$ continuous

$$a_0 = f(0), \quad a_n = f(n) - f(n_-), \quad n_- = n - n_{v-1} p^{v-1}$$

$$\sum_n a_n \rightarrow f \text{ uniform}$$

Functional analysis in reals $\mathbf{R}$, p-adics $\mathbf{Q}_p$

functional norm in $\mathbf{R}[x]$

$$\|f\|_I = \sqrt{\int_I f^2(x) dx} \quad L_2 \text{ norm on interval } I$$

$$\text{differential } f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$\text{differential operator } Df(x) = f'(x)$$

$$\text{mean value } f'(c) = \frac{f(b) - f(a)}{b - a}, c \in [a, b]$$

$$\text{fixed point } |f(x) - f(y)| \leq q|x - y|, q < 1 \rightarrow \\ \exists x_* : f(x_*) = x_*$$

$$\text{integral } \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^{n-1} f(x_i)(x_{i+1} - x_i)$$

$$\text{for } x_i \in [a, b]$$

$$\text{integral operator } (I_a f)(x) = \int_a^x f(t) dt$$

$$\text{fundamental theorem } D(I_a f)(x) = f(x)$$

functional norm in $\mathbf{Q}_p[x]$

$$\|f\|_s = \sup_{x \in \mathbf{Q}_p} |f(x)| \quad \text{max-norm}$$

$$\|f\|_G = \sup_k |a_k|, \quad f(x) = \sum_{k \geq 0} a_k x^k \quad \text{Gauss norm}$$

$$\text{strict differential } f'(a) = \lim_{(x,y) \rightarrow (a,a)} \frac{f(x) - f(y)}{x - y}$$

$$\text{differential operator } Df(x) = f'(x)$$

$$\text{forward diff.operator } f : \mathbf{Z}_p \rightarrow \overline{\mathbf{Q}}_p$$

$$f'(y) = \sum_{k \geq 1} (-1)^{k-1} (\nabla^k f)(y) / k$$

$$\text{mean value } |f(x+h) - f(x)| \leq |h| \|f'\|_G$$

$$\text{for } |x| \leq 1, |h| \leq r_p = |p|^{1/(p-1)}$$

$$\text{fixed point } \|f'\|_G < 1, \inf_{x \in B_{\leq 1}} |f(x) - x| \leq |p|^{1/(p-1)} \rightarrow$$

$$\exists x_* \in B_{\leq 1} : f(x_*) = x_*$$

Volkenborn integral

$$\int_{\mathbf{Z}_p} f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{j=0}^{p^n} f(j) = (Sf)'(0)$$

$$\text{sum operator } Sf(x) = \sum_{0 \leq i \leq x} f(i)$$

$$\text{integral operator } (If)(x) = \int_{\mathbf{Z}_p} f(x+t) dt$$

fundamental theorem

$$(If)(x) = (DSf)(x)$$

$$(\nabla S)(x) = f(\text{floor}(x+1))$$

$$S\nabla f = \nabla S f - f(0)$$

Functional analysis in complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$

Complex functional analysis

equivalent for $f(z) = u(x, y) + iv(x, y)$

- f is holomorphic in Ω , $f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$
- f satisfies in Ω the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$
- f is expandable in Taylor series

$f(z) = \sum_{k=0}^{\infty} a_k (z-c)^k$, converges in neighborhood

for every $c \in \Omega$

Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)} dz$$

Laurent series

$$a_k = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-c)^{k+1}} dz$$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-c)^k$$

Quaternionic functional analysis

$$q = x_0 + x_1 i + x_2 j + x_3 k$$

- $f : H \rightarrow H$ **left/right-regular** if satisfies Fueter eq.

$$\frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} + k \frac{\partial f}{\partial x_3} = 0$$

$$\frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1} i + \frac{\partial f}{\partial x_2} j + \frac{\partial f}{\partial x_3} k = 0$$

- f is left-differentiable in Ω ,

$$\lim_{h \rightarrow 0} \left(h^{-1} (f(q+h) - f(q)) \right) = f'_l(q)$$

f left-differentiable $\rightarrow f$ linear $f(q) = aq + b$

Cauchy-formula: f regular inside $\Omega \subset H$

$$f(q_0) = \frac{1}{2\pi^2} \int_{\partial\Omega} \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dq f(q)$$

Laurent-Fueter series

f is regular in open set U except in $q_0 \in U$, then

$$f(q) = \sum_{n=0}^{\infty} \sum_{v \in \sigma_n} (P_v(q-q_0) a_v + G_v(q-q_0) b_v)$$

$$a_v = \frac{1}{2\pi^2} \int_{\partial U} G_v(q-q_0) Dq f(q)$$

$$b_v = \frac{1}{2\pi^2} \int_{\partial U} P_v(q-q_0) Dq f(q)$$

$$G_v(q) = \frac{q^{-1}}{|q|^2}, \quad \partial_v = \frac{\partial^n}{\partial x_{i_1} \dots \partial x_{i_n}}$$

$G_v(q)$ is a rational function of q : $G_v(q) = \partial_v G(q)$

$P_v(q)$ is a polynomial in q :

$$P_v(q) = \frac{1}{n!} \sum_{\pi(v)=(i_1, \dots, i_n)} (te_{i_1} - x_{i_1}) \dots (te_{i_n} - x_{i_n})$$

Generalized differential operator

Taylor operators

Taylor series , analytical function f

$$f(x) = \sum_{k=0}^{\infty} a_k(f, x_0)(x - x_0)^k, \quad a_k(f, x_0) = \frac{f^{(k)}(x_0)}{k!}, \quad \text{convergence radius } R(f, x_0) = \frac{1}{\limsup_{k \rightarrow \infty} |a_k(f, x_0)|^{1/k}}$$

$$\text{Taylor polynomial } S(f, x_0, n) = \sum_{k=0}^n a_k(f, x_0)(x - x_0)^k, \quad \text{remainder } R_n(f, x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!},$$

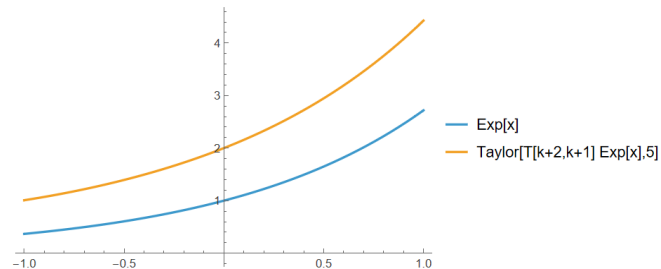
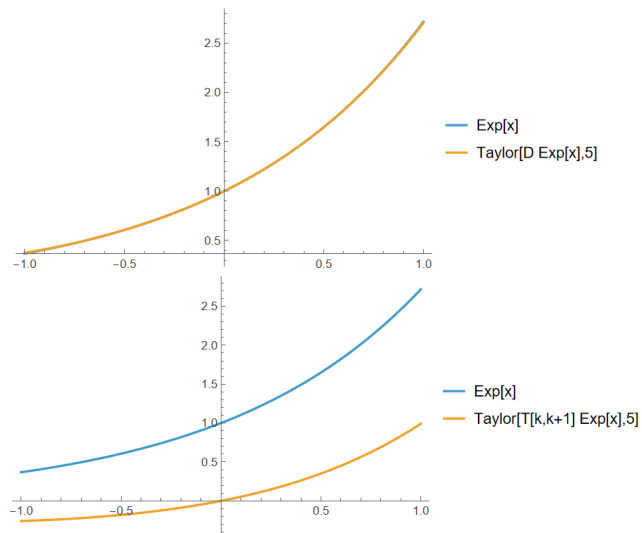
$$\text{Taylor operator } T(\alpha(k), \beta(k)) \text{ acts on Taylor coefficients } (a_k(f, x_0))_k : a_k(Tf, x_0) = \alpha(k) a_{\beta(k)}(f, x_0)$$

$$D = T(k+1, k+1)$$



Examples

$f(x) = \exp(x)$, $x_0 = 0$, truncation $n=5$



Riemann-Liouville general derivative

$$D_a^\alpha f(x) = \frac{d^n}{dx^n} I_a^{n-\alpha} f(x) = \frac{d^n}{dx^n} \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{n-\alpha-1} f(t) dt, n-1 < \alpha < n$$

$$D^n f(x) = \frac{d^n f(x)}{dx^n} \quad n \in \mathbb{N}, n \geq 1$$

approximated by a limit expression similar to derivative D

$$Df(x) = \frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$T(\alpha, a) f(x) = D_a^\alpha f(x), \text{ in an environment of } x$$

$$T(\alpha, a) f(x) = \lim_{h \rightarrow 0} \left((1-\alpha) f(x) + \alpha \frac{f(x+h(x-a)^{1-\alpha}) - f(x)}{h} \right)$$

application

Fractional quantum mechanics

$$E_{kin} = d_\alpha |\vec{p}|^\alpha, \quad d_\alpha = \frac{1}{(2m)^{\alpha-1} c^{\alpha-2}}$$

fractional Laplace

$$(-\hbar^2 \Delta)^{\alpha/2} \psi(\vec{p}, t) = \frac{1}{2\pi \hbar^3} \int \exp\left(-i \frac{\vec{p} \cdot \vec{r}}{\hbar}\right) |\vec{p}|^\alpha \varphi(\vec{p}, t) d^3 p$$

$$\text{fractional QM Hamiltonian } H = d_\alpha (-\hbar \Delta)^{\alpha/2} + V(\vec{r})$$

fractional Schrödinger eq.

$$i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = d_\alpha (-\hbar \Delta)^{\alpha/2} \psi(\vec{r}, t) + V(\vec{r}) \psi(\vec{r}, t)$$

$$\text{Coulomb } V(\vec{r}) = -Ze_0^2 / |\vec{r}|, \quad E_n = -(\alpha-1) E_0 \frac{1}{n^{\alpha/(\alpha-1)}}$$

$$\text{harm. oscillator } V(\vec{r}) = q^2 |\vec{r}|^\beta, \quad E_n = E_0 \left(n + \frac{1}{2}\right)^{\alpha\beta/(\alpha+\beta)}$$

application

Diffusion

Diffusion is described by Fick's equation

$$\frac{\partial \rho(\vec{r}, t)}{\partial t} = D \Delta \rho(\vec{r}, t), \text{ where } \rho(\vec{r}, t) \text{ is the density}$$

Anomalous diffusion processes in complex media is described by

$$\frac{\partial^\alpha \rho(\vec{r}, t)}{\partial t^\alpha} = D \Delta^\beta \rho(\vec{r}, t), \text{ with fractional order } \alpha, \beta$$

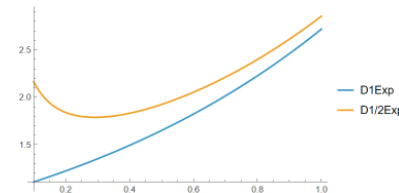
Examples

$$D^{1/2}(x) = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

$$D^{1/2}(1/x) = \frac{\text{EulerGamma} - \text{Log}[x] + \text{PolyGamma}[0, -\frac{1}{2}]}{2\sqrt{\pi} x^{3/2}}$$

$$D^{1/2}(x \wedge 2) = \frac{8x^{3/2}}{3\sqrt{\pi}}$$

$$D^{1/2}(1/x \wedge 2) = \frac{3(-1 + \text{EulerGamma} - \text{Log}[x] + \text{PolyGamma}[0, -\frac{3}{2}])}{4\sqrt{\pi} x^{5/2}}$$



1 P-adic number analysis

[1]

P-adic numbers

P-adic numbers $\mathbb{Q}_p$ are an extension of the field of rationals $\mathbb{Q}$ such that congruences modulo powers of a fixed prime p are related to proximity in the p-adic metric.

Any rational number $x = \frac{a}{b} \in \mathbb{Q}$ can be written uniquely in the form $p^n \frac{u}{v}$ where $p \nmid u$, $p \nmid v$ (u and v not divisible by p), where p is a prime number, r and s are integers not divisible by p .

The corresponding p-adic representation of x is then $x = \sum_{k=n}^{\infty} a_k p^k$, where $a_k \in \{0, 1, \dots, p-1\}$, and $a_n \neq 0$.

The integer n is the order of the p-adic number (index of the lowest non-zero coefficient), n can be negative.

The coefficient a_k obey the arithmetic mod(p), i.e. they are elements of the underlying Galois field F_p , $a_k \in F_p$.

Then we define the p-adic norm of x by $|x|_p = p^{-n}$, also, $|0|_p = 0$

For natural numbers $x \in \mathbb{N}$ we have the finite sum representation $x = \sum_{k=n}^m a_k p^k$ with basis p analogous to the usual decimal representation.

For negative integers $x \in \mathbb{Z}$, $x = -|x|$ we have the representation $x = \sum_{k=n}^m \tilde{a}_k p^k$, where the coefficients are the negative coefficients mod(p) $\tilde{a}_k + a_k = 0 \pmod{p}$ of

the positive number representation $|x| = \sum_{k=n}^m a_k p^k$.

The p-adic integers $\mathbb{Z}_p$ obey multiplication and addition mod(p) with carry, and they have no zero-divisors, i.e. they allow multiplicative inverse, and can be extended to a field $\mathbb{Q}_p$.

Rationals $1/q$, $q \neq p^k$, $k > 0$, are p-adic integers $q^{-1}_p \in \mathbb{Z}_p$

Rationals $1/q$, $q = p^k$, $k > 0$, are proper p-adic fractions e.g. $5^{-1}_5 = .1_5$

Arbitrary rationals $\frac{u}{v}$ (u and v not divisible by p) are formed by multiplication mod(p) $u * (1/v) \pmod{p}$

Valid identities are

$$-1 = \sum_{k=0}^{\infty} (p-1) p^k, \text{ e.g. } (p=5) \dots 4444444_5$$

$$\frac{1}{1-p} = \sum_{k=0}^{\infty} p^k$$

p-adics are written as

- fractions, from right to left, where the point signifies the order (and is skipped, when at beginning, i.e. when order $v=0$), e.g. $p=5$

$$1/5 = .1_5$$

$$1/11 = \dots 42330423304233042331_5$$

- lists, from left to right, where the order is the first element of the list, e.g. $p=5$

$$1/5 = \{-1, \{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}\}$$

$$1/11 = \{0, \{1, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4\}\}$$

The cardinality of $\mathbb{Z}_p$ (and of $\mathbb{Q}_p$) is the cardinality of continuum like $\mathbb{R}$, whereas $\mathbb{Q}$ is countable (cardinality of $\mathbb{N}$).

The p -adics $\mathbb{Q}_p$ were first introduced by Hensel (1897) in a paper which was concerned with the development of algebraic numbers in power series, p -adic numbers were then generalized to valuations by Kürschák in 1913.

Rationals as p -adic numbers

The p -adic valuation (=norm) $|x|_p$ on $\mathbb{Q}$ gives rise to the p -adic metric $d(x, y) = |x - y|_p$

which in turn gives rise to the p -adic topology.

It can be shown that the rationals $\mathbb{Q}$, together with the p -adic metric, do not form a complete metric space.

The completion of this space is the set of complete p -adic numbers $\bar{\mathbb{Q}}_p = \text{Comp}(\mathbb{Q}_p, |x - y|_p)$.

The real numbers $\mathbb{R}$ are the completion of the rationals $\mathbb{Q}$ with respect to the usual absolute value $|x|$,

$\bar{\mathbb{Q}} = \bar{\mathbb{Q}} = \text{Comp}(\mathbb{Q}, |x - y|)$ is the set of all convergent series in $\mathbb{Q}$ in the metric $d(x, y) = |x - y|$

A rational number with p -adic absolute value 1 has a purely periodic p -adic expansion if and only if it lies in the real interval $[-1, 0)$.

A genuine fractional rational number with p -adic absolute value 1, outside the real interval $[-1, 0)$ has a fixed initial expansion followed by a purely periodic p -adic expansion.

p -adic faculty $n!$

The p -adic faculty $n!$ (without p -powers) has p -adic valuation

$$|n!|_p = p^{-(n - A_p(n))/(p-1)}$$

where the p -adic expansion of n is $n = a_0 + a_1 p^1 + \dots + a_L p^L$

and $A_p(n) = a_0 + a_1 + \dots + a_L$

For sufficiently large n , $|n!|_p \leq p^{-n/(2p-2)}$

p -adic metric

The p -adic metric is a non-Archimedean distance function where two numbers are close if their difference is highly divisible by a prime p .

Any rational number $x = \frac{a}{b} \in \mathbb{Q}$ can be written uniquely in the form $p^n \frac{u}{v}$ where $p \nmid u$, $p \nmid v$ (u and v not divisible by p).

The p-adic norm is then $|x|_p = p^{-n}$, the metric is $d(x, y) = |x - y|_p$

The metric is non-archimedean (ultrametric), it obeys the strong triangle inequality $|x - y| \leq \max(|x|, |y|)$

instead of the simple (archimedean) triangle inequality $|x - y| \leq |x| + |y|$.

p-adic half-metric

The p-adic metric yields a distance function sufficient for convergence criteria, but it is little sensitive near zero, therefore not well-suited for numerical calculations.

A better metric function is $f_{rat}(x_p, n, L)$ used for recovering rational numbers from the p-adic numbers.

$|x_p|_{hp} = |f_{rat}(x_p, n, L)|$, where period start position $n=1$ and period length $L=N-1$, N =precision.

$|x_p|_{hp}$ is a half-norm, because it is not symmetric $|x_p|_{hp} \neq |-x_p|_{hp}$, and neither is the distance $|x_p - y_p|_{hp}$, but it satisfies the triangle relation

$|x_p - y_p|_{hp} \leq |x_p|_{hp} + |y_p|_{hp}$.

Examples

p-adic numbers p=5, 10 or 20 digits

p-adic fraction form (right to left), below: list form (left to right)

1=1₅

{0, {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

finite

-1=44444444444444444444₅

{0, {4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4}}

period 4

1/5=.1₅

{-1, {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

here order v=-1, norm $| \cdot |_5 = 1/5$, all others have v=0, $| \cdot |_5 = 1$

11= 21₅

{0, {1, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

finite

$$1/11 = 42330423304233042331_5$$

{0, {1, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4}}}

fixed=1, period 04233

$$3/11 = 233042330423304233043_5$$

{0, {3, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3}}}

fixed=3, period 23304

$$-3/11 = 4233042330423304232311402114021140211402_5$$

{0, {2, 0, 4, 1, 1, 2, 0, 4, 1, 1, 2, 0, 4, 1, 1, 2, 0, 4, 1, 1}}}

period 11402

$$12/11 = 2042330423304233042332_5$$

{0, {2, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4}}}

fixed=2, period 04233

Non-archimedean (ultra-metric) distance

p-adic distance is $d_p(x, y) = |x - y|_p$

A non-Archimedean distance, or ultrametric, is a distance metric $d(x, y)$ that satisfies a strict form of the triangle inequality:

$|x - y| \leq \max(|x|, |y|)$, whereas Euclidean distance obeys the simple triangle inequality $|x - y| \leq |x| + |y|$, i.e. it is Archimedean.

Unlike Euclidean distance, it implies distances do not "add up" or accumulate, often resulting in topologies that are totally disconnected, where all balls are clopen (both open and closed).

- Strong triangle inequality $|x - y| \leq \max(|x|, |y|)$, weaker in general $|x - y| \leq |x| + |y|$
- Strong multiplicativity $|x * y| = |x| * |y|$, weaker in general $|x * y| \leq |x| * |y|$
- Isosceles Triangles: All triangles are isosceles, with the two larger sides equal.
- Ball Structure: Two balls $B_{x,r} = \{x \mid |x| < r\}$ either are disjoint or one is contained within the other.
- Center Property: Any point within a ball can be considered its center.
- Topology: The resulting topology is totally disconnected, where all balls are clopen (open and close)
- Smallness: Numbers that are large in the real sense (like 5^{100}) are small in the p-adic sense mod 5, while small numbers (like $1/5^{100} \bmod 5$) are large.
- **Ostrowski's Theorem:** This theorem states that any non-trivial absolute value on the rational numbers $\mathbb{Q}$ is equivalent to either the usual real absolute value or a p-adic absolute value.

- **Ostrowski's Second Theorem:** The theorem states that any field, complete with respect to an Archimedean absolute value, is (algebraically and topologically) isomorphic to either the real numbers or the complex numbers.

Mapping $\mathbb{Q}_p \rightarrow \mathbb{R}$

There is **no canonical embedding** (a natural, structure-preserving map) from the p-adic numbers $\mathbb{Q}_p$ into the real numbers $\mathbb{R}$.

Here is a breakdown of why this is the case based on number theory principles:

- **Different Completions:** Both $\mathbb{Q}_p$ and $\mathbb{R}$ are completions of the field of rational numbers $\mathbb{Q}$, but they are completed with respect to completely different, non-equivalent metrics (absolute values).
- **Topological Incompatibility:** $\mathbb{R}$ is a connected, archimedean field, while $\mathbb{Q}_p$ is totally disconnected and non-archimedean. There is no continuous field homomorphism between them.
- **Algebraic Closure:** $\mathbb{R}$ is not algebraically closed (its algebraic closure is $\mathbb{C}$), whereas $\mathbb{Q}_p$ is also not algebraically closed (its algebraic closure is $\mathbb{Q}_p^{\text{a}}$).

Mapping $\mathbb{Q} \rightarrow \mathbb{Q}_p$

- **Definition:** The map is defined by $\iota_p : \mathbb{Q} \rightarrow \mathbb{Q}_p$, where $\iota_p(x) = x$
- **Completeness:** The p-adic field $\mathbb{Q}_p$ is defined as the completion of $\mathbb{Q}$ with respect to the p-adic absolute value $|\cdot|_p$

Therefore, $\mathbb{Q}$ acts as a dense subfield of $\mathbb{Q}_p$

- **Dense Subset:** The image $\iota_p(\mathbb{Q})$ is dense in $\mathbb{Q}_p$, meaning any p-adic number can be approximated by a sequence of rational numbers using the p-adic norm.
- **Field Structure:** This mapping is a injective ring homomorphism, preserving both addition and multiplication.
- **p-adic expansion**

Any rational number $\frac{a}{b} \in \mathbb{Q}$ can be written uniquely in the form $p^n \frac{u}{v}$ where $p \nmid u$, $p \nmid v$

The canonical embedding is then $\iota : p^n \frac{u}{v} \rightarrow \sum_{k=n}^{\infty} a_k p^k$, $\mathbb{Q} \rightarrow \mathbb{Q}_p$, with $a_k \in \{0, 1, \dots, p-1\}$

Extensions of $\mathbb{Q}_p$

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- The elements of $\mathbb{Q}_p$ are exactly those represented by series of the form $\sum_{r \in S} a_r p^r$, where S is a bounded-below subset of the integers and each a_r is a

multiplicative lift (i.e., a "Teichmüller lift") of an element of F_p .

- The Teichmüller character is a homomorphism (lift) of multiplicative groups:

$\omega : F_p^\times \rightarrow Z_p^\times$, such that $\omega(a)$ is the unique $(p-1)$ -th root of unity in Z_p which is congruent to a modulo p

- The elements of $\mathbb{Q}_{\text{nr}}$ (the maximal unramified extension of $\mathbb{Q}_p$) are exactly represented by series of the form $\sum_{r \in S} a_r p^r$,

where S is again a bounded-below subset of the integers, but the a_r can now be lifts of any element of F_q , where $q=p^n$ is a fixed power of p .

- The elements of $\bar{Q}_p$ and its completion C_p are represented by certain series of the form $\sum_{r \in S} a_r p^r$, where the a_r are again lifts of elements of F_q , but now S can be a more general well-ordered subset of $\mathbb{Q}$.

Theorem [Kedlaya] (Huang, Rayner, Stefanescu). Let L be the set of generalized power series of the form $f = \sum_{i \in S} x_i t^i$, where the set $S \subset \mathbb{Q}$ (which depends on f)

has the following properties:

1. Every nonempty subset of S has a least element (i.e. S is well-ordered).
2. There exists a natural number m such that every element of $m \cdot S$ has denominator a power of p .

Then L is an algebraically closed field

- The elements of Ω_p , the *spherical completion* of C_p , are represented exactly by series of the form $\sum_{r \in S} a_r p^r$, where the a_r are as before, but now S can be *any* well-ordered subset of $\mathbb{Q}$ (with no other restrictions).

And this is in some sense as far as we can go, since Ω_p is maximally complete: we can't extend it any further without adding geometry, in the sense that it's the unique largest field of characteristic zero (sum of 1's is always $\neq 0$) with value group $\mathbb{Q}$ and residue field the algebraic closure of F_p .

Visualization

[2]

The canonical mapping is $\iota: p^n \frac{u}{v} \rightarrow \sum_{k=n}^{\infty} a_k p^k$, $\mathbb{Q} \rightarrow \mathbb{Q}_p$, with $a_k \in \{0, 1, \dots, p-1\}$

The canonical p -adic scalar valuation is $v_c: \left(\sum_{k=n}^{\infty} a_k p^k \right)_p \rightarrow \left(p^{-n} \sum_{k=n}^{\infty} a_k p^{k-n} \right)$, $\mathbb{Q}_p \rightarrow \mathbb{Q}$

The canonical p -adic vector valuation length n_c is $v_v: \left(\sum_{k=n}^{\infty} a_k p^k \right)_p \rightarrow p^{-n} \{a_k p^{k-n}\}_{k=n}^{n+n_c-1}$, $\mathbb{Q}_p \rightarrow \mathbb{Q}^{n_c}$

Example 1 p -adic norm, prime base $p_0=5$

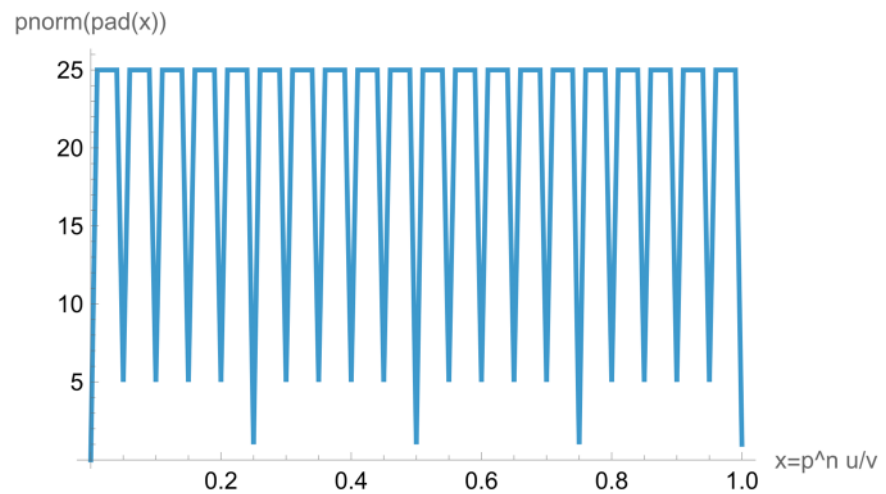
npts=100 number grid points

dint=1 interval $[0,1]$

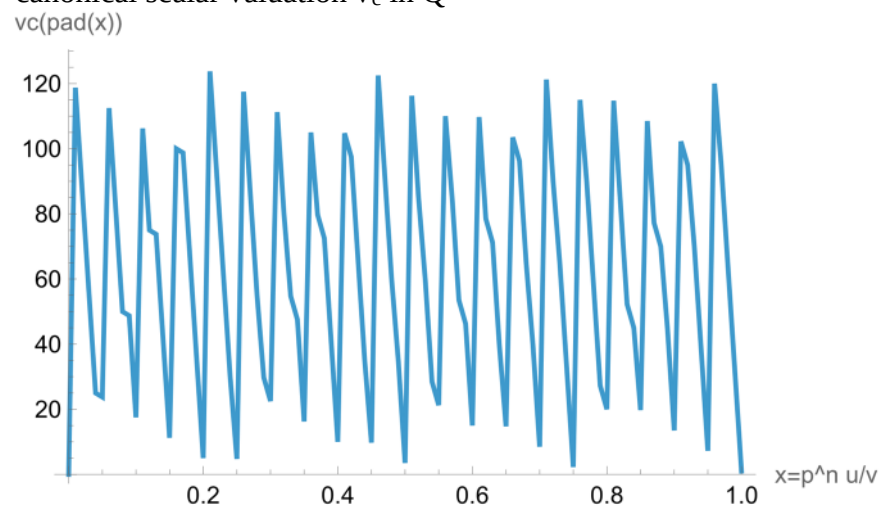
$p_0=5$ base prime $p=5$

Ndigits=100 number of elements in series expansion

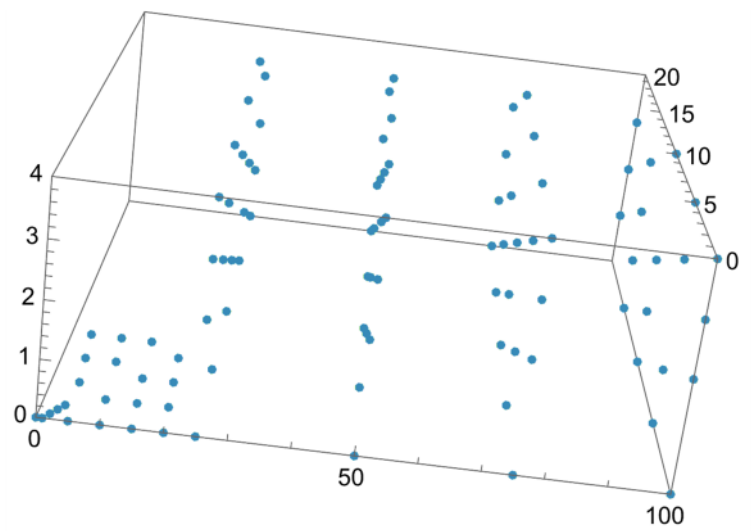
p -adic norm $|x|_p$



canonical scalar valuation v_c in Q

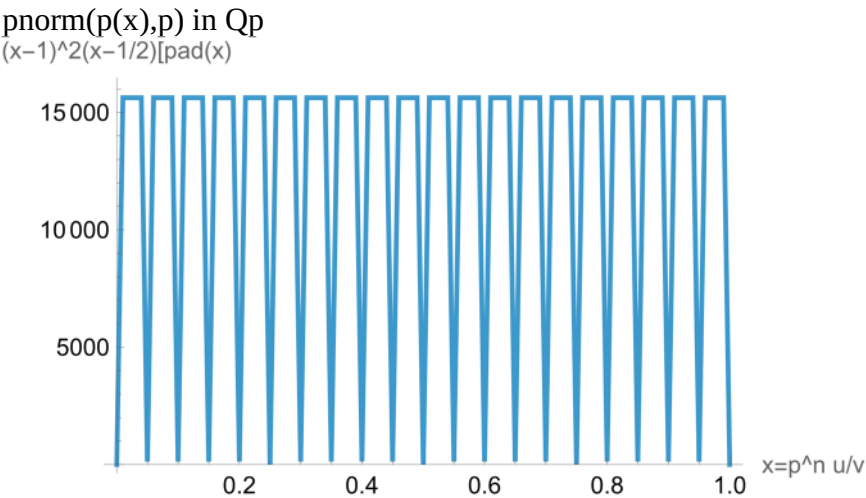
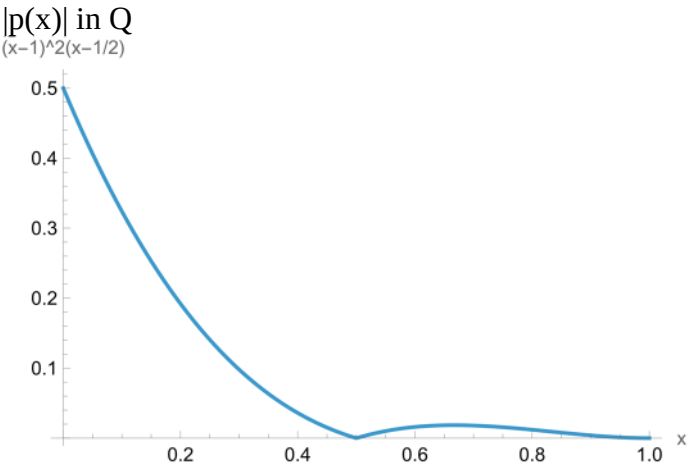


canonical vector valuation v_v with length 3 in Q^3

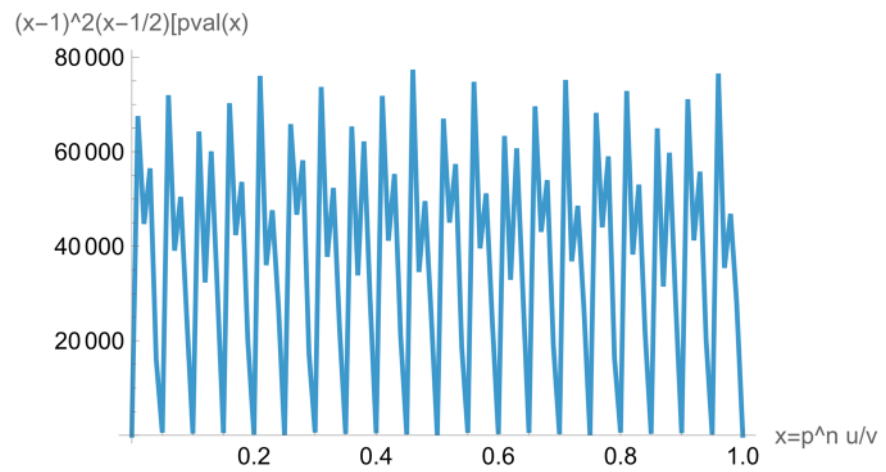


Example 2 polynomial $p(x)=(x-1)^2 (x-12)$, prime base $p_0=5$

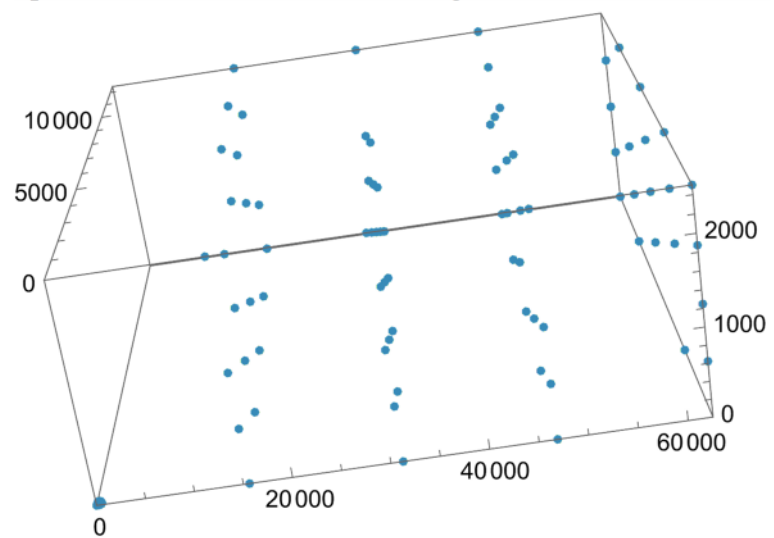
npts=100 number grid points
dint=1 interval [0,1]
 $p_0=5$ base prime
Ndigits=100 number of elements in series expansion
pnorm=p-adic norm $|x|_p$
pval=p-adic scalar valuation v_c
vpval vector valuation v_v with length 3 in Q^3



scalar valuation pval($p(x), p$) in Qp

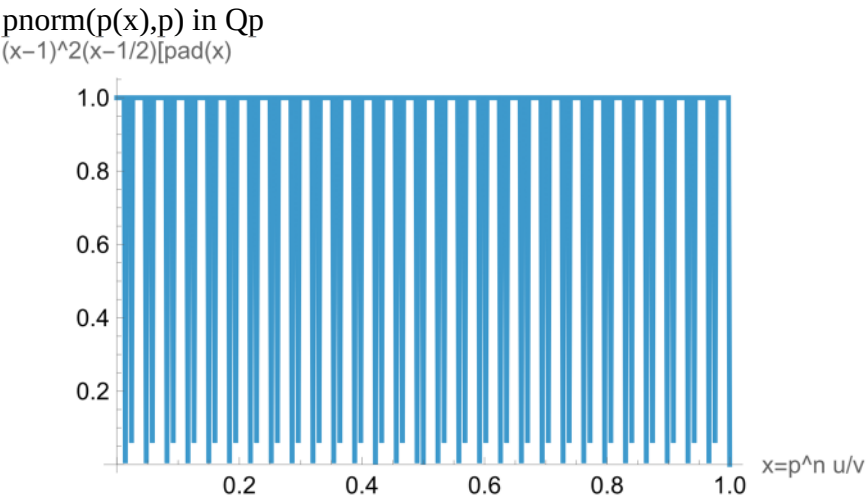
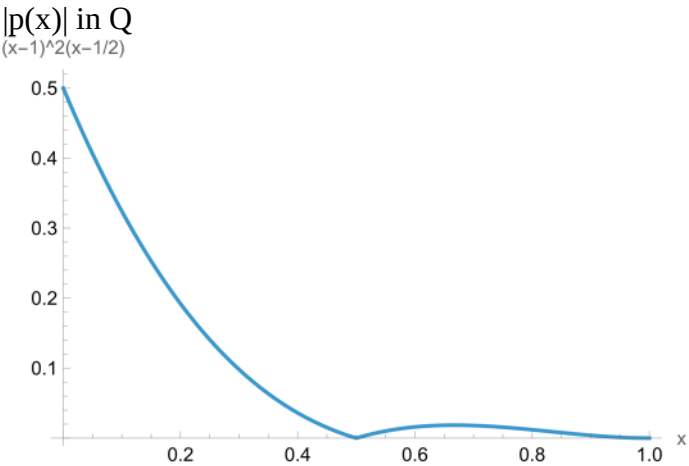


vpval vector valuation v_v with length 3 in Q^3



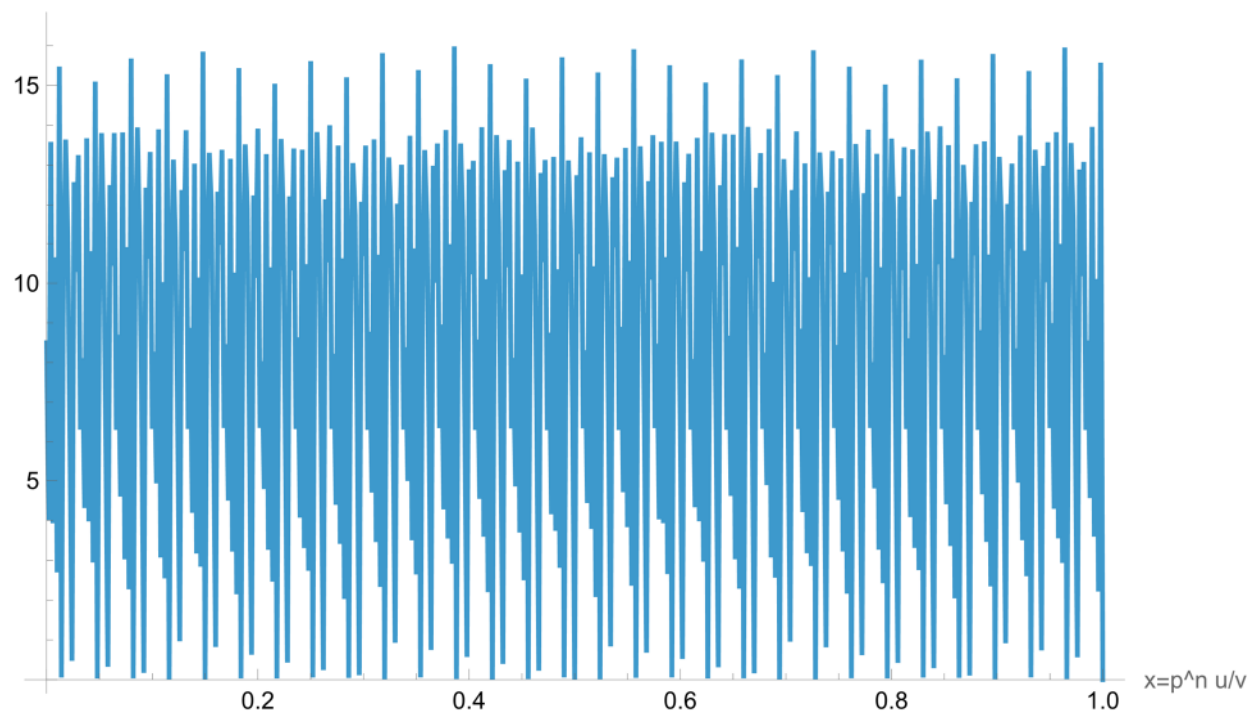
Example 3 polynomial $p(x)=(x-1)^2 (x-12)$, prime base $p_0=17$

npts=100 number grid points
dint=1 interval [0,1]
 $p_0=17$ base prime
Ndigits=500 number of elements in series expansion
pnorm=p-adic norm $|x|_p$
pval=p-adic scalar valuation v_c
vpval vector valuation v_v with length 3 in Q^3

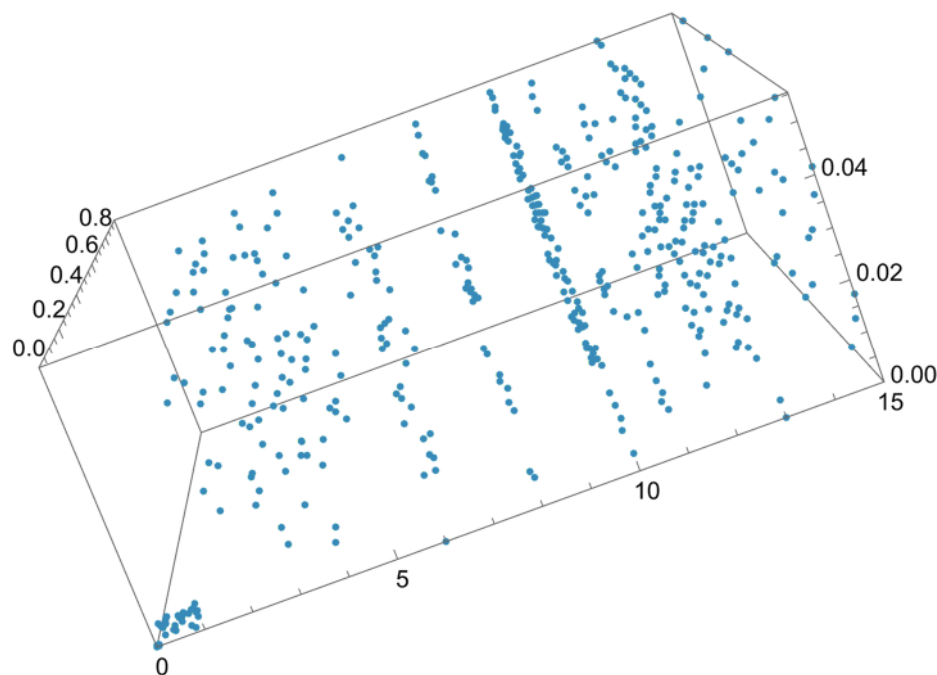


scalar valuation pval($p(x), p$) in Qp

$$(x-1)^2(x-1/2)[pval(x)]$$



vpval vector valuation v_v with length 3 in Q^3



P-adic analysis in brief

In mathematics, p-adic analysis is a branch of number theory that studies functions of p-adic numbers. Along with the more classical fields of real and complex analysis, which deal, respectively, with functions on the real and complex numbers, it belongs to the discipline of mathematical analysis.

The theory of complex-valued numerical functions on the p-adic numbers is part of the theory of locally compact groups (abstract harmonic analysis). The usual meaning taken for p-adic analysis is the theory of p-adic-valued functions on spaces of interest.

Applications of p-adic analysis have mainly been in number theory, where it has a significant role in diophantine geometry and diophantine approximation. Some applications have required the development of p-adic functional analysis and spectral theory. In many ways p-adic analysis is less subtle than classical analysis, since the ultrametric inequality means, for example, that convergence of infinite series of p-adic numbers is much simpler. Topological vector spaces over p-adic fields show distinctive features; for example aspects relating to convexity and the Hahn–Banach theorem are different.

Ostrowski's theorem

Ostrowski's theorem, (1916), states that every non-trivial absolute value on the rational numbers $\mathbb{Q}$ is equivalent to either the usual real absolute value or a p-adic absolute value.

Mahler's theorem

expresses continuous p-adic functions in terms of polynomials.

In any field of characteristic 0, one has the following result.

In mathematics, the characteristic of a ring R , often denoted $\text{char}(R)$, is defined to be the smallest positive number of copies of the ring's multiplicative identity (1) that will sum to the additive identity (0). If no such number exists, the ring is said to have characteristic zero.

Let $(\nabla f)(x) = f(x+1) - f(x)$ be the forward difference operator. Then for polynomial functions f we have the Newton series:

$$f(x) = \sum_{k=0}^{\infty} (\nabla^k f)(0) \binom{x}{k}$$

$$\text{where } \binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!}$$

is the k -th binomial coefficient polynomial.

Over the field of real numbers, the assumption that the function f is a polynomial can be weakened, but it cannot be weakened all the way down to mere continuity.

Mahler's theorem: If f is a continuous p -adic-valued function on the p -adic integers then the Newton series identity holds.

Hensel's lemma

Hensel's lemma, also known as Hensel's lifting lemma, named after Kurt Hensel, is a result in modular arithmetic, stating that

if a polynomial equation has a simple root modulo a prime number p , then this root corresponds to a unique root of the same equation modulo any higher power of p ,

which can be found by iteratively "lifting" the solution modulo successive powers of p . More generally it is used as a generic name for analogues for complete commutative rings (including p -adic fields in particular) of the Newton method for solving equations.

Since p -adic analysis is in some ways simpler than real analysis, there are relatively easy criteria guaranteeing a root of a polynomial.

To state the result, let $f(x)$ be a polynomial with integer (or p -adic integer) coefficients, and let m, k be positive integers such that $m \leq k$.

If r is an integer such that

$$f(r) \equiv 0 \pmod{p^k} \text{ and } f'(r) \not\equiv 0 \pmod{p}$$

then there exists an integer s such that

$$f(s) \equiv 0 \pmod{p^{k+m}} \text{ and } r \equiv s \pmod{p^k}$$

Furthermore, this s is unique modulo p^{k+m} , and can be computed explicitly as

$$s = r + t p^k \text{ where } t = -\frac{f(r)}{p^k f'(r)}$$

Hasse principle

Helmut Hasse's local–global principle, also known as the Hasse principle, is the idea that one can find an integer solution to an equation by using the Chinese remainder theorem to piece together solutions modulo powers of each different prime number. This is handled by examining the equation in the completions of the rational numbers: the real numbers and the p -adic numbers. A more formal version of the Hasse principle states that certain types of equations have a rational solution if and only if they have a solution in the real numbers and in the p -adic numbers for each prime p .

In mathematics, the *Chinese remainder theorem* states that if one knows the remainders of the Euclidean division of an integer n by several integers, then one can determine uniquely the remainder of the division of n by the product of these integers, under the condition that the divisors are pairwise coprime (no two divisors share a common factor other than 1).

p -adic exponential function

In mathematics, particularly p-adic analysis, the p-adic exponential function is a p-adic analogue of the usual exponential function on the complex numbers. As in the complex case, it has an inverse function, named the p-adic logarithm.

The usual exponential function on $\mathbb{C}$ is defined by the infinite series

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Entirely analogously, one defines the exponential function on $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$, by

$$\exp_p(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

However, unlike $\exp$ which converges on all of $\mathbb{C}$, $\exp_p$ only converges on the disc

$$|z|_p < p^{-1/(p-1)}$$

It follows from Legendre's formula that if $|z|_p < p^{-1/(p-1)}$ then $\left| \frac{z^n}{n!} \right|_p \rightarrow 0$, p-adically.

Legendre formula

For any prime number p and any positive integer n , let $v_p(n)$ be the exponent of the largest power of p that divides n (that is, the p-adic valuation of n).

Then

$$v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$$

where $\lfloor x \rfloor$ is the floor function.

Calculus over p-adic numbers in brief

Calculus over p-adic numbers ($\mathbb{Q}_p$) is a branch of analysis that operates on a field constructed from rational numbers using a non-Archimedean metric, where numbers are considered close if their difference is divisible by a high power of a prime p . Unlike real calculus, p-adic calculus uses an ultrametric, where convergence of series is much simpler, and the concept of "derivative" does not necessarily imply constant behavior for zero-derivative functions.

Fundamental Concepts in p-adic Calculus

Derivative

derivative of p-adic polynomial $f(x) \in \mathbb{Q}_p[x]$, $f(x) = \sum_{k=0}^n a_k x^k$

$$\text{is } f'(x) = \sum_{k=1}^n k a_k x^{k-1}$$

The derivative is computed by taking the limit

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \text{ for } h \rightarrow 0, \text{ i.e. } h(n) = \sum_{i \geq n} p^i$$

forward-difference operator ∇ on the binomial functions:

$$(\nabla f) = f(x+1) - f(x)$$

$$\nabla \binom{x}{0} = 0, \quad \nabla \binom{x}{i} = \binom{x}{i-1} \quad i \geq 1$$

compare: the **differential operator** D on x -powers $x^i / i!$

$$D(x^0) = 0, \quad D\left(\frac{x^i}{i!}\right) = \frac{x^{i-1}}{(i-1)!} \quad i \geq 1$$

Mahler series

Theorem (Mahler)

Let $f : Q_p \rightarrow Q_p$, then there is a unique sequence $(m_i)_{i \geq 0}$, such that

$$f(x) = \sum_{i=0}^x m_i \binom{x}{i}, \quad m_i = (\nabla^i f)(0)$$

With Pochhammer symbol $(x)_0 = 1$, $(x)_i = x(x-1)\dots(x-i+1) \quad i \geq 1$

$$\text{binomial polynomials are } \nabla (x)_i = i(x)_{i-1}, \quad \binom{x}{i} = \frac{(x)_i}{i!}$$

$$\text{Mahler series becomes } f(x) = \sum_{i=0}^x \frac{(\nabla^i f)(0)}{i!} (x)_i, \quad (\nabla^k f)(0) = \sum_{i \leq k} (-1)^{k-i} \binom{k}{i} f(i)$$

$$\text{compare: classical analysis Taylor series } f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i$$

Unlike real analysis, the p -adic derivative of a non-constant function can be zero.

However, for a polynomial $f(x) \in Q_p[x]$, the formal derivative f' is not zero, unless $f = \text{const}$.

Integration

sum operator $Sf(x) = F(x)$, $F(n) = \sum_{0 \leq i \leq n} f(i)$ is the inverse of ∇ in comparison to the integration operator $\int f(x) dx$

p -adic Metric: The p -adic absolute value, $|x|_p$ is defined by factoring out the highest power of a prime p from a number.

The distance between two numbers is $|x - y|_p$

Convergence: A p -adic series $\sum_i a_i$, $a_i \in Q_p$ converges if and only if $\lim_{i \rightarrow \infty} |a_i|_p = 0$

This is a major simplification compared to real analysis, as convergence is only determined by the size of the terms, not their alternating sign.

Ultrametric Inequality: p-adic numbers satisfy a stronger triangle inequality:

$$|x + y|_p \leq \max(|x|_p, |y|_p) \text{ instead of } |x + y| \leq |x| + |y|$$

Compactness: The space of p-adic integers $\mathbb{Z}_p$ is a compact ring, which makes it a natural domain for integration.

Derivatives in $\mathbb{Q}_p$

Definition: A function $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ is differentiable at x_0

if a p-adic linear map exists representing the local behavior of the function, similar to the real-valued case.

f is strictly differentiable at a point $a \in X$ if the difference quotient

$$\Phi f(a + h, a) = \frac{f(a + h) - f(a)}{h} \text{ has a limit } f'(a) = \lim_{h \rightarrow 0} \Phi f(a + h, a)$$

Definition: We say that f is strictly differentiable at a point $a \in X$ and denote this property by $f \in S^1(a)$ if the difference quotients

$$\Phi f(x, y) = \frac{f(x) - f(y)}{x - y} \text{ have a limit } f'(a) = \lim_{(x, y) \rightarrow (a, a)} \Phi f(x, y)$$

Zero Derivative Surprise: In p-adic analysis, a function can have a (strict) derivative of zero everywhere, but not be a constant function. This is because the p-adic numbers are totally disconnected, allowing functions to "jump" without being continuous in the real sense.

Hensel's Lemma: An important analytical tool, essentially a p-adic version of Newton's method, that allows for finding roots of polynomials by refining approximate roots.

Integration in $\mathbb{Q}_p$

Haar Measure: Because p-adic numbers are locally compact groups, they have a natural translation-invariant measure (the Haar measure), which is used to define integration.

p-adic Gamma Function: A specific function of interest defined using these principles.

Coleman's Theory: A powerful theory of p-adic line integrals (or "p-adic abelian integrals") that enables a "Fundamental Theorem of Calculus" analogue, which can demonstrate that the integral of a $1/x$ function over a closed loop is zero, despite the different topological structure.

Application of p-adic analysis in Planck-scale physics

[6]

In Planck-scale physics, the uncertainty is in the order of the measured value of physical variables, in case of length we have the relation

$$\Delta x \geq l_p = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-35} m, \text{ and the space-time is discrete with the Planck length as fundamental unit.}$$

There are several reasons for p-adic-discrete mathematics being the most appropriate description of Planck-scale physics.

- the triangle inequality is no longer valid, we have a non-Archimedean geometry, typical for p-adic numbers.
- the fractal and hierarchical character of quantum spin foam at Planck scale are well represented in p-adic analysis, but not in complex analysis
- space-time vectors are integer variables on an equidistant grid, no longer continuous variables in real and complex numbers

- all physical variables become discrete: space-time unit is l_p , energy-momentum unit is minimal energy $E_{\min} = \hbar c / R_\Lambda = 2.06 \times 10^{-34} eV$,
 $R_\Lambda = 1 / \sqrt{\Lambda_0} = 0.95 \times 10^{26} m$ Lambda-radius, $\Lambda_0 = 1.1 \times 10^{-52} m^{-2}$ cosmological constant
- the continuous differential operators in quantum mechanics become discrete difference operators at Planck scale, which is exactly the case in p-adic analysis (differential operator ∇ in real and complex analysis is replaced by the forward-difference operator ∇_p in p-adic analysis)
- the differential equations of quantum mechanics (Schrödinger, Dirac, Maxwell equations) become discrete difference equations
- the eigenfunctions of difference equations are discrete polynomials, based on binomial expressions, which corresponds to series expansions in p-adic analysis, which are based on binomial polynomials (Mahler series instead of Taylor series).
- a natural choice of p-adic base is p=2, i.e. bi-adic numbers, which fits well to the Pauli principle for spinors, and their bi-adic nature (Grassmann algebra) .

Schrödinger und Dirac equation as p-adic difference equations

Schrödinger equation in Planck units, $\hbar = G = c = 1$, $l_p = \sqrt{\frac{\hbar G}{c^3}} = 1$ is ,

$$i \frac{\partial \Psi(\vec{x}, t)}{\partial t} = \left(-\frac{1}{2m} \nabla^2 + V(\vec{x}) \right) \Psi(\vec{x}, t)$$

where space-time vectors are discrete in $l_p = 1$ -units , i.e. are consist of integers

with time-independent form

$$E \Psi(\vec{x}) = \left(-\frac{1}{2m} \nabla^2 + V(\vec{x}) \right) \Psi(\vec{x})$$

Dirac equation in Planck units with electromagnetic field is

$$\left(i \gamma^\mu \left(\partial_\mu + ie A_\mu \right) - m \right) \Psi(x^\mu) = 0$$

We can formulate both equations with p-adic functions $\Psi : Z_p \rightarrow C_p$, field vector $A_\mu : Z_p \rightarrow C_p$, and the p-adic forward-difference operator

$\nabla_p(x) f(x) = f(x+1) - f(x)$, Z_p p-adic integers, C_p p-adic complex numbers.

P-adic Schrödinger equation is

$$i \nabla_p(t) \Psi(\vec{x}, t) = \left(-\frac{1}{2m} \left(\vec{\nabla}_p(\vec{x}) \cdot \vec{\nabla}_p(\vec{x}) \right) + V(\vec{x}) \right) \Psi(\vec{x}, t) \text{ p-adic Schrödinger}$$

with 3-dim forward-difference-gradient $\vec{\nabla}_p(\vec{x}) = (\nabla_p(x_1), \nabla_p(x_2), \nabla_p(x_3))$

P-adic discrete Dirac equation in Planck units with electromagnetic field is

$$\left(i \gamma^\mu \left(\nabla_p^\mu(x^\mu) + ie A_\mu \right) - m \right) \Psi(x^\mu) = 0 \text{ p-adic Dirac}$$

with 4-dim forward-difference-gradient $\nabla_p^\mu(x^\mu) = (\nabla_p(x_0), \nabla_p(x_1), \nabla_p(x_2), \nabla_p(x_3))$

Here, $|\Psi|^2$ is the probability density, normalized appropriately over volume V :

$$\sum_{x^\mu \in V} \left(|\Psi(x^\mu)| \right)^2 = 1 .$$

The time-independent p-adic Schrödinger equation results from the time-dependent continuous form:

$$E \Psi(\vec{x}) = \left(-\frac{1}{2m} \left(\vec{\nabla}_p(\vec{x}) \cdot \vec{\nabla}_p(\vec{x}) \right) + V(\vec{x}) \right) \Psi(\vec{x}), \text{ with eigenvalues } E = E_n$$

P-adic Schrödinger equation with harmonic oscillator

Time-independent Schrödinger eq. for harmonic oscillator in Planck units is [7]

$$E \Psi(x) = \left(-\frac{1}{2m} \nabla^2 + \frac{1}{2} m \omega^2 x^2 \right) \Psi(x)$$

with energy eigenvalues $E = (n + 1/2) \hbar \omega$ and eigenfunctions

$$\Psi_n(x) = \sqrt{\frac{1}{2^n n!}} \left(\frac{m\omega}{\pi \hbar} \right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\left(x \sqrt{\frac{m\omega}{\hbar}}\right), \text{ Hermite polynomials } H_n$$

reformulated $m = 1$, $\omega = 1$

$$E \Psi(x) = \left(-\frac{1}{2} \nabla^2 + \frac{1}{2} x^2 \right) \Psi(x)$$

Hermite polynomials have the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) , H_0(x) = 1$$

For the p-adic discrete version we consider x on a grid with N points, stepsize $h = \sqrt{2/N}$,

$$x_k = kh , k = -N, ..., 0, ..., N$$

Time-independent P-adic Schrödinger equation for harmonic oscillator is

$$E \Psi(x) = \left(-\frac{1}{2} \left(\nabla_p(x) \cdot \nabla_p(x) \right) + \frac{1}{2} x^2 \right) \Psi(x) ,$$

reformulated

$$E \Psi(x_k) = \left(-\frac{1}{2} \left(\Psi(x_{k+2}) - 2\Psi(x_{k+1}) + \Psi(x_k) \right) + \frac{1}{2} x_k^2 \Psi(x_k) \right)$$

This difference equation has the solution in Kravchuk functions [7], based on scaled Kravchuk polynomials $k_{n,h}(x)$ with the recurrence relation analogous to Hermite polynomials

$$k_{n+1,h}(x) = 2xk_{n,h}(x) - 2n \left(1 - h^2 \frac{n-1}{2} \right) k_{n-1,h}(x) , k_{0,h} = 1$$

discrete Kravchuk polynomials

$$K_n(k, N) = \frac{1}{2^n} \sum_{j=0}^n (-1)^{n-j} \binom{k}{j} \binom{N-k}{n-j}$$

$$k_{n,h}(x) = \frac{\alpha_{n,h}}{d_n} K_n \left(\frac{1}{h} \left(\frac{1}{h} + x \right), N \right)$$

$$\text{with the norm factor } \alpha_{n,h} = \frac{1}{h^N} \sqrt{\frac{(N-n)!}{N!n!}}, \text{ and } d_n = \frac{1}{2^n} \sqrt{\frac{N!}{n!(N-n)!}}$$

$$d_n = \frac{1}{2^n} \sqrt{\frac{N!}{n!(N-n)!}},$$

Kravchuk functions are

$$\varphi_{n,h}(k) = \alpha_{n,h} k_{n,h}(hk) \sqrt{\rho_h(hk)}$$

$$\text{with norm function } \rho_h(k) = \frac{1}{h 2^N} \binom{N}{k}, \quad a = kh$$

with the eigenvalues $E = (n+1/2)$.

Theorem: Kravchuk polynomials converge uniformly to Hermite polynomials

For given N and $h = \sqrt{2/N}$, for all $0 < \delta < 1$, there is a constant $C(\delta)$, with the following bound

$$\text{for all } n \leq \frac{\delta}{3} \left| \log h \right| \quad x \in R, \quad \left| H_n(x) - k_{n,h}(x) \right| \leq C(\delta) h^{2-\delta} e^{x^2/2} 2^{n/2} \sqrt{n!}$$

2 Real analysis

[3]

Real analysis studies real numbers $\mathbb{R}$, focusing on continuity, limits, and integration on a line, while complex analysis explores functions over complex numbers $\mathbb{C} = a + bi$, which form a 2D plane.

Real analysis is the branch of mathematical analysis that is concerned with properties of real numbers and real-valued functions. Classical real analysis provides the rigorous foundations of calculus, specifically convergent series, limits, continuity, differentiability, smoothness, and integrability. https://en.wikipedia.org/wiki/Real_analysis_-_cite_note-1 Central to the subject is the completeness of the real numbers, which distinguishes the real numbers from the rational numbers and underlies many basic results about limits and continuous functions.

Basics

Limit

Definition. Let f be a real-valued function defined on $E \subset \mathbb{R}$. We say that $f(x)$ tends to L as x approaches x_0 , or that the limit of $f(x)$ as x approaches x_0 is L if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in E$, $0 < |x - x_0| < \delta$ implies that $|f(x) - L| < \varepsilon$. We write this symbolically as $f(x) \rightarrow L$ as $x \rightarrow x_0$.

Definition. Let (a_n) be a real-valued sequence. We say that (a_n) is a Cauchy sequence if, for any $\varepsilon > 0$, there exists a natural number N such that $m, n \geq N$ implies that $|a_m - a_n| < \varepsilon$.

It can be shown that a real-valued sequence is Cauchy if and only if it is convergent.

Uniform and pointwise convergence

Roughly speaking, pointwise convergence of functions f_n to a limiting function $f: E \rightarrow \mathbb{R}$, denoted $f_n \rightarrow f$, simply means that given any $x \in E$, $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$. In contrast, uniform convergence is a stronger type of convergence, in the sense that a uniformly convergent sequence of functions also converges pointwise, but not conversely.

Uniform convergence requires members of the family of functions, f_n , to fall within some error $\varepsilon > 0$ of f for every value of $x \in E$, whenever $n \geq N$, for some integer N .

Compactness

Definition. A set $E \subset \mathbb{R}$ is compact if it is closed and bounded.

Definition. A set E in a metric space is compact if every sequence in E has a convergent subsequence.

Continuity

Definition. If $I \subset \mathbb{R}$ is a non-degenerate interval, we say that $f: I \rightarrow \mathbb{R}$ is continuous at $p \in I$ if

$$\lim_{x \rightarrow p} f(x) = f(p)$$

We say that f is a continuous map if f is continuous at every $p \in I$.

Definition. If X is a subset of the real numbers, we say a function $f: X \rightarrow \mathbb{R}$ is uniformly continuous on X if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$, $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

Definition. Let $I \subset \mathbb{R}$ be an interval on the real line. A function $f: I \rightarrow \mathbb{R}$ is said to be absolutely continuous on I if for every positive number ε , there is a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ of I satisfies

$$\sum_{k=1}^n (y_k - x_k) < \delta$$

then

$$\sum_{k=1}^n |f(y_k) - f(x_k)| < \varepsilon$$

Differentiability

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at a if the limit

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \text{ exists}$$

Series convergence

$$\sum_{k=1}^n \frac{1}{2^k} = 1$$

$$\sum_{k=1}^n \frac{1}{k} = \infty$$

$$\sum_{k=1}^n \frac{(-1)^k}{k} = \log 2$$

Taylor series

The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!} (x-a)^i$$

Fourier series

The Fourier series of a complex-valued P -periodic function $f(x)$, integrable over the interval $[0, P]$ on the real line, is defined as a trigonometric series of the form

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \exp(i 2\pi k x / P)$$

$$c_k = \frac{1}{P} \int_0^P f(x) \exp(-i 2\pi k x / P) dx$$

real form

$$f_N(x) = a_0 + \sum_{k=1}^N (a_k \cos(2\pi k x / P) + b_k \sin(2\pi k x / P))$$

$$a_0 = \frac{1}{P} \int_0^P f(x) dx, \quad a_k = \frac{2}{P} \int_0^P f(x) \cos(2\pi k x / P) dx, \quad b_k = \frac{2}{P} \int_0^P f(x) \sin(2\pi k x / P) dx$$

Integration

We say that the Riemann integral of f on $[a, b]$ is S if for any $\varepsilon > 0$ there exists $\delta > 0$ such that,

for partition $\Delta_i = (x_{i+1} - x_i)$ $x_i \in [a, b]$ with mesh $\|\Delta_i\| < \delta$, we have

$$\left| S - \sum_{k=1}^n f(x_i) \Delta_i \right| < \varepsilon$$

$$\text{i.e. } \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^{n-1} f(x_i) (x_{i+1} - x_i) \text{ for } x_i \in [a, b]$$

Important results

Bolzano-Weierstraß

The theorem states that each infinite bounded sequence in $\mathbb{R}_n$ has a convergent subsequence, i.e. $\mathbb{R}_n$ is locally compact.

Heine-Borel theorem

For a subset S of Euclidean space $\mathbb{R}_n$, the following two statements are equivalent:

- S is compact, that is, every open cover of S has a finite subcover
- S is closed and bounded.

L'Hôpital's rule

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

where f' and g' are the derivatives of f and g .

Mean value theorem

Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function on the closed interval $[a, b]$, and differentiable on the open interval (a, b) , where $a < b$.

Then there exists $\xi \in (a, b)$ such that $c \in [a, b]$

Taylor's theorem

Let $k \geq 1$ be an integer and let the function $f: \mathbb{R} \rightarrow \mathbb{R}$ be k times differentiable at the point $a \in \mathbb{R}$.

Then there exists a function $h_k: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) = \sum_{i=0}^k \frac{f^{(i)}(a)}{i!} (x-a)^i + h_k(x)$$

$$\text{and remainder } h_k(x) = \frac{f^{(k+1)}(\xi)}{(k+1)!} (x-a)^{k+1}, \quad \xi \in [a, x]$$

$$\lim_{x \rightarrow a} h_k(x) = 0$$

Weierstrass approximation theorem

Let $p_n(x) = \sum_{i=0}^n f\left(a + \frac{i}{n}(b-a)\right) B_{ni}^n(x)$ be a continuous function.

If $\varepsilon > 0$ is given, then there exists a polynomial function p_n such that $|f(x) - p_n(x)| < \varepsilon$ for all $x \in [a, b]$

$$p_n(x) \rightarrow f(x)$$

Here, the approximating polynomials are the Bernstein polynomials

$$B_{i,n}^{a,b}(x) = \frac{1}{(b-a)^n} \binom{n}{i} (x-a)^i (x-b)^{n-i}$$

with the error

$$\|f - p_n\| \leq \frac{\pi}{2} \frac{1}{(n+1)^k} \|f^{(k)}\|$$

Fundamental theorem of calculus

Let f be a continuous real-valued function defined on a closed interval $[a, b]$.

Let $F = I_a f$ be the function defined, for all x in $[a, b]$, by

$$(I_a f)(x) = \int_a^x f(t) dt$$

Then F is uniformly continuous on $[a, b]$ and differentiable on the open interval (a, b) , and

$$(I_a f)'(x) = f(x)$$

for all x in (a, b) so F is an antiderivative of f .

Newton–Leibniz theorem

Let f be a real-valued function on a closed interval $[a, b]$ and F a continuous function on $[a, b]$

which is an antiderivative (inverse-derivative) of f in (a, b) : $F'(x) = f(x)$

If f is Riemann integrable on $[a, b]$ then $F(b) - F(a) = \int_a^b f(t) dt$

Dini theorem

If X is a compact topological space, and $(f_n)_{n \in \mathbb{N}}$ is a monotonically increasing sequence

(meaning $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in X$) of continuous real-valued functions on X which converges pointwise to a continuous function $f: X \rightarrow \mathbb{R}$, then the convergence is uniform.

Banach fixed-point theorem

Let (X, d) be a non-empty complete metric space with a contraction mapping $f: X \rightarrow X$

$$d(f(x), f(y)) \leq q d(x, y), \quad q < 1, \quad x, y \in X$$

Then f admits a unique fixed point $x_* \in X$, meaning $f(x_*) = x_*$

Furthermore, x_* can be found as follows:

start with $x_0 \in X$ and iterate

$$x_{n+1} = f(x_n), \text{ then } x_n \rightarrow x_*$$

Hahn-Banach theorem: functional with given values

Let $(x_i)_{i \in I}$ be vectors in a real or complex normed space X and let $(c_i)_{i \in I}$ be scalars also indexed by $I \neq \emptyset$.

There exists a continuous linear functional f on X such that $f(x_i) = c_i$ for all $i \in I$

if and only if there exists a $K > 0$ such that for any choice of scalars $(s_i)_{i \in I}$ where all but finitely many s_i are 0, the following holds:

$$\left| \sum_{i \in I} s_i c_i \right| < K \left| \sum_{i \in I} s_i x_i \right| \quad (\text{linear form is low-bounded})$$

Stokes theorem

Let Σ be a smooth oriented surface in $\mathbb{R}_3$, parametrized by $\Sigma(u, v)$,
with boundary $\partial \Sigma \equiv \Gamma$, parametrized by $\Gamma(t)$.

If a vector field $F(x, y, z)$ has continuous first-order partial derivatives in Σ ,

then

$$\iint_{\Sigma} (\nabla \times F) \cdot d\Sigma = \oint_{\partial \Sigma} F \cdot d\Gamma$$

Newton-Raphson approximation method

The Newton-Raphson method is a solution procedure for a zero α of a differentiable function f

$$\varepsilon_n = |\alpha - x_n|$$

the error at n steps $\xi_n \in [x_n, \alpha]$, $|\varepsilon_{n+1}| = \varepsilon_n^2 \frac{|f''(\xi_n)|}{2|f'(\xi_n)|}$

is $|\varepsilon_{n+1}| = \varepsilon_n^2 \frac{|f''(\xi_n)|}{2|f'(\xi_n)|}$, so the convergence is quadratic.

3 Complex analysis

[4]

Complex analysis, traditionally known as the **theory of functions of a complex variable**, is the branch of mathematical analysis that investigates functions of complex numbers.

Complex analysis features superior rigidity in comparison with real analysis—differentiability implies infinite differentiability (analyticity)—unlike real analysis where derivatives may not be continuous.

Complex functions that are differentiable at every point of an open subset Ω of the complex plane are said to be holomorphic on Ω . In the context of complex analysis, the derivative of f at z_0 is defined to be

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

The following properties are equivalent for $(z) = u(x, y) + iv(x, y)$

- f is holomorphic in Ω
- f satisfies in Ω the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- f is expandable in Taylor series $f(z) = \sum_{k=0}^{\infty} a_k (z-c)^k$, which converges in a neighborhood of c for every $c \in \Omega$

Important results

Great Picard's Theorem

If an analytic function f has an essential singularity at a point w , then on any punctured neighborhood of w , $f(z)$ takes on all possible complex values, with at most a single exception, infinitely often.

Cauchy integral formula

If $f: U \rightarrow \mathbb{C}$ is a holomorphic function, and γ is a curve oriented counterclockwise, forming the boundary of D .

Then for every a in D ,

$$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)} dz$$

Laurent series

The Laurent series for a complex function $f(z)$ about a point c is given by

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-c)^k$$

where a_n and c are constants, with a_n defined by a line integral that generalizes Cauchy's integral formula:

$$a_k = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-c)^{k+1}} dz$$

The path of integration γ is counterclockwise around a curve enclosing c and lying in an annulus A in which $f(z)$ is holomorphic (=analytic).

Liouville's theorem

A bounded function that is holomorphic in the entire complex plane must be constant.

Riemann mapping theorem

If U is a non-empty simply connected open subset of the complex number plane $\mathbb{C}$ which is not all of $\mathbb{C}$, then there exists a biholomorphic mapping f from U onto the open unit disk

$$D = \{z \in \mathbb{C} \mid |z| < 1\}$$

The map f is essentially unique: if z_0 is an element of U and ϕ is an arbitrary angle, then there exists precisely one f as above such that $f(z_0) = 0$ and such that the argument of the derivative of f at the point z_0 is equal to ϕ .

Runge's approximation theorem : approximation by rational functions

Denoting by $\mathbb{C}$ the set of complex numbers, let K be a closed subset of $\mathbb{C} \cup \{\infty\}$ and let f be a function which is holomorphic on an open set containing K . If A is a set containing at least one complex number from every connected component of $\mathbb{C} \cup \{\infty\} \setminus K$, then there exists a sequence (r_n) of rational functions which converges uniformly to f on K and such that all the poles of the functions (r_n) are in A .

4 Quaternionic analysis

Basics

Quaternionic analysis is an extension of complex analysis to the setting of quaternions, which retains the central concepts of analyticity, differentiation, and integration, but is profoundly more intricate due to the noncommutative nature of quaternions.

Quaternionic analysis extends complex analytic concepts into four-dimensional quaternionic space, replacing the Cauchy-Riemann equations with the Fueter equation. It reconciles analyticity, harmonicity, and integral structures within a noncommutative setting, forming the foundation for further explorations in both pure and applied mathematics.

Quaternions

A quaternion $q \in \mathbb{H}$ can be expressed as:

$$q = q_0 + q_1 i + q_2 j + q_3 k, \text{ or in vector representation } q = (q_0, q_1, q_2, q_3)$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and the imaginary units satisfy:

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

Quaternions form a non-commutative division algebra over $\mathbb{R}$, and a non-commutative field, with multiplication

$$pq = (p_0, p_1, p_2, p_3)(q_0, q_1, q_2, q_3) \\ = \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

It is also possible to present the quaternion multiplication in vector form

$$pq = (p_0, \vec{p})(q_0, \vec{q}) = (p_0 q_0 - \vec{p} \cdot \vec{q}, p_0 \vec{q} + q_0 \vec{p} + \vec{p} \times \vec{q})$$

For a quaternion in vector form $q = q_0 + \vec{q} = q_0 + q_1 i + q_2 j + q_3 k$

the conjugate is $\bar{q} = q_0 - q_1 i - q_2 j - q_3 k, \quad \bar{\bar{q}} = q_0 + \vec{q}$

The conjugate has the properties

$$\bar{\bar{q}} = q, \quad \overline{(p+q)} = \bar{p} + \bar{q}, \quad \overline{(pq)} = \bar{q} \bar{p}, \quad q\bar{q} = \bar{q}q$$

The norm of quaternions is $N(q) = |q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ with the properties

$$N(\bar{q}) = N(q), \quad N(pq) = N(p)N(q), \quad N(rq) = |r|N(q), \quad r \in \mathbb{R}$$

The inverse of $q \neq 0$ is $q^{-1} = \frac{\bar{q}}{|q|^2}$.

There is a isomorphic mapping from quaternion to Pauli-matrices

$$(1, i, j, k) \rightarrow (1, i\sigma_x, i\sigma_y, i\sigma_z)$$

Quaternions

A quaternion $q \in \mathbb{H}$ can be expressed as:

$q = q_0 + q_1 i + q_2 j + q_3 k$, or in 4-vector representation $q = (q_0, q_1, q_2, q_3)$ in $\mathbb{R}^4$ with the basis

$$\{e_0, e_1, e_2, e_3\} = \{1, i, j, k\}$$

where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and the imaginary units satisfy:

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

Quaternions also have a representation as complex 2-vector in $\mathbb{C}^2$

$$q = z + we_2, \quad q = z + we_2 = (q_0 + q_1 e_1) + (q_2 + q_3 e_1) e_2 \quad \text{in the basis } \{e_0, e_2\} = \{1, j\}$$

$$\text{or in 2x2 matrix representation } Q = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}$$

Quaternions form a non-commutative division algebra over $\mathbb{R}$, and a non-commutative field, with multiplication

$$pq = (p_0, p_1, p_2, p_3)(q_0, q_1, q_2, q_3)$$

$$= \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

It is also possible to present the quaternion multiplication in vector form

$$pq = (p_0, \vec{p})(q_0, \vec{q}) = (p_0 q_0 - \vec{p} \cdot \vec{q}, p_0 \vec{q} + q_0 \vec{p} + \vec{p} \times \vec{q})$$

For a quaternion in vector form $q = q_0 + \vec{q} = q_0 + q_1 i + q_2 j + q_3 k$

the conjugate is $\bar{q} = q_0 - q_1 i - q_2 j - q_3 k$, $\bar{q} = q_0 - \vec{q}$

The conjugate has the properties

$$\bar{\bar{q}} = q, \quad \overline{(p+q)} = \bar{p} + \bar{q}, \quad \overline{(pq)} = \bar{q} \bar{p}, \quad q\bar{q} = \bar{q}q$$

The norm of quaternions is $N(q) = |q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ with the properties

$$N(\bar{q}) = N(q), \quad N(pq) = N(p)N(q), \quad N(rq) = |r|N(q), \quad r \in \mathbb{R}$$

The inverse of $q \neq 0$ is $q^{-1} = \frac{\bar{q}}{|q|^2}$.

There is an isomorphic mapping from quaternion to Pauli-matrices

$$(1, i, j, k) \rightarrow (1, i\sigma_x, i\sigma_y, i\sigma_z)$$

Biquaternions

A biquaternion $q \in H_C$ can be expressed as:

$q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$, or in 4-vector representation $q = (q_0, q_1, q_2, q_3)$ in C^4 with the basis $\{e_0, e_1, e_2, e_3\}$

where $q_0, q_1, q_2, q_3 \in C$ and the imaginary units are the same Hamilton units as for quaternions, but now $e_i \neq i$.

Biquaternions form a 4-dimensional associative, non-commutative algebra over the complex numbers.

The algebra of biquaternions is isomorphic to the ring of 2×2 complex matrices $M_2(C)$.

Biquaternion algebra is not a division algebra: unlike standard real quaternions, not every non-zero biquaternion has a multiplicative inverse (some non-zero elements have a zero norm).

There are zero-divisors: $(1 + ie_1)(1 - ie_1) = 1 - 1 = 0$

Biquaternions have two conjugations

-biconjugate $\bar{q} = q_0 e_0 - q_1 e_1 - q_2 e_2 - q_3 e_3$

-complex-conjugate $q^* = q_0^* e_0 + q_1^* e_1 + q_2^* e_2 + q_3^* e_3$

with the properties: $q\bar{q} = \bar{q}q$, $q\bar{q} \in C$, $q^{-1} = \{\bar{q}(q\bar{q})\}^{-1}$ if $(q\bar{q}) \neq 0$

Biquaternions also have a representation as complex 2-vector in H^2

$q = z + we_2$, $q = z + we_2 = (q_0 + q_1 e_1) + (q_2 + q_3 e_1)e_2$ in the basis $\{e_0, e_2\} = \{1, j\}$

or in 2x2 matrix representation $Q = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}$

The norm-square is $N(q) = |q|^2 = q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$, where $N(pq) = N(p)N(q)$, but $N(q)$ can be zero or negative, so $|q|$ is only a semi-norm.

Application in quantum mechanics

Basic particles of Standard Model (leptons, quarks) with spin $S=1/2$ are represented by Dirac-bispinor wavefunctions $\Psi = \begin{pmatrix} q_L \\ q_R \end{pmatrix}$, with left-handed 2-component

Weyl spinor q_L and right-handed 2-component Weyl spinor q_R .

Dirac-bispinor transform under spacetime rotation angles $(\vec{\varphi}, \vec{v})$ with the spinorial matrix S

$$S = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right) = \begin{pmatrix} A_L & 0 \\ 0 & A_R \end{pmatrix} = \begin{pmatrix} \exp\left(-\frac{i}{2}(\vec{\varphi} - i\vec{v})\vec{\sigma}\right) & 0 \\ 0 & \exp\left(-\frac{i}{2}(\vec{\varphi} + i\vec{v})\vec{\sigma}\right) \end{pmatrix}$$

These Dirac-bispinors are isomorphic to biquaternions, and the Dirac equation and their solutions can be equivalently formulated in biquaternions. This has the advantage, that different wavefunctions can be multiplied without losing the spinor transformation property, and they form a (non-commutative) semi-norm non-division quaternionic algebra. Sum of products of Dirac-wavefunctions emerge in representation of many-particle systems.

Quaternions and rotation

A unit quaternion has the vector form $q = \cos \theta + \hat{u} \sin \theta$, which is a 3D-rotation by angle 2θ around the axis $\hat{u}$, the rotated vector is

$R(\hat{v}) = q\hat{v}q^{-1}$, where the vector $v = (v_1, v_2, v_3)$ with the quaternion vector $\hat{v} = v_1i + v_2j + v_3k$.

The polynomial ring $H[x]$ has polynomials without zeros in H , so H is not algebraically closed.

$p(x) = ix + xi - j$ has no root in H , because $ix + xi$ always lies in the plane spanned by $\{1, i\}$.

H is topologically complete (all Cauchy sequences converge in H).

Quaternionic Functions

A function $f : H \rightarrow H$ assigns a quaternion to each quaternion.

Unlike complex functions, left- and right-multiplication need to be distinguished:

The Fueter regularity

A quaternionic function f is (left) regular if it satisfies the Fueter operator equation:

for $q = x_0 + x_1i + x_2j + x_3k$

$$\frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} + k \frac{\partial f}{\partial x_3} = 0$$

This is the quaternionic analogue of the Cauchy-Riemann equations in complex analysis

$$\frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1}$$

A right-regular function satisfies $fD=0$.

Properties

Linearity: The Fueter operator is linear over R , but not over H due to noncommutativity.

Analyticity analogue: Regular functions possess analogues of complex differentiability, though the structure is richer in four real dimensions.

Harmonicity: Components of regular functions satisfy the Laplace equation:

$$\Delta f_l = 0, \quad l = 0, 1, 2, 3$$

$$\Delta f \equiv \frac{\partial^2 f}{\partial x_0^2} + \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$$

Δ is the 4-dimensional Laplacian

Cauchy-Type Integral Formula

Quaternionic analysis admits an analogue of the Cauchy integral formula.

If f is left-regular inside a bounded domain $\Omega \subset H$ with smooth boundary $\partial\Omega$, then:

$$f(q) = \frac{1}{2\pi} \int_{\partial\Omega} (p - q)^{-1} dS(p) f(p)$$

where $dS(p)$ is the outward-oriented surface measure on $\partial\Omega$.

The order of multiplication matters in these integral formulas.

There are distinct left- and right-Cauchy formulas corresponding to left- or right-regular functions.

Properties

A general quaternion is $q = t + ix + jy + kz$, $t, x, y, z \in \mathbb{R}$, choosing an orthonormal positively oriented basis $e_i = \{i, j, k\}$,

we obtain $q = t + e_i x_i$ with $e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k$.

The coordinates of q are polynomials in q (as opposed to complex numbers $\mathbb{C}$)

$$t = \frac{1}{4}(q - iqi - jqj - kqk)$$

$$x = \frac{1}{4i}(q - iqi + jqj + kqk)$$

$$y = \frac{1}{4j}(q + iqi - jqj + kqk)$$

$$z = \frac{1}{4k}(q + iqi + jqj - kqk)$$

We have the basic relations

$$\bar{q} = t - ix - jy - kz \text{ conjugate}$$

$$|q| = \sqrt{q\bar{q}} = \sqrt{t^2 + x^2 + y^2 + z^2} \text{ norm}$$

$$\text{Re } q = t = \frac{1}{2}(q + \bar{q}) \text{ real part}$$

$$\text{Pu } q = \frac{1}{2}(q - \bar{q}) = ix + jy + kz \text{ pure quaternionic part}$$

$$\text{Un } q = \frac{q}{|q|} \in S \text{ unit quaternion, where } S \text{ is the unit sphere in } H$$

$$\langle q_1, q_2 \rangle = \text{Re}(q_1 \bar{q}_2) = t_1 t_2 + x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1, |q_1 q_2| = |q_1| |q_2|, \text{Re}(q_1 q_2) = \text{Re}(q_2 q_1)$$

$$q^{-1} = \frac{\bar{q}}{|q|^2}$$

Quaternionic differential forms

The differential df is a quaternion-valued 1-form

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Exterior product of θ an r -form and φ an s -form,
the exterior product $\theta \wedge \varphi$ is a sum of terms with index permutations

$$\theta \wedge \varphi(h_1, \dots, h_{r+s}) = \frac{1}{r!s!} \sum_{\rho} \varepsilon(\rho) \theta(h_{\rho(1)}, \dots, h_{\rho(r)}) \varphi(h_{\rho(r+1)}, \dots, h_{\rho(r+s)})$$

with the properties

$$a(\theta \wedge \varphi) = (a\theta) \wedge \varphi$$

$$(\theta \wedge \varphi)a = \theta \wedge (\varphi a)$$

$$(\theta a) \wedge \varphi = \theta \wedge (\varphi a)$$

The basic differential is $dq = dt + i dx + j dy + k dz$

with the property

$$dq \wedge dq = \frac{1}{2} \varepsilon_{ijk} e_i dx_j \wedge dx_k = i dy \wedge dz + j dz \wedge dx + k dx \wedge dy$$

The pure quaternionic differential 3-form is

$$Dq = dx \wedge dy \wedge dz - \frac{1}{2} \varepsilon_{ijk} e_i dt \wedge dx_j \wedge dx_k =$$

$$dx \wedge dy \wedge dz - i dt \wedge dy \wedge dz - j dt \wedge dz \wedge dx - k dt \wedge dx \wedge dy$$

$Dq(a, b, c)$ is a quaternion which is perpendicular to vectors a, b and c in $\mathbb{R}^4$, and has magnitude equal to the volume $V(a, b, c)$ of the 3-dimensional parallelepiped with edges (a, b, c)

Properties

$$\langle h_1, Dq(h_2, h_3, h_4) \rangle = v(h_1, h_2, h_3, h_4)$$

$$Dq(a, b, c) = \frac{1}{2}(c\bar{a}b - b\bar{a}c)$$

Examples

$$v(1, i, j, k) = 1$$

$$Dq(1, i, j) = 1, \quad Dq(1, e_i, e_j) = -\varepsilon_{ijk} e_k$$

Quaternionic differential forms and regularity

There is a relation between regularity and quaternionic differential forms:

Theorem A quaternionic function $f : H \rightarrow H$ is left regular at q if and only if

$$Dq \wedge df = 0$$

$$\text{and } Dq \wedge df = -4\bar{\partial}_l f (dt \wedge dx \wedge dy \wedge dz)$$

Theorem A quaternionic function $f : H \rightarrow H$ is left regular at q if it is real-differentiable at q and there exists a quaternion $f'_l(q)$ such that $d(dq \wedge dq f) = Dq \bar{\partial}_l f$

Quaternionic functions

rotation by a versor u ($|u| = 1$): $f(q) = uqu^{-1}$

multiplicative inverse $f(q) = q^{-1}$

affine transform $f(q) = aq + b$

linear fractional transformations $f(q) = uqv$ with versors u, v

quaternion conjugation $f(q) = \bar{q} = -(q + iqi + jqj + kqk) / 2$

extension of a complex map $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$

generalized extension

$f(q) = u(x, y) + cv(x, y)$, u even, v odd, extension to a quaternion variable $q = x + yc$, $c^2 = -1$

for conjugate $\bar{q} = x - y\bar{c}$, we obtain $f_5(\bar{q}) = f_5(q)$

Differentiability of quaternion functions

A function $f : H \rightarrow H$, $f(q)$, $q = t + ix + jy + kz$, $t, x, y, z \in R$ is real-differentiable, if derivatives

$\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ exist.

A function $f : H \rightarrow H$ is quaternion-left-differentiable at q if the limit

$\lim_{h \rightarrow 0} (h^{-1} (f(q+h) - f(q)))$ exists,

$f : H \rightarrow H$ is quaternion-right-differentiable at q if the limit

$\lim_{h \rightarrow 0} ((f(q+h) - f(q))h^{-1})$ exists.

Example $f(q) = q$, $\lim_{h \rightarrow 0} (h^{-1} (f(q+h) - f(q))) = \lim_{h \rightarrow 0} ((f(q+h) - f(q))h^{-1}) = f'(q) = 1$,

so f is left-differentiable and right-differentiable

Example $f(q) = iq + 1$,

left-derivative $\lim_{h \rightarrow 0} (h^{-1} (f(q+h) - f(q))) = \lim_{h \rightarrow 0} h^{-1}ih$

for $h = it$, t real, $\lim_{h \rightarrow 0} h^{-1}ih = i$,

for $h = jt$, t real, $\lim_{h \rightarrow 0} h^{-1}ih = -i$,

so the left-derivative limit does not exist.

right-derivative $\lim_{h \rightarrow 0} (h^{-1} (f(q+h) - f(q))) = \lim_{h \rightarrow 0} i h h^{-1} = i$, so f is right-differentiable

The quaternion-differentiability is too strong to be a useful property, according to the theorem by Fueter.

Theorem If a function $f : H \rightarrow H$ is left-differentiable on a connected open set $U \subset H$, then it is linear

$$f(q) = aq + b \text{ for all } q \in U$$

Regularity

A quaternionic function f is left-regular if

$$\frac{\partial f}{\partial t} + i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} = 0$$

and right-regular if

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = 0$$

The left-derivative operator is defined by

$$\bar{\partial}_l = \frac{1}{2} \left(\frac{\partial f}{\partial t} + i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \right)$$

the right-derivative operator is defined by

$$\bar{\partial}_r = \frac{1}{2} \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k \right)$$

Using derivative operators, quaternionic function f is left-regular if

$$\bar{\partial}_l f = 0, \text{ function } f \text{ is right-regular if } \bar{\partial}_r f = 0$$

Example: $f(q) = q = t + ix + jy + kz$, $\bar{\partial}_l f = -1/2$, f is not left-regular

Example: $f(q) = q = t + ix - jy + kz$, $\bar{\partial}_l f = 0$, $\bar{\partial}_r f = 0$, so f is left-regular and right-regular

Example: $f(q) = q = jx + iy$, $\bar{\partial}_l f = 0$, so f is left-regular

But $\bar{\partial}_l(if) \bar{\partial}_l(if) = -j/2$, so if is not left-regular.

Properties of regular functions

- The sum of two left-regular functions is left-regular
- If f is left-regular and a is a quaternion then fa is left-regular, but af may not be left-regular.

Regular and harmonic functions

For quaternionic function $f : H \rightarrow H$, we define the following derivative operators

$$\partial_i f = \frac{1}{2} \left(\frac{\partial f}{\partial t} - i \frac{\partial f}{\partial x} - j \frac{\partial f}{\partial y} - k \frac{\partial f}{\partial z} \right)$$

$$\bar{\partial}_i f = \frac{1}{2} \left(\frac{\partial f}{\partial t} + i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \right)$$

$$\partial_r f = \frac{1}{2} \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} i - \frac{\partial f}{\partial y} j - \frac{\partial f}{\partial z} k \right)$$

$$\bar{\partial}_r f = \frac{1}{2} \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k \right)$$

and the Laplace operator

$$\Delta f = \left(\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right)$$

Theorem For every 2x real-differentiable function $f : H \rightarrow H$, we have $\Delta f = 4\partial_i \bar{\partial}_i f = 4\bar{\partial}_i \partial_i f$

Corollary

- if f is left-regular, then it is harmonic
- If f is harmonic, then $\partial_i f$ is left-regular.

Quaternionic integrals

Let $f : H \rightarrow H$ be a real-differentiable quaternionic function, and

$\gamma : [0,1] \rightarrow H$ a piecewise-smooth path.

The left line integral is defined as:

$$\int_{\gamma} f(q) dq = \int_0^1 f(\gamma(t)) \gamma'(t) dt$$

Here, the differential $\gamma'(t) dt$ multiplies on the right, and the integral is order-sensitive due to non-commutativity

Example

$$\gamma(t) = t(1+i+j+k), \quad t \in [0,1], \quad f(q) = q$$

$$\text{compute } \int_{\gamma} q dq = \int_0^1 t(1+i+j+k)(1+i+j+k) dt = \frac{1}{2}(1+i+j+k)$$

Quaternionic Cauchy-Type Integral

Theorem (Cauchy–Fueter integral formula)

If f is regular on every point of a (positively oriented) 4-parallelepiped Ω and q_0 is a point within Ω ,

$$f(q_0) = \frac{1}{2\pi^2} \int_{\partial\Omega} \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dq f(q)$$

, which generalizes the complex integral $\oint_{\partial\Omega} \frac{z^n}{z} dz = 0$, $n > 1$

A practical approach is to integrate component functions:

$$f(q) = f_0 + f_1 i + f_2 j + f_3 k, \quad q = x_0 + x_1 i + x_2 j + x_3 k$$

$$\int_{\gamma} f(q) dq = \sum_{\mu=0}^3 \sum_{\nu=0}^3 f_{\mu} dx_{\nu} (i^{\mu} i^{\nu})$$

which is a sum of 16 integrals.

Example

$$\int_0^1 t(1+i+j+k) dt = \int_0^1 t dt + i \int_0^1 t dt + j \int_0^1 t dt + k \int_0^1 t dt = \frac{1}{2}(1+i+j+k)$$

Power series expansion

THEOREM (Laurent series). Suppose f is regular in an open set U except possibly at $q_0 \in U$.

Then there is a neighbourhood N of q_0 such that if $\mathbf{q} \in \mathbf{N}$, $f(q)$ can be represented by a series

$$f(q) = \sum_{n=0}^{\infty} \sum_{\nu \in \sigma_n} (P_{\nu}(q-q_0) a_{\nu} + G_{\nu}(q-q_0) b_{\nu})$$

which converges uniformly in any hollow ball $\{q; r \leq |q-q_0| \leq R\}$ inside N

The coefficients a_{ν} and b_{ν} are given by

$$a_{\nu} = \frac{1}{2\pi^2} \int_{\partial U} G_{\nu}(q-q_0) Dq f(q)$$

$$b_{\nu} = \frac{1}{2\pi^2} \int_{\partial U} P_{\nu}(q-q_0) Dq f(q)$$

Here

$$G_{\nu}(q) = \frac{q^{-1}}{|q|^2}$$

Let ν be an unordered set of n integers $\{i_1, \dots, i_n\}$ with $1 \leq i_r \leq 3$;

there are $(n+1)(n+2)/2$ such sets ν ; we will denote the set of all of them by σ_n .

$$\partial_{\nu} = \frac{\partial^n}{\partial x_{i_1} \dots \partial x_{i_n}}$$

$G_{\nu}(q)$ is a rational function of q : $G_{\nu}(q) = \partial_{\nu} G(q)$

$P_v(q)$ is a polynomial in q : $P_v(q) = \frac{1}{n!} \sum_{\pi(v)=(i_1, \dots, i_n)} (te_{i_1} - x_{i_1}) \dots (te_{i_n} - x_{i_n})$, summed over all different orderings $\pi(v)$

5 Generalization of differential operators

Fractional calculus

The differential operator $Df(x) = \frac{df(x)}{dx}$ is defined by a limit of the quotient $\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, and its multiple application defines the exponential

expression $D^n f(x) = \frac{d^n f(x)}{dx^n}$.

It can be generalized to fractional exponent by the Riemann-Liouville formula [8]

$$D_a^\alpha f(x) = \frac{d^n}{dx^n} I_a^{n-\alpha} f(x) = \frac{d^n}{dx^n} \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{n-\alpha-1} f(t) dt, \quad n-1 < \alpha < n$$

where I_a^α is the Riemann-Liouville fractional integral

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

For $a=0$, the lower index is skipped: $I_0^\alpha = I^\alpha$, $D_0^\alpha = D^\alpha$.

For integer $\alpha = n \geq 0$, the fractional derivative and integral become the ordinary derivative D^n and integral I^n .

I_a^α is based on the Cauchy formula for the integral I_a , $(I_a f)(x) = \int_a^x f(t) dt$, with $I_0^1 = I$

$$(I^n f)(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t) dt$$

$$f(x) = \int_0^x (x-t)^{-1} f(t) dt, \quad n=0$$

$$(I f)(x) = \int_0^x f(t) dt, \quad n=1$$

Here $\Gamma(x)$ is the gamma-function, which generalizes the factorial: $\Gamma(n) = (n-1)!$ for $n > 0$.

The operator I^α is additive in the exponential $I^\alpha I^\beta = I^{\alpha+\beta}$

$$I^\alpha (I^\beta f)(x) = I^\beta (I^\alpha f)(x) = (I^{\alpha+\beta} f)(x) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^x (x-t)^{\alpha+\beta-1} f(t) dt$$

However, D^α is not additive: $D^\alpha D^\beta \neq D^{\alpha+\beta}$.

The Riemann-Liouville generalized derivative can be approximated by a limit expression similar to derivative D [9]:

$$T(\alpha,a)f(x)=\lim_{h\rightarrow 0}\left((1-\alpha)f(x)+\alpha\frac{f\left(x+h(x-a)^{1-\alpha}\right)-f(x)}{h}\right),$$

with $T(\alpha,a)f(x)=D_a^\alpha f(x)$, if the limit exists in a environment of x .

Examples

$$D^\alpha\left(x^\lambda\right)=\frac{\Gamma\left(\lambda+1\right)x^{\lambda-\alpha}}{\Gamma\left(\lambda+1-\alpha\right)}$$

$$D^\alpha\left(c\right)=\frac{cx^{-\alpha}}{\Gamma\left(1-\alpha\right)}\text{ for constant }c$$

$$D^{1/2}\left(x\right)=\frac{2\sqrt{x}}{\sqrt{\pi}}$$

$$D^{1/2}\left(1/x\right)=\frac{\text{EulerGamma}-\text{Log}[x]+\text{PolyGamma}[0,-\frac{1}{2}]}{2\sqrt{\pi}x^{3/2}}=\frac{0.2821(0.6137-1.0\text{Log}[x])}{x^{3/2}}$$

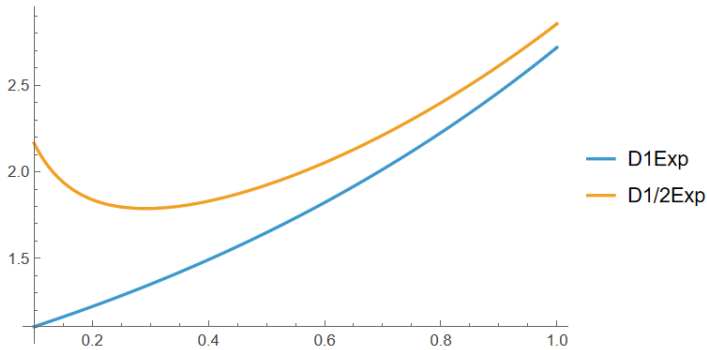
$$D^{1/2}\left(1/x^{\wedge}2\right)=\frac{3(-1+\text{EulerGamma}-\text{Log}[x]+\text{PolyGamma}[0,-\frac{3}{2}])}{4\sqrt{\pi}x^{5/2}}=\frac{0.4231(0.2803-1.0\text{Log}[x])}{x^{5/2}}$$

$$D^{1/2}\left(x^{\wedge}2\right)=\frac{8x^{3/2}}{3\sqrt{\pi}}$$

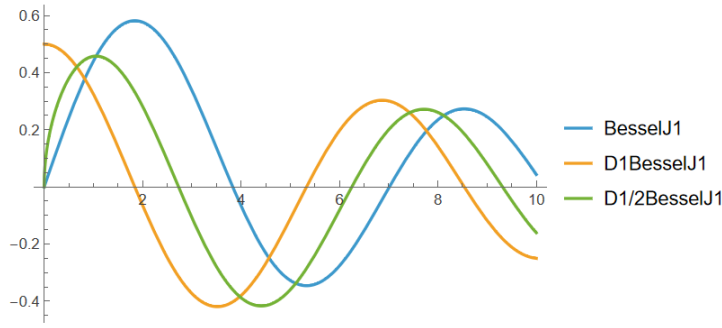
$$D^{1/2}\left(x^{\wedge}3\right)=\frac{16x^{5/2}}{5\sqrt{\pi}}$$

$$D^{1/2}\left(x^{\wedge}4\right)=\frac{128x^{7/2}}{35\sqrt{\pi}}$$

$$D^{1/2}\exp(x)=-e^x(-1+\text{GammaRegularized}[-\frac{1}{2},x])$$



$$D^{1/2}BesselJ1(x)=\sqrt{\frac{\pi}{2}(-1+\text{HypergeometricPFQ}[\{\frac{1}{2}\},\{-\frac{1}{4},\frac{1}{4}\},-\frac{x^2}{4}]})}{4x^{3/2}\text{Gamma}[\frac{3}{4}]\text{Gamma}[\frac{5}{4}]}$$



Application of fractional calculus

Diffusion

Diffusion is described by Fick's equation

$$\frac{\partial \rho(\vec{r}, t)}{\partial t} = D \Delta \rho(\vec{r}, t), \text{ where } \rho(\vec{r}, t) \text{ is the density of diffusing substance}$$

Anomalous diffusion processes in complex media can be well characterized by using fractional-order diffusion equation

$$\frac{\partial^\alpha \rho(\vec{r}, t)}{\partial t^\alpha} = D \Delta^\beta \rho(\vec{r}, t), \text{ with fractional order } \alpha, \beta$$

Fractional Schrödinger equation

Fractional quantum mechanics is based on the fractional Laplace operator [37] for $1 < \alpha \leq 2$

$$(-\hbar^2 \Delta)^{\alpha/2} \psi(\vec{p}, t) = \frac{1}{2\pi \hbar^3} \int \exp\left(-i \frac{\vec{p} \cdot \vec{r}}{\hbar}\right) |\vec{p}|^\alpha \varphi(\vec{p}, t) d^3 p$$

with the Fourier transform wave function $\varphi(\vec{p}, t) = \int \exp\left(-i \frac{\vec{p} \cdot \vec{r}}{\hbar}\right) \psi(\vec{p}, t) d^3 p$.

The fractional classical kinetic energy is $E_{kin} = d_\alpha |\vec{p}|^\alpha$, with the constant $d_\alpha = \frac{1}{(2m)^{\alpha-1} c^{\alpha-2}}$.

For $\alpha = 2$, the classical formula is recovered: $E_{kin} = \frac{|\vec{p}|^2}{2m}$.

In quantum mechanics, the momentum becomes the operator $\vec{p} = \frac{\hbar}{i} \nabla$, $|\vec{p}|^2 = -\hbar^2 \Delta$,

so the fractional Hamiltonian reads $H = d_\alpha (-\hbar \Delta)^{\alpha/2} + V(\vec{r})$.

The fractional Schrödinger equation is then

$$i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = d_\alpha (-\hbar \Delta)^{\alpha/2} \psi(\vec{r}, t) + V(\vec{r}) \psi(\vec{r}, t),$$

For $\alpha = 2$, it becomes the ordinary Schrödinger equation.

When applied to the Coulomb potential $V(\vec{r}) = -\frac{Ze_0^2}{|\vec{r}|}$, we obtain the Bohr energy levels [37] for $1 < \alpha \leq 2$

$$E_n = -(\alpha - 1) E_0 \frac{1}{n^{\alpha/(\alpha-1)}}, \quad E_0 = \left(\frac{(Ze_0^2)^\alpha}{\alpha^\alpha d_\alpha \hbar^\alpha} \right)^{1/(\alpha-1)}$$

When applied to the general harmonic oscillator with the potential $V(\vec{r}) = q^2 |\vec{r}|^\beta$, we obtain the energy levels [37] for $1 < \alpha \leq 2, 1 < \beta \leq 2$

$$E_n = E_0 \left(n + \frac{1}{2} \right)^{\alpha\beta/(\alpha+\beta)}, \quad E_0 = \left(\frac{\pi \hbar \beta d_\alpha^{1/\alpha} q^{2/\beta}}{2B(1/\beta, 1+1/\alpha)} \right)^{\alpha\beta/(\alpha+\beta)},$$

we recover the ordinary harmonic oscillator energies for $\alpha = 2, \beta = 2$, with the Beta-function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Taylor-operators as generalized derivatives

Differential operator D is linear and shift-independent, i.e. $\frac{d(f(x-x_0))}{dx} = \frac{(df)(x-x_0)}{dx}$.

An analytic (Taylor-expandable, infinitely differentiable) function f has a Taylor power series expansion in an environment of x_0 ,

$$f(x) = \sum_{k=0}^{\infty} a_k(f, x_0)(x-x_0)^k, \quad a_k(f, x_0) = \frac{f^{(k)}(x_0)}{k!},$$

with convergence radius $R(f, x_0) > 0$ and with n-term-truncation remainder $R_n(f, x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!}$.

The Taylor polynomial $S(f, x_0, n) = \sum_{k=0}^n a_k(f, x_0)(x-x_0)^k$ is the n-term-truncated Taylor series with the remainder $R_n(f, x_0)$:

$$f(x) = S(f, x_0, n) + R_n(f, x_0).$$

The convergence radius follows from the Cauchy-Hadamard formula $R(f, x_0) = \frac{1}{\limsup_{k \rightarrow \infty} |a_k(f, x_0)|^{1/k}}$.

For analytic function f , the Taylor series determines the function uniquely (independent of x_0) within its convergence radius.

Furthermore, the Taylor series converges absolutely and uniformly on compact sets within its convergence radius.

The operator D acts on the Taylor series $(a_k(f, x_0))_k$ in the form:

$$a_k(Df, x_0) = (k+1)a_{k+1}(f, x_0)$$

A Taylor operator T is an extension of a differential operator, it acts on the Taylor (coefficient) sequence $(a_k(f, x_0))_k$ in the form:

$$T = T(\alpha(k), \beta(k)),$$

$a_k(Tf, x_0) = \alpha(k) a_{\beta(k)}(f, x_0)$, where $\alpha(k)$ and $\beta(k)$ are polynomial maps in integers.

For a Taylor sequence $(a_k(f, x_0))_k$ with convergence radius $R(f, x_0) > 0$, the resulting sequence $a_k(Tf, x_0)$ has the same convergence radius $R(f, x_0)$ and

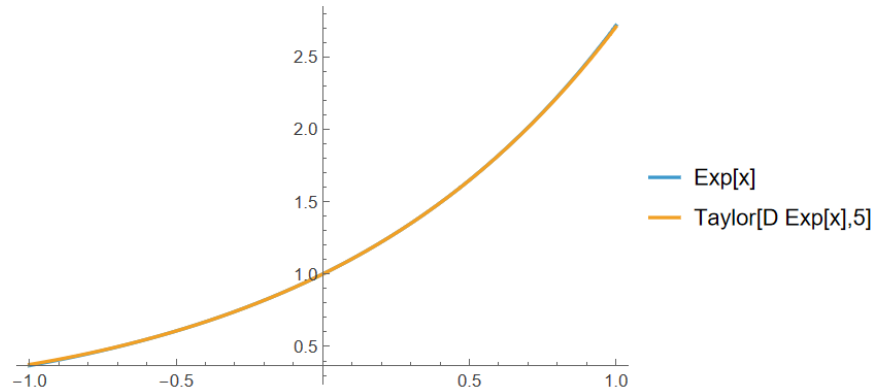
defines an analytic function $(Tf)(x) = \sum_{k=0}^{\infty} a_k(Tf, x_0)(x - x_0)^k$.

For the differential operator D we have $\alpha(k) = k + 1$, $\beta(k) = k + 1$, i.e. $D = T(k + 1, k + 1)$.

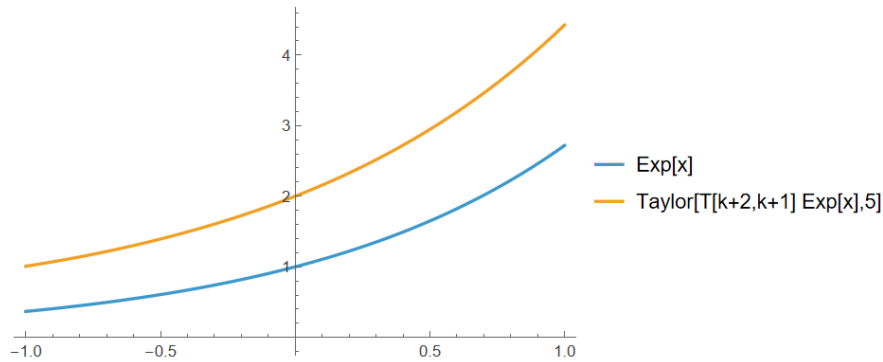
Examples

We present 3 examples with the exponential $f(x) = \exp(x)$, $x_0 = 0$, Taylor polynomials truncated at $n=5$.

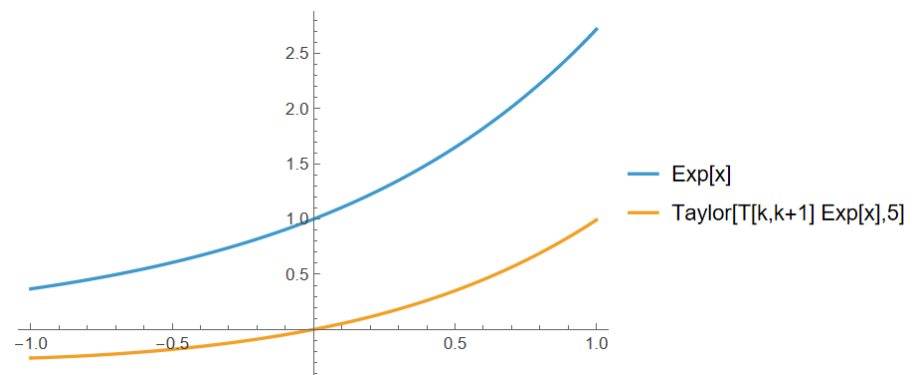
- $f(x) = \exp(x)$ and $DS(f, x_0, n)$ with the differential operator $D = T(k + 1, k + 1)$



- $f(x) = \exp(x)$ and $T(k + 2, k + 1)S(f, x_0, n)$ with the Taylor operator $T(k + 2, k + 1)$



- $f(x) = \exp(x)$ and $T(k, k + 1)S(f, x_0, n)$ with the Taylor operator $T(k, k + 1)$



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