Mathematical foundations of physics and physical experience

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Abstract

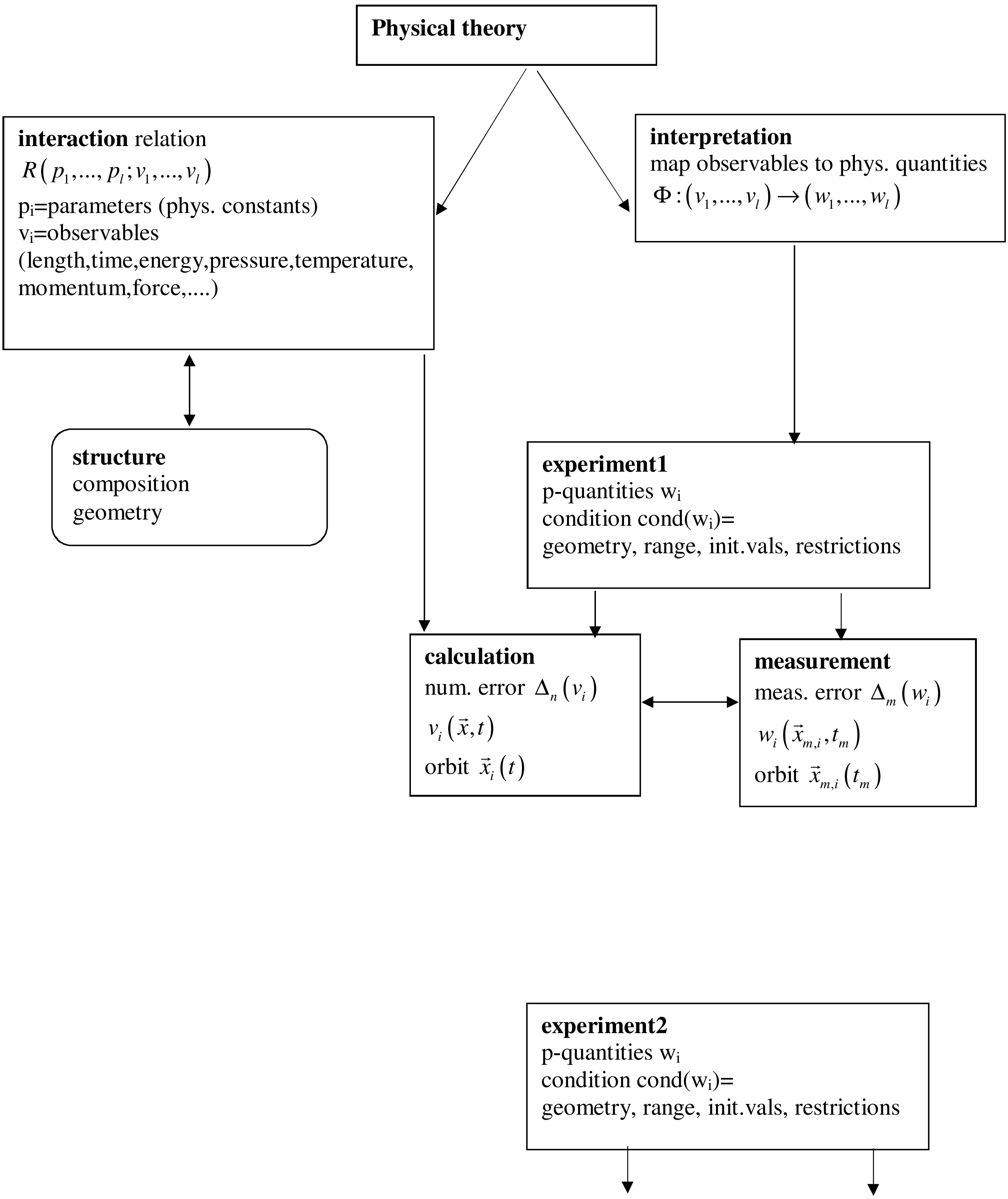
This paper presents a precise formulation of Popper's concept of a physical theory, with detailed discussion of its mathematical roots.
One can hardly overestimate the importance of symmetries in physics and of their corresponding algebraic structures.
It starts with the fundamental role of the Lorentz group and algebra, which is valid in all areas of physics.
It continues with the Heisenberg group-algebra, which defines the transition from classical to quantum physics and is at the core of quantum mechanics.
All four quantum interactions are carried by gauge fields described by SU(n) Lie algebras.
Furthermore, the spinors of the standard model rely on the Dirac-Clifford algebras, which define the fundamental Dirac equation.
Apart from symmetries, the other two fundamental pillars of physics are geometry and functional analysis.
Then, there is the probability theory, which plays an eminent role in statistical mechanics and thermodynamics.
Finally, there are mathematical logic and set theory, which form the axiomatic fundament of all other mathematical disciplines, and indirectly, of all branches of physics.

The paper consists of seven chapters:
0 Mathematical fundament of physics
1 Symmetry in physics
2 Mathematical models of physical cognition
3 Geometry in physics: Euclidean, hyperbolic and spherical geometry
4 Logic in physics: first-order logic
5 Functional analysis in physics
6 Lie and Clifford algebras in physics

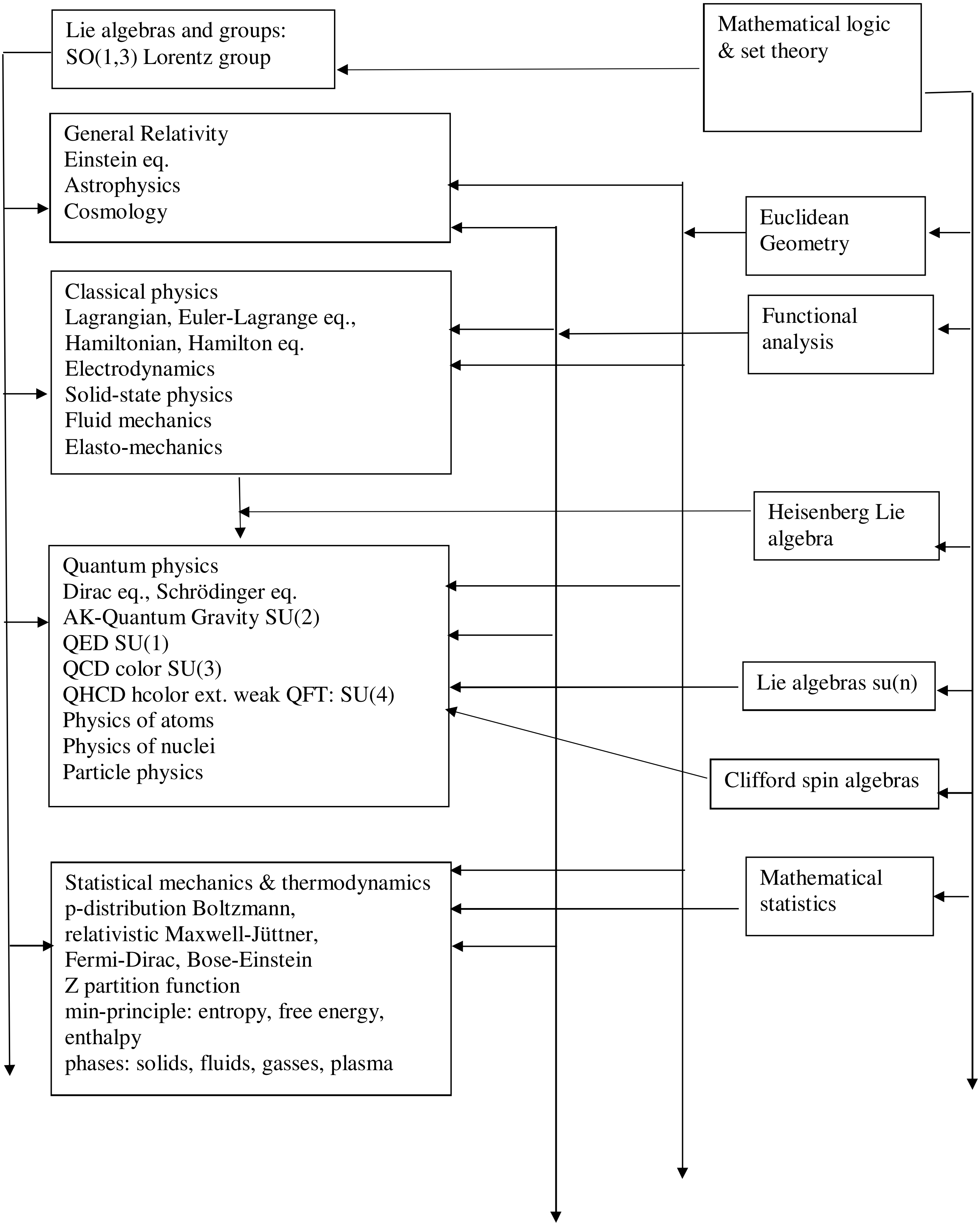
The contents are summarized in form of flowcharts at the beginning.

Flowcharts

Physics: mathematical model and measurement



Mathematical fundament of physics



Scheme of a simple physical model: falling object in constant gravitational field

Structure:
falling object mass m
geometry: spherical

Interaction: constant gravitation
Euler-Lagrange eom

$$\frac{\partial}{\partial t} p_i = \frac{\partial L}{\partial x_i}$$

$$L = T - V = \frac{1}{2} p^2 - m\,g\,z\,,\;\; p = m\dot{z}$$

Model map

$z \to h_m$ height

$x \to l_m$ hor. distance

$t \to t_m$ time

$\dot{x} \to v_{x,m}$ hor. velocity

$\dot{z} \to v_{z,m}$ vert. vlocity

$E \to E_m$ kinetic energy

Interaction: equation

$m\ddot{z} = -mg$

$T = \frac{m\dot{z}^2}{2},\; E = \frac{m}{2}\left(\dot{z}^2 + \dot{x}^2\right)$

Experiment
observables

$\left(h_m, l_m, t_m, v_{x,m}, v_{z,m}, E_m\right)$

init. conditions $\left(h_0, v_{x,0}, v_{z,0} = 0\right)$

restriction: no air resistance
units

$$\begin{pmatrix} h_m(m), l_m(m), t_m(s), \\ v_{x,m}(m/s), v_{z,m}(m/s), E_m(J) \end{pmatrix}$$

measurement

orbit $l_m(t_m),\; h_m(t_m)$

velocity $v_{x,m}(t_m), v_{z,m}(t_m)$

(vert.) kin. energy

$E_{m,1} = E_m\left(t_{m,1}\right)$ at end-time

$t_{m,1}$ (~impact imprint volume)

calculation

orbit $x(t) = v_{x,0}t\,,\; z(t) = h_0 - \frac{gt^2}{2}$

velocity $v_x = v_{x,0}\,,\; v_z = -gt$

(vert.) kin. energy

$E = \frac{m}{2}\left(\left(gt\right)^2 + {v_{x,0}}^2\right)$

Scheme of a quantum physical model: hadrons

Structure: quarks
U-quarks=(u,c,t), s=1/2, color=rgb,q=+2/3
D-quarks=(d,s,b), s=1/2, color=rgb,q=-1/3

Interaction: Lie algebra SU(3)
8 generators $T_a = \lambda_a / 2$,
$\left[T_a, T_b\right] = i\, f^{abc} T_c$, λ_a 3x3 Gell-Mann

Structure: hadrons (gen1,gen2)
$\Psi_h = \left(q_1, q_2, q_3\right)$

(1,1) nucleons
$p = \left(u, u, d\right)$
$n = \left(d, d, u\right)$

(1,1) v-mesons
$\rho^+ = \left(u\bar{d}, u\bar{d}, u\bar{d}\right)$
$\rho^0 = \left(u\bar{u} - d\bar{d},,\right) / \sqrt{2}$
$\omega^0 = \left(u\bar{u} + d\bar{d},,\right) / \sqrt{2}$

(1,1) ps-mesons
$\pi^+ = \left(u0, 0\bar{d}, 00\right)$
$\pi^0 = \left(u0 + 0\bar{d}, u0 - 0\bar{d}, 00\right) / \sqrt{2}$

Dirac eq.
$\left(i\hbar\gamma^\mu D_\mu - mc\right)\Psi = 0$
Feynman cross-section
$M_{fi} = \Gamma\left(\varepsilon, \kappa_i\right) = \int_0^\infty \frac{d^4 p_1}{(2\pi)^4} \ldots \int_0^\infty \frac{d^4 p_n}{(2\pi)^4} i\,\Delta(p_1) \ldots i\,\Delta(p_n)$

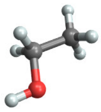
measured
mass spectrum
$m\left(\Psi_h\right)$

measured
cross-section $\Gamma\left(\kappa_i\right)$

Structure: Dirac-Clifford algebra
Cl(1,3)(C)
4 generators Dirac matrices γ^μ
$\left\{\gamma^\mu, \gamma^\mu\right\} = 2\eta_{\mu\nu} I$

Scheme of a classical physical model: solid-fluid-gas phases

Structure: molecule
molecule data (ethanol CH3-CH2-OH)



Interaction: orbital correlation calc.
electron orbital correlation MP2

Structure: intermolecular pot.
$u(\vec{r}, m_k, \varepsilon_0, \sigma_0) = \varepsilon_0 u_0 (\vec{r} / \sigma_0, m_k)$,
$m_k = (m, d_0)$, mag.dip.mom. d_0,
mass m, energy ε_0, radius σ_0

Statistical variables
N, V, T
$\beta = 1 / k_B T$, $\lambda = v^{1/3}$,
$v = V / N$

Structure: distribution function φ
$\varphi(r, \theta, \lambda; m_k, c_k) = \exp\left(-\frac{r}{\lambda}\right) \varphi_{rdf}(r, \theta; m_k, c_k)$
$\varphi_{rdf} = \varphi_{rdf}(r, \theta; m_k, c_k)$

Statistical mechanics
p-distribution Boltzmann
min-principle: free energy

Structure: partition function Z
$Z = \int dV \exp\left(-\frac{r}{\lambda}\right) \varphi_{rdf}(\vec{r}) \exp(-\beta u(\vec{r}))$
free energy
$F(\lambda, \beta; m_k, c_k) = -\frac{1}{\beta} \log Z(\lambda, \beta; m_k, c_k)$
Gibbs energy
$G = F + PV = \langle E \rangle - TS + PV = -\frac{1}{\beta} \frac{\partial (V \log Z)}{\partial V}$

Landau minimal principle
$F(\lambda, \beta; m_k, c_k) = -\frac{1}{\beta} \log Z(\lambda, \beta; m_k, c_k)$
$F(\lambda, \beta; c_{0,k}) = \min(c_k) \underset{\rightarrow}{} \varphi(c_{0,k}) = \varphi_0$
$Z(\lambda, \beta; c_{0,k}) = \max(c_k) \underset{\rightarrow}{} \varphi(c_{0,k}) = \varphi_0$
$c_{0,k} = c_{0,k}(\lambda, \beta; m_k)$

Peng-Robinson eos gas-fluid (rel. coord. $\lambda / \lambda_c, \beta / \beta_c$)
$p = p(\beta, v, \omega) = \frac{1}{\beta(v-b)} - \frac{\alpha(\omega, \beta) a}{v(v+b) + b(v-b)}$,
$a = 0.45723$, $b = 0.077796$
$\alpha = \left(1 + (0.480 + 1.574\omega - 0.176\omega^2)(1 - \sqrt{\beta})\right)^2$
ω = acentric factor $\omega = -\log_{10}\left(p_{sat}\left(\frac{10}{7}\right)\right)$

Mie-Grüneisen eos solid

reference point (ρ_0, β_0), Young modulus $Y(\eta) = y_1\eta - y_2$

$$p(\rho,\beta) = \frac{Y(\eta-1)\left(\eta - \frac{\Gamma_0}{2}(\eta-1)\right)}{(\eta - s(\eta-1))^2} + \Gamma_0 \frac{n_{dof}}{2}\left(\frac{1}{\beta} - \frac{1}{\beta_0}\right) + p_0 \quad , \eta = \frac{\rho}{\;}$$

Saturation curve fluid-gas $p_{fg}(T)$:

Maxwell-area-rule $p = p_{PR}(v_f, T)$,

$p = p_{PR}(v_g, T)$, $G_{PR}(v_f, T) = G_{PR}(v_g, T)$

Melting curve solid-fluid $p_{sf}(T)$:

Maxwell-area-rule $p_{PR} = p(v_f, T)$,

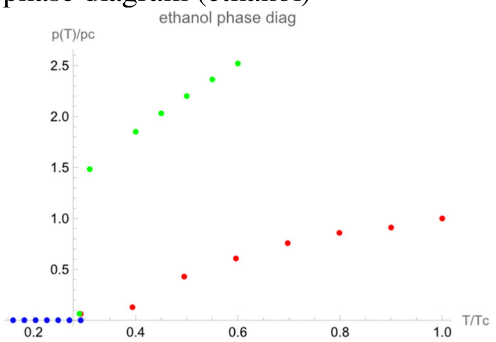
$p = p_{MG}(v_s, T)$, $G_{PR}(v_f, T) = G_{MG}(v_s, T)$

Sublimation curve solid-gas $p_{sg}(T)$:

Maxwell-area-rule $p_{PR} = p(v_g, T)$,

$p = p_{MG}(v_s, T)$, $G_{PR}(v_g, T) = G_{MG}(v_s, T)$

phase diagram (ethanol)



phase diagram measured

Fundamental metric vector rotation group

SO(1,3) Lorentzian rotation group, complex 4x4 matrices

flat metric $\eta_{\mu\nu} = diag(+1,-1,-1,-1)$

vector rotation $x' = \Lambda x$

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right) = \exp(-i(\vec{\phi}\,\vec{J} + \vec{v}\,\vec{K}))$$

$\vec{\varphi}$ =3 spatial rotation angles $\vec{v}$ =3 boost angles

$M^{\mu\nu}$ $=(\mu,\nu)$ coordinate swaps

$\vec{J}$ =3 elementary spatial rotations, $\vec{K}$ =3 elementary boosts

$$(M^{\mu\nu})^{\rho}_{\sigma} = -i\,(g^{\mu\rho}\delta^{\nu}_{\sigma} - g^{\nu\rho}\delta^{\mu}_{\sigma})$$

$$J^k = \frac{1}{2}\varepsilon^{lmk} M^{lm} \qquad K^k = M^{0k}$$

$$J^1 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad J^2 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad J^3 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K^1 = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K^2 = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K^3 = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Fundamental metric spin rotation group

Spin(1,3) spin rotations of Dirac spinors, quaternion 2x2 matrices

flat metric $\eta_{\mu\nu} = diag(+1,-1,-1,-1)$

rotation S of $\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$, $\Psi' = S\Psi$

$$S = exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right),$$

$$S = \begin{pmatrix} A_L & 0 \\ 0 & A_R \end{pmatrix} = \begin{pmatrix} exp\left(-\frac{i}{2}(\vec{\varphi}-i\vec{v})\vec{\sigma}\right) & 0 \\ 0 & exp\left(-\frac{i}{2}(\vec{\varphi}+i\vec{v})\vec{\sigma}\right) \end{pmatrix}$$

$\vec{\varphi}$ =3 spatial rotation angles $\vec{v}$ =3 boost angles

spinor anticommutator $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

spin generators $\Sigma^{\mu\nu}$, $\Sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \tilde{\sigma}^{\mu\nu} \end{pmatrix}$

with Pauli spin-tensor $\sigma^{\mu\nu} = \frac{i}{4}[\sigma^\mu, \tilde{\sigma}^\nu]$, $\tilde{\sigma}^{\mu\nu} = \frac{i}{4}[\tilde{\sigma}^\mu, \sigma^\nu]$, $\sigma^\mu = \frac{1}{2}(1, \vec{\sigma})$, $\tilde{\sigma}^\mu = \frac{1}{2}(1, -\vec{\sigma})$

and Pauli 3-vector $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$

equivalence {1, i, j, k} $\rightarrow \{I, i\sigma_3, i\sigma_2, i\sigma_1\}$ between Pauli matrices and quaternions

Interaction gauge symmetry groups

Electromagnetism

classical electrodynamics
Maxwell equations $\partial_\mu F^{\mu\nu} = j^\nu$
electric 3-vector-field $\vec{E}$
magnetic 3-pseudo-v-field $\vec{B}$, $\vec{B} = \vec{\nabla} \times \vec{A}$
four-vector potential $A_\mu = \left(\frac{\phi}{c}, \vec{A} \right)$
$E_i = -cF^{0i}$, $B_k = -\varepsilon_{kij} F^{ij}$ duality

quantum electrodynamic

quantum electrodynamics
scale $r_B = \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2} = 5.29 * 10^{-11} m$,
E- scale $E_{em} = \frac{\hbar c}{r_B} = 3.7 keV$
$D_\mu = \partial_\mu + ieA_\mu$ SU(1)
Dirac eq. $\left(i\gamma^\mu D_\mu - m \right) \Psi = 0$
field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
Maxwell equations $\partial_\mu F^{\mu\nu} = 0$
field eq. $\partial_\mu F^{\mu\nu} = j^\nu = -e \bar{\psi} \gamma^\nu \psi$
Lagrangian $L = \bar{\psi} \left(i\gamma^\mu D_\mu - m \right) \psi - \frac{1}{4} F^a{}_{\mu\nu} F_a^{\mu\nu}$
classical Coulomb potential $E_{el}(r) = \frac{e_0^2}{r}$

Lie group SU(1)
={real numbers}

Gravity

Ashtekar-Kodama gravitation classical

field carrier graviton $A_\mu^{\ \nu}$ S=2

32 AK-equation for graviton $A_\mu^{\ \nu}$ and tetrad $E^{\mu\nu}$

$$H_{(\mu,\nu)}^{\ \ \kappa} \equiv F_{\mu\nu}^{\ \ \kappa} + \frac{\Lambda}{3}\varepsilon_{\mu\nu\rho}E^{\rho\kappa} = 0$$

$$G^{\mu} \equiv \partial_\nu E^{\nu\mu} + \varepsilon^{\mu}_{\ \kappa\lambda}A_\nu^{\ \kappa}E^{\nu\lambda} = 0$$

$$I_\mu \equiv E^{\kappa}{}_{\nu}F_{\mu\kappa}^{\ \ \nu} = 0$$

with field tensor $F_{\mu\nu}^{\ \ \kappa}$

$$F_{\mu\nu}^{\ \ \kappa} = \partial_\mu A_\nu^{\ \kappa} - \partial_\nu A_\mu^{\ \kappa} + \varepsilon^{\kappa}{}_{\kappa_1\kappa_2}A_\mu^{\ \kappa_1}A_\nu^{\ \kappa_2}$$

boundary cond. at $x \to \infty$ for tetrad and external metric $g_{\mu\nu}$

$$E^{\mu\kappa}E^{\nu}{}_{\kappa} = g^{\mu\nu}/(-\det(g))^{3/4}$$

$\Lambda \to 0$

GR with Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

with $\kappa = \dfrac{8\pi G}{c^4}$

at the horizon $r = r_s$ $E^{\mu\nu} >> 1$ and $\Lambda|E| \sim 1$

singularity becomes a finite peak

AK-quantum-gravity

$r_{gr} = \sqrt{l_P\sqrt{\dfrac{1}{\Lambda}}} = 39\mu m$, E-scale $E_{gr} = \dfrac{\hbar c}{r_{gr}} = 5.04 * 10^{-3}\, eV$

covariant derivative $D_\mu = \partial_\mu - i\sqrt{\alpha_{gr}}\left(\tau_a \bullet A^a\right)_\mu$, $\tau^a = i\varepsilon_\nu^{\ a}{}_\lambda$

gauge group Lie(ε) ≅ $SU(2)$, $[\tau^a, \tau^b] = i\varepsilon_c^{\ ab}\tau^c$

Dirac eq. $\left(i\hbar\gamma^\mu D_\mu - mc\right)\Psi = 0$

field eq.= $AK(A_\mu^{\ \nu}, E^{\mu\nu}, \Lambda)$, graviton tensor $A_\mu^{\ \nu}$, tetrad $E^{\mu\nu}$

boundary cond. r→∞ $E\eta E^t = g_{ext}^{-1}/\left(-det\, g_{ext}\right)^{3/4}$

external metric g_{ext}

quantum metric $g = g_{Schw}(r_s = r_\Lambda)$ $g_{11} = 1/\left(1 - \dfrac{r_\Lambda}{r}\right)$

approx. quantum grav. potential $E_{gr}(r) = \dfrac{r_\Lambda}{r}mc^2$, $r_\Lambda = 2.03 * 10^{-57} m$

grav. quantum noise $P_p = \dfrac{r_\Lambda}{r_p}m_p c^2 \dfrac{c}{\lambda_0} \sim 10^{-27} eV/s$

(r_n, m_n particle radius, mass, λ_0 char. length)

SU(2)e $\tau^a = i\varepsilon_\nu^{\ a}{}_\lambda$

$[\tau^a, \tau^b] = i\varepsilon_c^{\ ab}\tau^c$

Color interaction

color (strong) interaction SU(3)
field carrier gluon $A^a{}_\mu$ a=1..8 S=1
charge color=(r,g,b)
energy scale $E_{col} = 220 MeV$, conf-scale

$$r_{col} = \frac{\hbar c}{E_{col}} = 0.89*10^{-15} m \approx r_{pr}$$

cov. derivative $D_\mu = \partial_\mu - igA^a{}_\mu T^a$

Dirac eq. $\left(i\, D_\mu \gamma^\mu - m\right)\psi = 0$

field eq.

$$\frac{1}{2}\partial_\nu\left(F^{a\mu\nu} - F^{a\nu\mu}\right) + \frac{g}{2}\left(f^{kac}A^c_\nu F^{k\mu\nu} + f^{kba}A^b_\nu F^{k\nu\mu}\right) = j^a{}_\nu = g\,\bar{\psi}\gamma^\mu T^a \psi$$

$\psi = \left(u_1, u_2, u_3\right)$, e.g. proton $p = \left(u, u, d\right)$

baryon $b:8A_g$ meson $m:2A_g$ v-meson $m:4A_g$

field tensor $F^a{}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu A^c_\nu$

Lagrangian $L = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}F^a{}_{\mu\nu}F_a^{\mu\nu}$

Yukawa-boson π^0, π^+, π^-

color Lie group SU3)
generators Gell-Mann matrices λ_a
a=1...8

π^0, π^+, π^- pions
low-energy strong force

Extended weak interaction

hyper-color interaction SU(4)
4 charges (L-, L+, R-, R+)
15 hc-bosons $A^a{}_\mu$ a=1...15, S=1
cov. derivative $D_\mu = \partial_\mu - ig\, A^a_\mu T_a$, $T_a = \lambda_a / 2$
Dirac eq. $\left(i\gamma^\mu D_\mu - m\right)\Psi = 0$
field eq.
$$\frac{1}{2}\partial_\nu\left(F^{a\mu\nu} - F^{a\nu\mu}\right) + \frac{g}{2}\left(f^{kac}A^c_\nu F^{k\mu\nu} + f^{kba}A^b_\nu F^{k\nu\mu}\right) = j^a{}_\nu = g\,\bar{\psi}\gamma^\mu T^a\psi$$
$\psi = \left(u_{L-}, u_{L+}, u_{R-}, u_{R+}\right)$, e.g. electron $e^- = \left(r_{L-}, 0, r_{R-}, 0\right)$
generations $f_1 : 1A_h$ $f_2 : 4A_h$ $f_3 : 15A_h$
field tensor $F^a{}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu A^c_\nu$
Lagrangian $L = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}F^a{}_{\mu\nu}F_a^{\mu\nu}$
coupling Callan-Symanzik
$$g_{hc}(\mu) = 4\pi\sqrt{3 / \left(76\sqrt{\left(\log(\frac{\mu}{\Lambda_{hc}})\right)^2 + c_{GE1}{}^2}\right)}$$
E-scale $\Lambda_{hc} = 2m(Z_0) = 180GeV$
$r_{hc} = \frac{\hbar c}{\Lambda} = 1.08 * 10^{-18} m$

extended weak
hyper-color Lie group SU(4)
generators SU(4)-matrices λ_a
a=1...15

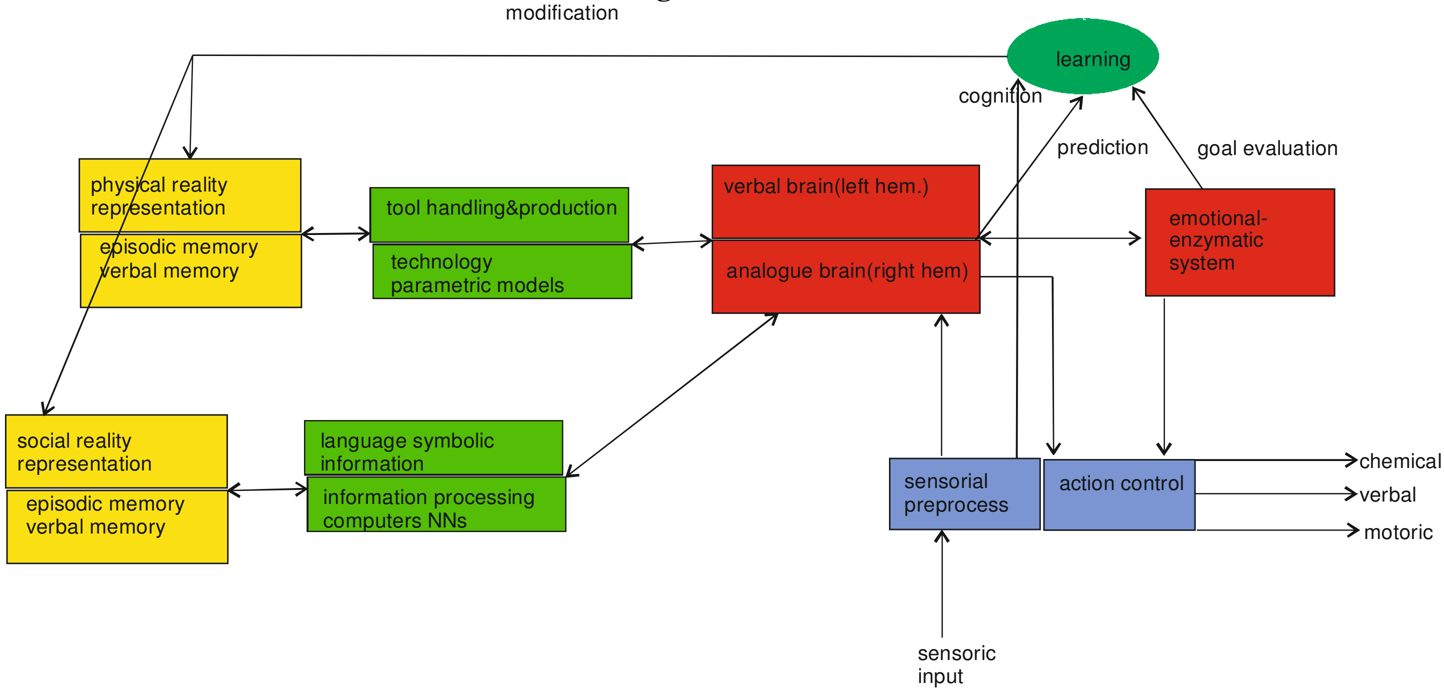
W, Z, H boson

symmetry breaking $SU(4) = SU(2)_L \otimes SU(1)_R \otimes SU(1)_{em}$
field Yukawa boson Z_0, W^+, W^-, H
interaction $L(W^\pm) \otimes Y(Z) \otimes em(A^\mu)$
doublet $SU(2)_L$ interaction $L(W^\pm) = W^\mu J_W{}^\mu$
current $J_W{}^\mu = \bar{\nu}_L \gamma^\mu e_L + ...$
singlet $SU(1)_R$ interaction $Y(Z) = Z^\mu J_Z{}^\mu$
current $J_Z{}^\mu = c_{eL}\bar{e}_L\gamma^\mu e_L + c_{eR}\bar{e}_R\gamma^\mu e_R + ...$
singlet $SU(1)_{em}$ interaction $em(A^\mu) = A^\mu J_A{}^\mu$
em-current $J_A{}^\mu = q(e)\,\bar{e}\,\gamma^\mu e + ...$
SU(1) right-handed derivative
$$D_\mu R = \left(\partial_\mu + i\,g' B_\mu\right)R$$
SU(2) left-handed derivative
$$D_\mu L = \left(\partial_\mu + \frac{i}{2}g' B_\mu - \frac{i}{2}g\sigma_i W^i_\mu\right)L$$

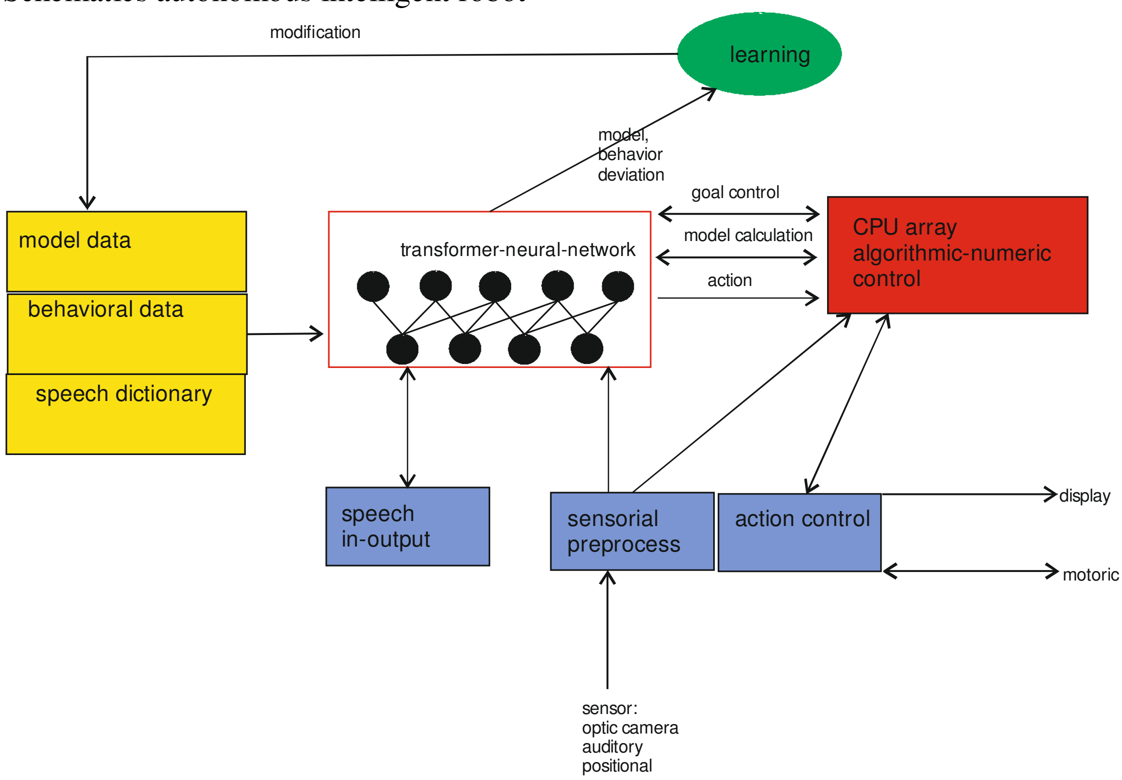
Pauli weak interaction
Lie group SU(2)
generators Pauli-matrices $i\sigma_k$
k=1...3

Models of cognition

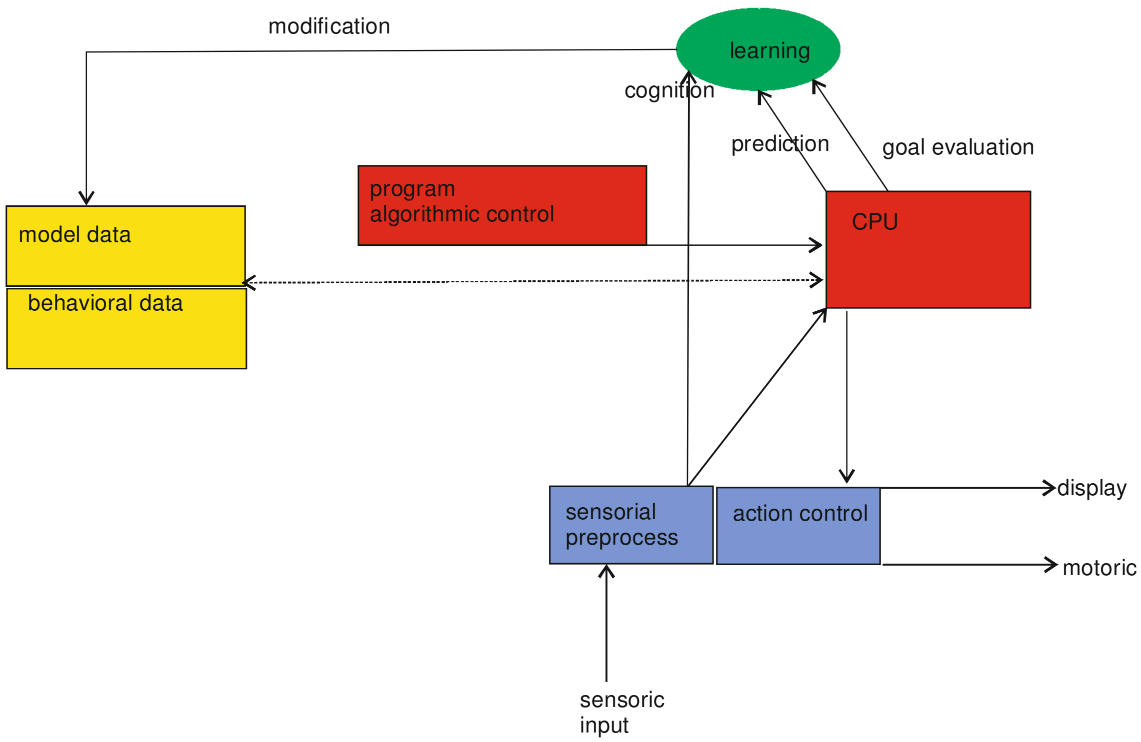
Schematic of a human being



Schematics autonomous intelligent robot

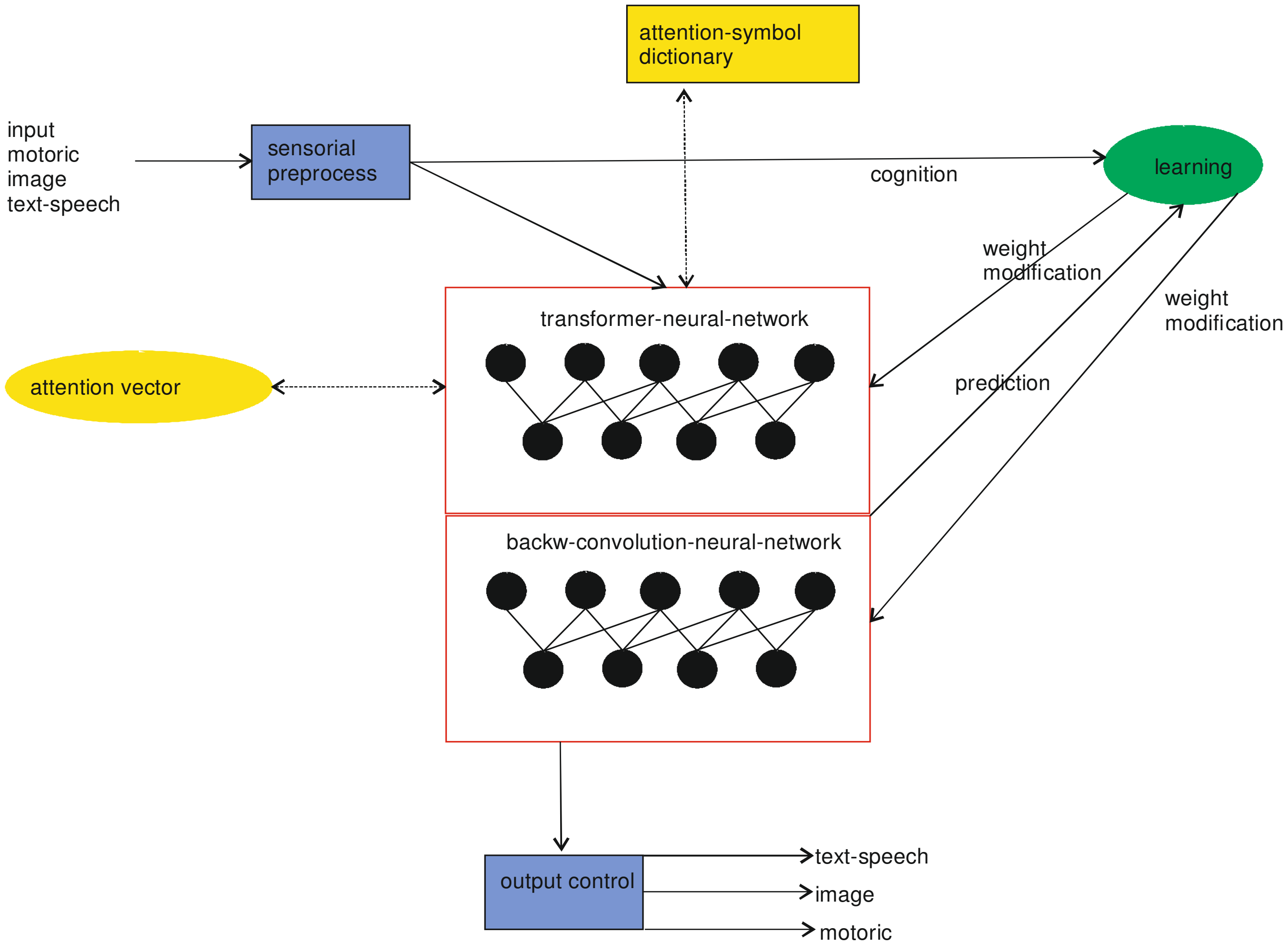


Schematic of analytic cognition
learning Neumann computer-robot



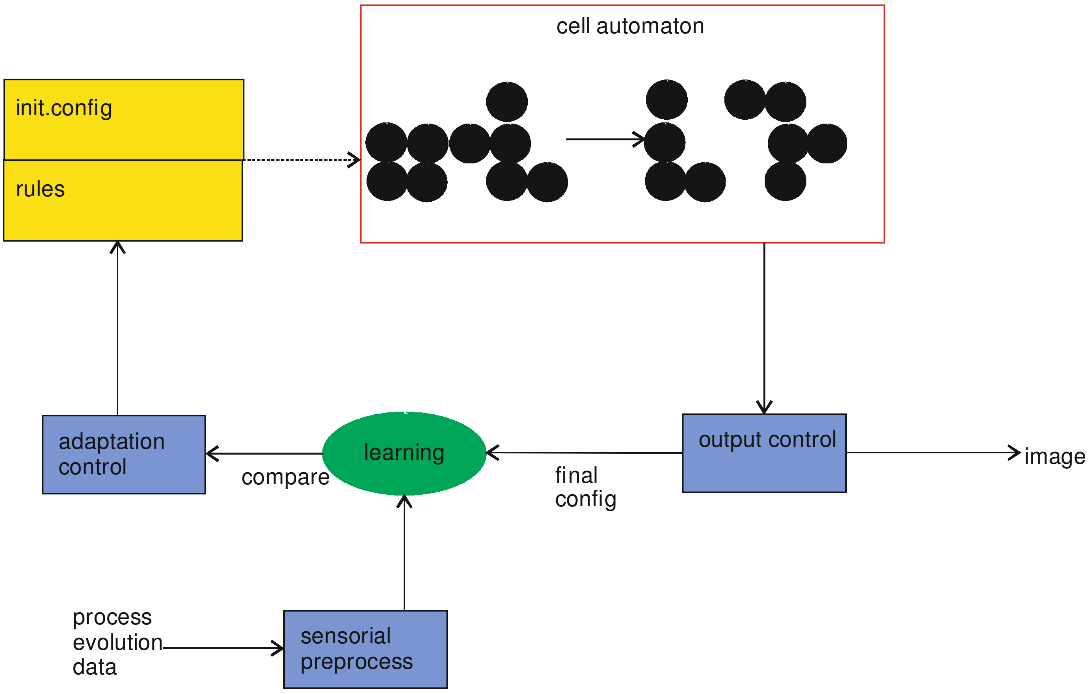
Schematic of pragmatic-adaptive cognition

adaptive transformer neuralnetwork-robot



Schematic of hermeneutic cognition

adaptive cell-automaton process simulator



Geometry axiomatics

Tarski's axioms (T-Euclidean, 2d)
Primitive relations:

- Betweenness, Bxyz
- *Congruence*, Cwxyz

Axioms (first-order logic) 10+3
Congruence 3, Betweenness 2, Continuity 1, LUdimension 2, (Parallel 3), Segment 2

T-Euclidean geometry is decidable:

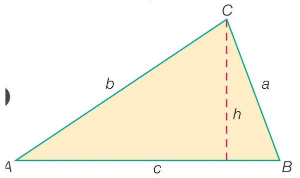
Euclid's axioms
1 "To draw a straight line from any point to any point."
2 "To produce [extend] a finite straight line continuously in a straight line."
3 "To describe a circle with any centre and distance [radius]."

Hilbert's axioms (H-Euclidean, 3d)
Primitive relations:

- *Betweenness*, $Bx_1x_2x_3$
- Containment Cxl, Cxp,Clp
- *Congruence*, Kx_1x_2 $K\alpha_1\alpha_2$

Elements: point x, line l, plane p, angle α
Axioms (second-order logic) 19+1
Combination 7, Order 4, (Parallel 1), Congruence 6, Continuity 2

Euclidean (curvature κ=0)

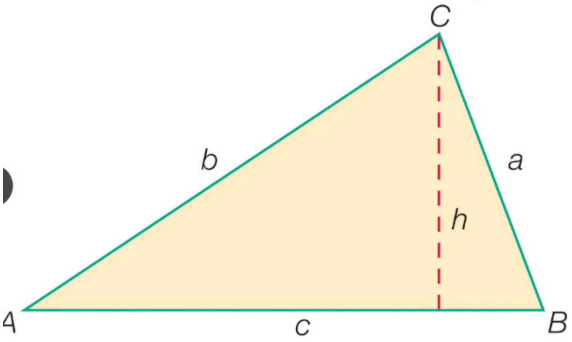


one parallel

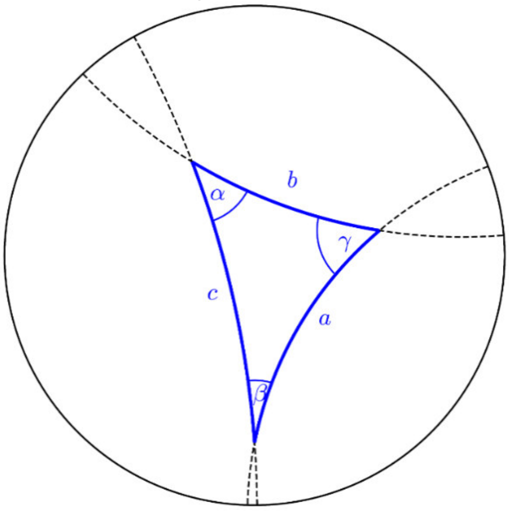
Properties of Euclidean, hyperbolic, and spherical geometries

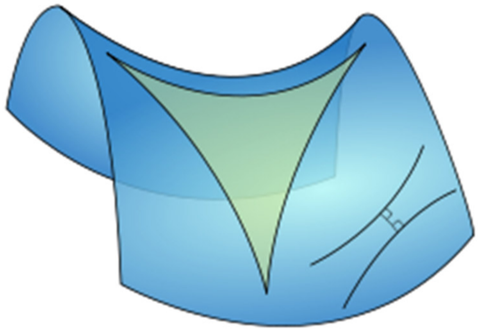
Euclidean	Hyperbolic	Spherical
Two distinct lines intersect in one point.	Two distinct lines intersect in one point.	Two distinct lines intersect in two points.
The sum of the measures of the angles of a triangle is 180.	The sum of the measures of the angles of a triangle is less than 180.	The sum of the measures of the angles of a triangle is more than 180.
Similar triangles that are not congruent exist.	Similar triangles are congruent.	Similar triangles are congruent.
Triangles are finite	Infinite triangles exist	Triangles are finite
Rectangles exist.	No quadrilateral is a rectangle.	No quadrilateral is a rectangle.
A line does not have finite length and is unbounded.	A line does not have finite length.	A line has finite length and is unbounded.
Two parallel lines are equidistant.	No two parallel lines are equidistant.	Parallel lines do not exist.
Two distinct lines do not enclose a finite area.	Two distinct lines do not enclose a finite area.	Two distinct lines enclose a finite area.
The area of a triangle is equal to half the product of the base and height.	The area of a triangle is proportional to its defect.	The area of a triangle is proportional to its excess.
A unique line perpendicular to a given line through a point not on the line.	A unique line perpendicular to a given line through a point not on the line.	All lines perpendicular to a given line intersect at two antipodal points.
A line separates a plane.	A line separates a plane.	A line separates a plane.

Euclidean, hyperbolic, spherical trigonometry

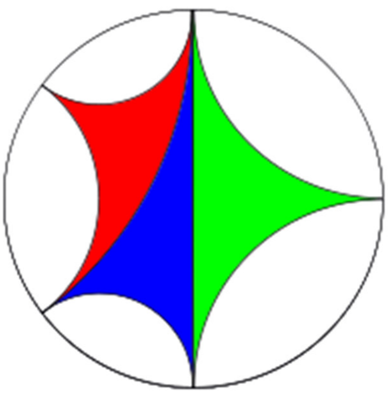


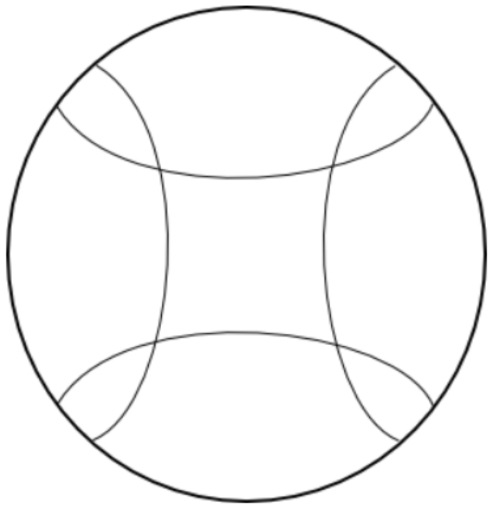
Euclidean triangle





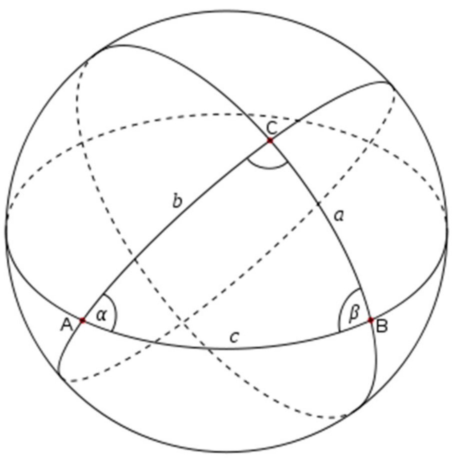
Hyperbolic triangle





Hyperbolic infinite triangle

Hyperbolic square $A=\alpha$, $B=\beta$, $C=\gamma$



Spherical triangle $A=\alpha$, $B=\beta$, $C=\gamma$

First-order logic FOL

Zermelo-Fraenkel set theory
(I) (Axioms of extensionality)
(II) (Axiom of subsets)
(III) (Axiom of pairing)
(IV) (Axiom of union)
(V) (Axiom of infinity)
(VI) (Axiom of the power set)
(VII) (Axiom of choice)

Structure (e.g. Tarski-Euclidean geometry)
Primitive relations:
- Betweenness, Bxyz
- *Congruence*, Cwxyz

Axioms
Congruence 3, Betweenness 2, Continuity 1, LUdimension 2, Parallel 3, Segment 2

First-order logic FOL
Connectives
$\wedge \vee \neg \rightarrow \forall \exists$
Formulas
e.g. $(p_1 \wedge (p_2 \vee \neg p_1)),\ldots$

FOL(model) (e.g. FOL(T-Euclidean))
Formulas
e.g. $(p_1 \wedge (p_2 \vee \neg p_1)),\ldots$
p_1 =Bxyz,…

Calculus syntactic deduction
Γ is a set of sentences, $\phi \psi \theta$ are sentences , $\Gamma |- \phi$ syntactic consequence ϕ from *Γ*

(As) $\phi \in \Gamma \quad \Rightarrow \quad \Gamma |- \phi$ (assumption)

($\wedge$I) $\Gamma_1 |- \theta$ and $\Gamma_2 |- \psi \quad \Rightarrow \quad \Gamma_1, \Gamma_2 |- (\theta \wedge \psi)$ (AND introduction)

($\wedge$E) $\Gamma |- (\theta \wedge \psi) \quad \Rightarrow \quad \Gamma |- \theta$, $\Gamma |- (\theta \wedge \psi) \quad \Rightarrow \quad \Gamma |- \psi$ (AND elimination)

($\vee$I) $\Gamma |- \theta \quad \Rightarrow \quad \Gamma |- (\theta \vee \psi)$, $\Gamma |- \psi \quad \Rightarrow \quad \Gamma |- (\theta \vee \psi)$ (OR introduction)

($\vee$E) $\Gamma_1 |- (\theta \vee \psi)$, Γ_2 , $\theta |- \phi$ and Γ_3 , $\psi |- \phi \quad \Rightarrow \quad \Gamma_1, \Gamma_2, \Gamma_3 |- \phi$ (OR elimin.)

($\rightarrow$ I) Γ , $\theta |- \psi \quad \Rightarrow \quad \Gamma |- (\theta \rightarrow \psi)$ (implication introduction)

($\rightarrow$E) $\Gamma_1 |- \theta \rightarrow \psi$ and $\Gamma_2 |- \theta \quad \Rightarrow \quad \Gamma_1, \Gamma_2 |- \psi$ (implication elimination)

($\neg$ I) $\Gamma_1, \theta |- \psi$ and $\Gamma_2, \theta |- \neg \psi \quad \Rightarrow \quad \Gamma_1, \Gamma_2 |- \neg \theta$ (negation introduction)

(DNE) $\Gamma |- \neg\neg \theta \quad \Rightarrow \quad \Gamma |- \theta$ (double negation elimination)

Calculus semantic model deduction (e.g. FOL(T-Euclidean))

Name	Sequent
Modus Ponens	
Modus Tollens	
Hypothetical Syllogism	
Disjunctive Syllogism	
Constructive Dilemma	
Destructive Dilemma	
Bidirectional Dilemma	
Simplification	
Conjunction	
Addition	
Composition	
De Morgan's Theorem (1)	$\dashv\models$
De Morgan's Theorem (2)	$\dashv\models$
Commutation (1)	$\dashv\models$
Commutation (2)	$\dashv\models$
Commutation (3)	$\dashv\models$
Association (1)	$\dashv\models$
Association (2)	$\dashv\models$
Distribution (1)	$\dashv\models$
Distribution (2)	$\dashv\models$
Double Negation	$\dashv\models$
Transposition	$\dashv\models$
Material Implication	$\dashv\models$
Material Equivalence (1)	$\dashv\models$
Material Equivalence (2)	$\dashv\models$
Material Equivalence (3)	$\dashv\models$
Exportation	
Importation	$\dashv\models$
Idempotence of disjunction	$\dashv\models$
Idempotence of conjunction	$\dashv\models$
Tertium non datur (Law of Excluded Middle)	
Law of Non-Contradiction	
Explosion	

Gödel’s FOL completeness
semantic_deduction(P,model)=T for all models
$\Rightarrow$ $\exists$ syntactic_deduction(model axioms,P)

Gödel’s first incompleteness theorem
F system with axioms algorithmically listable
F encompasses Peano arithmetic
$\Rightarrow$
$\exists$ statement A(n) about natural numbers
• A(n) true for all n
• A(n) unprovable in F

Gödel’s second incompleteness theorem
F system with axioms algorithmically listable
F encompasses Peano arithmetic
$\Rightarrow$
Consistency of F unprovable in F

Functional analysis
Numbers

Natural numbers N
n$\in$ {0,1,2,...}

Galois field F_p , p prime
a$\in$ {0,1,...,p-1}
addition, multiplication, inverse$\rightarrow$
finite non-ordered commutative field

Integers Z
n$\in$ {0,±1, ±2,...}
addition, multiplication$\rightarrow$
ordered metric commutative ring
w/o zero-divisors, compact

P-adic integers Z_p
$\left\{x=\sum_{k=n}^{m} a_k p^k\right\},\ a_k \in F_p\,,a_n \neq 0\,,\ n \geq 0$
norm $|x|_p = p^{-n}$
addition, multiplication$\rightarrow$
non-ordered commutative metric ring
w/o zero-divisors, compact

Rationals Q
x$\in$ {a/b, $a \in Z$, $b \in N$, $b > 0$ }
norm $|x|$=abs.value
metric $d(x,y)=|x-y|$
addition, multiplication, inverse$\rightarrow$
ordered commutative field
countable, algebraically incomplete,
topologically incomplete

P-adic rationals Q_p
$\left\{x=\sum_{k=n}^{\infty} a_k p^k\right\},\ a_k \in F_p\,,a_n \neq 0$
3/11=...23304 23304 $3._5$
norm $|x|_p = p^{-n}$, metric $d(x,y)=|x-y|_p$
ultrametric $|x-y| \leq \max(|x|,|y|)$
addition, multiplication, inverse$\rightarrow$
non-ordered commutative field
cardinality=continuum, algebraically incomplete,
topologically incomplete, locally compact
disconnected, fractal

Reals R=$Comp(Q,|x-y|)$
$R=\{x \mid (a_i) \rightarrow x, a_i \in Q\}$
Q dense in R
norm $|x|$=abs.value
metric $d(x,y)=|x-y|$
addition, multiplication, inverse$\rightarrow$
ordered commutative field
cardinality=continuum
algebraically incomplete
(needs $i=\sqrt{-1}$)
topologically complete,
locally compact

P-adic reals $\bar{C}_p = Comp(\bar{Q}_p,|x-y|)$
$\bar{Q}_p$ alg. closure Q_p
$\bar{Q}_p = \cup\left(Q_p[\xi] \mid \xi = root(P(x)), P(x) \in pol(Q_p)\right)$
cardinality=continuum
algebraically complete , topologically complete
locally non-compact, disconnected, fractal

Complexes C= $R[i]$
x∈ {a+b I, $a, b \in R$ }
Q dense in R
norm $|x| = \sqrt{a^2 + b^2}$ =abs.value
cardinality=continuum
algebraically complete
topologically complete
locally compact

Quaternions q∈H
$q = q_0 + \vec{q} = q_0 + q_1 i + q_2 j + q_3 k$
$i^2 = j^2 = k^2 = i\,jk = -1,\ i\,j = -ji = k$
$jk = -k\,j = i\,, ki = -ik = j$
conjugate $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$
norm $|q| = \sqrt{q\overline{q}} = \sqrt{q_0^{\,2} + q_1^{\,2} + q_2^{\,2} + q_3^{\,2}}$
inverse $q^{-1} = \dfrac{\overline{q}}{|q|^2}$
mapping to Pauli-matrices
$(1, i, j, k) \rightarrow (1, i\sigma_x, i\sigma_y, i\sigma_z)$
H is not closed algebraically
H is topologically complete

Polynomials and power series

sums in R: Cauchy convergence

$$\left(\sum_{i=0}^{\infty} a_i \mid a_i \in R\right) \to a$$

iff

$$\sum^{n} |a_i| < \varepsilon, \forall m, n$$

↓

polynomials in Q: $P(x) \in Q[x]$

algebraically solvable in C

zeros: Newton-Raphson iteration

$$x_{n+1} = x_n - f(x_n) / f'(x_n)$$

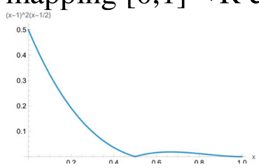
error $\varepsilon_n = |\alpha - x_n|$, $\xi_n \in [x_n, \alpha]$

$$|\varepsilon_{n+1}| = \varepsilon_n^{\,2} \frac{|f''(\xi_n)|}{2|f'(\xi_n)|}$$

converges

example $P(x) = (x-1)^2 (x-1/2)$

mapping [0,1]→R compact, connected



↓

polynomial approximation in R

Taylor series

$$f(x) = \sum_{i=0}^{k} \frac{f^i(a)}{i!}(x-a)^i + h_k(x)$$

$$h_k(x) = \frac{f^{k+1}(\xi)}{(k+1)!}(x-a)^{k+1}$$

Weierstraß approximation

f continuous $f: [a,b] \to R$

$$p_n(x) = \sum_{i=0}^{n} f\left(a + \frac{i}{n}(b-a)\right) B_{i,n}^{a,b}(x)$$

$p_n(x) \to f(x)$ uniform

$B_{i,n}^{a,b}(x)$ Bernstein polynomials

sums in Q_p: coefficients →0

$$\left(\sum_{i=0}^{\infty} a_i \mid a_i \in Q_p\right) \to a$$

iff

$$|a_i|_p \to 0$$

↓

polynomials in Q_p: $P(x) \in Q_p[x]$

algebraically solvable in C_p

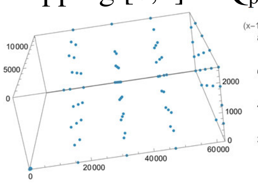
zeros: Hensel iteration

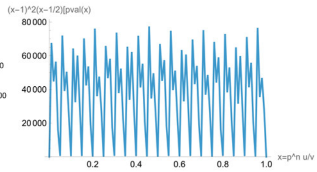
$$\hat{x}(x) = x - P(x) / P'(x)$$

converges if order $v(P'(x)) < n/2$

example $P(x) = (x-1)^2 (x-1/2)$

mapping [0,1]→ Q_p fractal, disconnected





↓

polynomial approximation in Q_p

Mahler series

$$f(x) = \sum_{x \geq k \geq 0} \nabla^k f(0) \binom{x}{k}$$ uniform conv.

$$\binom{x}{i} = \frac{x(x-1)\dots(x-i+1)}{i!}$$ binomial polynomials

forward difference operator $\nabla f(x) = f(x+1) - f(x)$

van-der-Putt approximation

$f: Z_p \to \bar{Q}_p$ continuous

$a_0 = f(0)$, $a_n = f(n) - f(n_-)$, $n_- = n - n_{v-1} p^{v-1}$

$$\sum a_n \to f$$

Functional analysis in R , Q_p

functional norm in R[x]

$\|f\|_I = \sqrt{\int_{x\in I} f^2(x)\,dx}$ L_2 norm on interval I

differential $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

differential operator $Df(x) = f'(x)$

mean value $f'(c) = \frac{f(b)-f(a)}{b-a}, c\in[a,b]$

fixed point $|f(x)-f(y)| \le q|x-y|$, q<1 $\rightarrow$

$\exists x_* : f(x_*) = x_*$

integral $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n-1} f(x_i)(x_{i+1}-x_i)$

for $x_i \in [a,b]$

integral operator $(I_a f)(x) = \int_a^x f(t)\,dt$

fundamental theorem $D(I_a f)(x) = f(x)$

functional norm in Q_p[x]

$\|f\|_s = \sup_{x\in Q_p} |f(x)|$ max-norm

$\|f\|_G = \sup_k |a_k|$, $f(x) = \sum_{k\ge 0} a_k x^k$ Gauss norm

strict differential $f'(a) = \lim_{(x,y)\to(a,a)} \frac{f(x)-f(y)}{x-y}$

differential operator $Df(x) = f'(x)$

forward diff.operator $f : Z_p \rightarrow \overline{Q}_p$

$f'(y) = \sum_{k\ge 1} (-1)^{k-1} (\nabla^k f)(y)/k$

mean value $|f(x+h)-f(x)| \le |h| \|f'\|_G$

for $|x|\le 1$, $|h| \le r_p = |p|^{1/(p-1)}$

fixed point $\|f'\|_G < 1$, $\inf_{x\in B_{\le 1}} |f(x)-x| \le |p|^{1/(p-1)}$ $\rightarrow$

$\exists x_* \in B_{\le 1} : f(x_*) = x_*$

Volkenborn integral

$\int_{Z_p} f(x)\,dx = \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n} f(j) = (Sf)'(0)$

sum operator $Sf(x) = \sum_{0\le i\le x} f(i)$

integral operator $(If)(x) = \int_{Z_p} f(x+t)\,dt$

fundamental theorem

$(If)(x) = (DSf)(x)$

$(\nabla S)(x) = f(floor(x+1))$

$S\nabla f = \nabla S f - f(0)$

Functional analysis in complex numbers C , quaternions H

Complex functional analysis

equivalent for $f(z) = u(x, y) + iv(x, y)$

- f is holomorphic in Ω, $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$
- f satisfies in Ω the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

- f is expandable in Taylor series

$f(z) = \sum_{k=0}^{\infty} a_k (z-c)^k$, converges in neighborhood for every $c \in \Omega$

Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)} dz$$

Laurent series

$$a_k = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-c)^{k+1}} dz$$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-c)^k$$

Quaternionic functional analysis

$q = x_0 + x_1 i + x_2 j + x_3 k$

- $f : H \to H$ **left/right-regular** if satisfies Fueter eq.

$$\frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2} + k \frac{\partial f}{\partial x_3} = 0$$

$$\frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1} i + \frac{\partial f}{\partial x_2} j + \frac{\partial f}{\partial x_3} k = 0$$

- f is left-differentiable in Ω,

$$\lim_{h \to 0} \left(h^{-1} \left(f(q+h) - f(q) \right) \right) = f_l'(q)$$

f left-differentiable → f linear $f(q) = aq + b$

Cauchy-formula: f regular inside Ω⊂H

$$f(q_0) = \frac{1}{2\pi^2} \int_{\partial\Omega} \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dq\, f(q)$$

Laurent-Fueter series

f is regular in open set U except in $q_0 \in U$, then

$$f(q) = \sum_{n=0}^{\infty} \sum_{\nu \in \sigma_n} \left(P_\nu (q-q_0) a_\nu + G_\nu (q-q_0) b_\nu \right)$$

$$a_\nu = \frac{1}{2\pi^2} \int_{\partial U} G_\nu (q-q_0) Dq\, f(q)$$

$$b_\nu = \frac{1}{2\pi^2} \int_{\partial U} P_\nu (q-q_0) Dq\, f(q)$$

$$G_\nu(q) = \frac{q^{-1}}{|q|^2}, \quad \partial_\nu = \frac{\partial^n}{\partial x_{i_1} \ldots \partial x_{i_n}}$$

$G_\nu(q)$ is a rational function of q : $G_\nu(q) = \partial_\nu G(q)$

$P_\nu(q)$ is a polynomial in q:

$$P_\nu(q) = \frac{1}{n!} \sum_{\pi(\nu) = (i_1, \ldots, i_n)} \left(te_{i_1} - x_{i_1} \right) \ldots \left(te_{i_n} - x_{i_n} \right)$$

Structure of Lie algebras

field K=R, C

↓

Lie algebra *g* over K

generators $X_1,...,X_n$, commutator $\left[X_i, X_j\right] = c_{ij}^k X_k$

elements $X \in g$, $X = \sum_i \alpha_i X_i$

g=ring with operations + *

additive commutative group

multiplicative non-commutative half-group

representation: kxk matrices over K

g=vector space over K, basis { $X_1,...,X_n$ }

$g = \langle X_1, ..., X_n \rangle$ dim(g)=n

↓

Cartan subalgebra *h*

$h \approx$ maximal commuting subalgebra of *g*

$h = \langle H_1, ..., H_k \rangle$, rank(g)=dim(h)=k

non-commuting basis $\left\{\tilde{X}_1, ..., \tilde{X}_{n-k}\right\}$

commutators $\left[h_i, \tilde{X}_j\right] = \sum_k \alpha_{ijk} \tilde{X}_k$

$\vec{\alpha}_{ij} = \left(\alpha_{ijk}\right)$

↓

Killing matrix κ

$\kappa_{ij} = c_{ib}^a c_{ja}^b$

eigenvalues $\lambda_i(\kappa) \neq 0 \rightarrow$ g semisimple $g = g_1 \oplus .... \oplus g_m$

$\exists h \subset g$, h Cartan subalgebra, $h \neq 0$

eigenvalues $\lambda_i(\kappa) = 0 \rightarrow$ g solvable $g \supset [g,g] \supset \left[g,[g,g]\right] \supset ... \supset 0$

eigenvalues $\lambda_i(\kappa) < 0 \rightarrow$ g compact (bounded)

Roots in ei basis

Cartan subalgebra h

basis $E^1_{jk} = E_{jk} - E_{k+n,j+n}$

Cartan subalgebra $h = \langle E_{ii} \rangle$, remaining subspace $h_\perp = \langle E^1_{ij} \rangle \ \ i \neq j$

ansatz $H = \sum_{i=1}^{n} \lambda_i E_{ii}$ for $H \in h$

↓

Commutators

simple roots $e_1 = E^1_{12}$, $e_2 = E^1_{23}$, ..., $e_n = f\left(E_{ij}\right)$

with the commutators

$r_1 = [H, e_1] = (\lambda_1 - \lambda_2) e_1$

$r_2 = [H, e_2] = (\lambda_1 - \lambda_2) e_2$

...

$r_n = [H, e_n] = (c_{n-1}\lambda_{n-1} + c_n \lambda_n) e_n$

for su(n+1) $r_n = (\lambda_n - \lambda_{n+1}) E_{n,n+1}$

for so(2n) $r_n = (\lambda_{n-1} + \lambda_n)(E_{2n-1,n} - E_{2n,n-1})$

for so(2n+1) $r_n = \lambda_n (E_{n,2n+1} - E_{2n+1,2n})$

for sp(2n) $r_n = 2\lambda_n (E_{n,2n} - E_{2n,n})$

↓

Roots

simple roots

$\alpha_1 = (\lambda_1 - \lambda_2)$

$\alpha_2 = (\lambda_2 - \lambda_3)$

...

$\alpha_n = (c_{n-1}\lambda_{n-1} + c_n \lambda_n)$

all roots $\pm\alpha_i \pm \alpha_j$

Types of Lie algebras

Lie algebra g over K
generators $X_1,\dots,X_n$, commutator $\left[X_i, X_j\right]=c_{ij}^k X_k$
elements $X \in g$, $X=\sum_i \alpha_i X_i$
g=ring with operations + *
g=vector space over K, basis { $X_1,\dots,X_n$ }
$g=\langle X_1,\dots,X_n\rangle$, dim(g)=n

su(n+1)
$\{x\in gl(n+1,C), \bar{x}^T=-x, tr(x$

dim=$(n+1)^2$-1, rank=n
generators Ti , i=1...$(n+1)^2$-1
commutator relation
$(T_a)_{jk}=c_{jk}^a$
simple roots $\alpha_i=\lambda_i-\lambda_{i+1}$
i=1,...,n+1, n(n+1) roots α_i

so(2n+1) odd orthogonal
$\{x\in gl(2n+1), x^T=-x\}$
dim= n(2n+1), rank=n
generators M_{kl} , $(M_{kl})_{mn}=\delta_{km}\delta_{ln}-\delta_{kn}\delta_{lm}$,
k,l=1,...2n, k<l
commutator relation
$[M_{mn}, M_{rs}]=\delta_{nr}M_{ms}-\delta_{mr}M_{ns}+\delta_{ms}M_{nr}-\delta_{ns}M_{mr}$
n simple roots $\alpha_i=(\lambda_i-\lambda_{i+1})$ *i=1...n-1*, $\alpha_n=\lambda_n$
$2n^2$ roots: $\pm\lambda_i\pm\lambda_j$, *1<=i<j<=n* , $\pm\lambda_i$ *1<=i<=n*

sp(2n) symplectic
2nx2n matrices of the form
$X=\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$, with symmetric B, C
dim= n(2n+1), rank=n
generators $A_{ij}=\begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}$ i,j=1...n
$B_{ij}=\begin{pmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{pmatrix}$, i<=j $C_{ij}=\begin{pmatrix} 0 & 0 \\ E_{ij}+E_{ji} & 0 \end{pmatrix}$, i<=j
commutator relation $\left[A_{ij}, A_{kl}\right]=\delta_{jk}A_{il}-\delta_{li}A_{kj}$,
$\left[A_{ij}, B_{kl}\right]=\delta_{jk}B_{il}-\delta_{li}B_{kj}$, $\left[A_{ij}, C_{kl}\right]=-\delta_{ik}C_{lj}-\delta_{il}C_{kj}$,
$\left[B_{ij}, C_{kl}\right]=\delta_{ik}A_{lj}+\delta_{jk}A_{li}+\delta_{il}A_{kj}+\delta_{jl}A_{ki}$
n simple roots $\alpha_i=(\lambda_i-\lambda_{i+1})$ for i=1,...,n-1 ;
$\alpha_n=2\lambda_n$
$2n^2$ roots $\pm(\lambda_i\pm\lambda_j)$, $\pm\lambda_j$ $1\le i< j\le n$

so(2n) even orthogonal
$\{x\in gl(2n), x^T=-x\}$
dim= n(2n-1), rank=n, $H_i=\langle M_{2i-1,2i}\rangle$
generators M_{kl} , $(M_{kl})_{mn}=\delta_{km}\delta_{ln}-\delta_{kn}\delta_{lm}$,
k,l=1,...2n, k<l
commutator relation
$[M_{mn}, M_{rs}]=\delta_{nr}M_{ms}-\delta_{mr}M_{ns}+\delta_{ms}M_{nr}-\delta_{ns}M_{mr}$
n simple roots $\alpha_i=(\lambda_i-\lambda_{i+1})$, *i=1...n-1* ,
$\alpha_n=(\lambda_{n-1}+\lambda_n)$
2n(n-1) roots $\pm(\lambda_i\pm\lambda_j)$ 1<=i<j<=n

Five exceptional algebras
g_2 (Dimension 14)
f_4 (Dimension 52)
e_6 (Dimension 78)
e_7 (Dimension 133)
e_8 (Dimension 248)

Special linear Lie algebras su(n+1) , Cartan type An

su(n+1)= $\left\{x\in gl\left(n+1,C\right),\overline{x}^{T}=-x,tr\left(x\right)=0\right\}$
dim=$(n+1)^2$-1, su(n+1) semi-simple compact,
Cartan subalgebra dim=n
generators Ti , i=1...$(n+1)^2$-1
commutator relation $\left[T_a,T_b\right]=c_{ab}^{k}T_k$
simple roots $\alpha_i=\lambda_i-\lambda_{i+1}$ for i=1,...,n+1, n(n+1) roots α_i
Killing form is $\kappa\left(x,y\right)=2\left(n+1\right)tr\left(x\ y\right)$

su(2)=gauge algebra of AK-quantum gravity
generators σ_i Pauli matrices,
commutator relation $\left[\sigma_a,\sigma_b\right]=2i\varepsilon_{abc}\sigma_c$
dim=3, , Cartan subalgebra dim=1
simple root is $\alpha_1=\lambda_1$
roots are $\pm\lambda_1$

su(3)=gauge algebra of (strong) color interaction
generators $T_a=i\,\lambda_a/2$, a=1...8,
commutator relation $\left[T_a,T_b\right]=c_{ab}^{k}T_k$
where λ_a are the 8 3x3 Gell-Mann matrices
dim=8 , Cartan subalgebra dim=2
3 simple roots are $\alpha_1=\lambda_1-\lambda_2$, $\alpha_2=\lambda_2-\lambda_3$, $\alpha_3=\lambda_1-\lambda_3$
6 roots are $\pm\alpha_1$, $\pm\alpha_2$, $\pm\alpha_3$
Killing matrix $\kappa_{ij}=-diag_8\left(\{3,...,3\}\right)$

su(4)=gauge algebra of extended weak interaction
generators $T_a=i\,\lambda_a/2$, a=1...15,
commutator relation $\left[T_a,T_b\right]=c_{ab}^{k}T_k$
where λ_a are the 15 extended 4x4 Gell-Mann matrices
dim=15 , Cartan subalgebra dim=3
the 3 simple roots are $\alpha_1=\lambda_1-\lambda_2$, $\alpha_2=\lambda_2-\lambda_3$, $\alpha_3=\lambda_3-\lambda_4$,
the 12 roots are $\pm\left(\lambda_1-\lambda_2\right)$, $\pm\left(\lambda_1-\lambda_3\right)$, $\pm\left(\lambda_1-\lambda_4\right)$, $\pm\left(\lambda_2-\lambda_3\right)$,
$\pm\left(\lambda_2-\lambda_4\right)$, $\pm\left(\lambda_3-\lambda_4\right)$
Killing matrix $\kappa_{ij}=-diag_{15}\left(\{4,...,4\}\right)$

so(1,3)=Lie algebra of the Lorentz group

so(1,3)= $\left\{x\in gl\left(3,R\right),x^T\eta=-\eta x\right\}$, η Minkowski $\eta = diag\left(\{-1,1,1,1\}\right)$

so(1,3) is semi-simple non-compact

generators rotations J_i , boosts K_i , i=1,2,3

commutators $\left[J_i,J_j\right]=\varepsilon_{ijk}J_k$, $\left[K_i,K_j\right]=-\varepsilon_{ijk}J_k$, $\left[J_i,K_j\right]=\varepsilon_{ijk}K_k$

dim=6 , Cartan subalgebra dim=2, Cartan type A1

2 simple roots $\alpha_1=\lambda_1-\lambda_2$, $\alpha_2=\lambda_1+\lambda_2$

4 roots $\pm\lambda_1\pm\lambda_2$,

Killing matrix is $\kappa_{ij}=diag_6\left(\{-4,-4,-4,4,4,4\}\right)$

$$J_1=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\quad J_2=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}\quad J_3=\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$$

$$K_1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad K_2=\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}\quad K_3=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}$$

Odd orthogonal Lie algebras so(2n+1) , Cartan type Bn

so(2n+1)= $\{x\in gl(2n+1), x^T=-x\}$
dim= n(2n+1), so(2n+1) simple compact,
Cartan subalgebra dim=n
generators M_{kl} , $(M_{kl})_{mn}=\delta_{km}\delta_{ln}-\delta_{kn}\delta_{lm}$, k,l=1,...2n, k<l
commutator relation $[M_{mn},M_{rs}]=\delta_{nr}M_{ms}-\delta_{mr}M_{ns}+\delta_{ms}M_{nr}-\delta_{ns}M_{mr}$
n simple roots $\alpha_i=(\lambda_i-\lambda_{i+1})$, *i=1...n-1* , $\alpha_n=\lambda_n$
$2n^2$ roots: $\pm\lambda_i\pm\lambda_j$, *1<=i<j<=n* , $\pm\lambda_i$ *1<=i<=n*

so(3) 3x3 skew-symmetric matrices, tangent space 3-dim rotations
dim= 3, simple compact,
Cartan subalgebra dim=1
generators infinitesimal rotations (J_1, J_2, J_3) about x,y,z axes

$$J_1=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\quad J_2=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\quad J_3=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}$$

commutator relation $[J_i,J_j]=\varepsilon_{ijk}J_k$
1 simple root λ_1
2 roots $\pm\lambda_1$

so(5)= 5x5 skew-symmetric matrices, tangent space 5-dim rotations
dim= 10, simple compact, so(5) sp(4) $so(5)\cong sp(4)$
Cartan subalgebra dim=2
generators 10 matrices S_{kl} k,l=1,...5, k<l
$(S_{kl})_{mn}=\delta_{km}\delta_{ln}-\delta_{kn}\delta_{lm}$
commutator relation $[S_{mn},S_{rs}]=\delta_{nr}S_{ms}-\delta_{mr}S_{ns}+\delta_{ms}S_{nr}-\delta_{ns}S_{mr}$
2 simple roots $\alpha_1=\lambda_1-\lambda_2$, $\alpha_2=\lambda_2$
8 roots $\pm\lambda_1$, $\pm\lambda_2$, $\pm\lambda_1\pm\lambda_2$

Symplectic Lie algebras sp(2n) , Cartan type Cn

sp(2n) 2nx2n matrices of the form

$$X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$$, with symmetric B, C

dim= n(2n+1), sp(2n) simple non-compact

Cartan subalgebra dim=n

generators $A_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}$ i,j=1...n

$$B_{ij} = \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix}$$, i<=j

$$C_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{pmatrix}$$, i<=j

commutator relation $\left[A_{ij}, A_{kl}\right] = \delta_{jk} A_{il} - \delta_{li} A_{kj}$,

$\left[A_{ij}, B_{kl}\right] = \delta_{jk} B_{il} - \delta_{li} B_{kj}$, $\left[A_{ij}, C_{kl}\right] = -\delta_{ik} C_{lj} - \delta_{il} C_{kj}$,

$\left[B_{ij}, C_{kl}\right] = \delta_{ik} A_{lj} + \delta_{jk} A_{li} + \delta_{il} A_{kj} + \delta_{jl} A_{ki}$

n simple roots $\alpha_i = (\lambda_i - \lambda_{i+1})$ for i=1,...,n-1 ; $\alpha_n = 2\lambda_n$

$2n^2$ roots $\pm(\lambda_i \pm \lambda_j)$ $\pm\lambda_j$ $1 \le i < j \le n$

sp(4)= 4x4 block matrices X of the form $X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$, symmetric *B,C*

dim= 10, simple non-compact , $J_2 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$

Cartan subalgebra dim=2

4 generators A_{ij} , 3 generators B_{ij} , 3 generators C_{ij}

commutator relation $\left[A_{ij}, A_{kl}\right] = \delta_{jk} A_{il} - \delta_{li} A_{kj}$, $\left[A_{ij}, B_{kl}\right] = \delta_{jk} B_{il} - \delta_{li} B_{kj}$,

$\left[A_{ij}, C_{kl}\right] = -\delta_{ik} C_{lj} - \delta_{il} C_{kj}$, $\left[B_{ij}, C_{kl}\right] = \delta_{ik} A_{lj} + \delta_{jk} A_{li} + \delta_{il} A_{kj} + \delta_{jl} A_{ki}$

2 simple roots $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = 2\lambda_2$

$\pm\lambda \pm \lambda \quad \pm 2\lambda \quad \pm 2\lambda$

Even orthogonal Lie algebras so(2n) , Cartan type Dn

so(2n)= $\{x \in gl(2n), x^T = -x\}$
dim= n(2n-1), so(2n+1) simple compact (n>=2),
Cartan subalgebra dim=n, spanned by $H_i = M_{2i-1,2i}$
generators M_{kl} , $(M_{kl})_{mn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$, k,l=1,...2n, k<l
commutator relation $[M_{mn}, M_{rs}] = \delta_{nr}M_{ms} - \delta_{mr}M_{ns} + \delta_{ms}M_{nr} - \delta_{ns}M_{mr}$
n simple roots $\alpha_i = (\lambda_i - \lambda_{i+1})$, *i=1...n-1* , $\alpha_n = (\lambda_{n-1} + \lambda_n)$
2n(n-1) roots $\pm(\lambda_i \pm \lambda_j)$ 1<=i<j<=n

so(4) 4x4 skew-symmetric matrices $X = \begin{pmatrix} 0 & -b_3 & b_2 & -a_1 \\ b_3 & 0 & -b_1 & -a_2 \\ -b_2 & b_1 & 0 & -a_3 \\ a_1 & a_2 & a_3 & 0 \end{pmatrix}$, tangent space 4-dim rotations

dim= 6, simple compact
Cartan subalgebra dim=2
generators 6 rotations of so(3) A_1, A_2, A_3

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3 boost rotations B_1, B_2, B_3

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

commutator relation $[A_i, A_j] = \varepsilon_{ijk}A_k$, $[B_i, B_j] = \varepsilon_{ijk}A_k$, $[A_i, B_j] = \varepsilon_{ijk}B_k$
2 simple roots $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_1 + \lambda_2$
4 roots $\pm\lambda_1 \pm \lambda_2$

Exceptional Lie algebras

- g_2 (Dimension 14): is the automorphism group of the octonions and the stabilizer of a generic 3-form in 7 dimensions, rank=2
- f_4 (Dimension 52): The Lie algebra of the exceptional group F_4, is the automorphism group of the Albert algebra (a 27-dimensional exceptional Jordan algebra), rank=4

Albert algebra is the space of 3×3 self-adjoint 3x3 matrices over the octonions .

- e_6 (Dimension 78): is the stabilizer of a certain cubic form and appears in the classification of certain projective planes over division algebras, rank=6
- e_7 (Dimension 133): Leaves invariant a symmetric rank-4 tensor in its fundamental 56-dimensional representation, rank=7
- e_8 (Dimension 248): Its root diagram is the E_8 lattice, which models the densest possible sphere packing in 8-dimensional space, rank=8

Structure of Clifford algebras

field K=R, C

↓

Clifford algebra g over K

generators $X_1,...,X_n$, anticommutator $\left\{X_i, X_j\right\} = c_{ij} I$

elements $q \in g$,

$$q = q_0 + \sum_{i_1 < i_2} q_{i_1 i_2} e_{i_1} e_{i_2} + \sum_{i_1 < i_2 < i_3} q_{i_1 i_2 i_3} e_{i_1} e_{i_2} e_{i_3} + ... + q_{i_1 ... i_n} e_{i_1} ... e_{i_n}$$

g=ring with operations + *

additive commutative group

multiplicative non-commutative half-group

representation: kxk matrices over K

g=vector space over K, basis { $X_1,...,X_n$ }

$g = \left\langle X_1, ..., X_n \right\rangle$, dim(g)=n

↓

Quaternion field

$$H = \left\{ q = a + bi + c\,j + d\,k \middle| a,b,c,d \in R \right\}$$

$i^2 = j^2 = k^2 = -1$, $i\,j = k = -j\,i$,

$j\,k = i = -k\,j$, $k\,i = j = -i\,k$

$$q^{-1} = \left(a + bi + c\,j + d\,k\right)^{-1} = \frac{a - bi - c\,j - d\,k}{a^2 + b^2 + c^2 + d^2}$$

$$q^* = -\frac{1}{2}\left(q + iqi + jq\,j + kqk\right)$$

$$\|q\| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

equivalence {1, i, j, k} $\rightarrow \{I, i\sigma_3, i\sigma_2, i\sigma_1\}$

↓

Dirac-Clifford spinor algebra

Cl(1,3)(C) generators γ^μ $\left\{\gamma^\mu, \gamma^\mu\right\} = 2\eta_{\mu\nu} I$

Dirac form $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

Weyl form $\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $\gamma^j = \Big(0 \quad \sigma^j \Big)$, $\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$

↓

General Dirac-Clifford spinor algebra dim=d

General Dirac matrices Γ_a are nxn matrices, $n = \begin{cases} 2^{d/2} & ,d = 2m \\ 2^{(d-1)/2} & ,d = 2m+1 \end{cases}$

$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} I$ in Cl(p,q)(C)

Weyl basis Euclidean metric d=4 Dirac matrices

$\gamma_E^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $\gamma_E^j = i\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$, $\gamma_E^5 = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

in Cl(d)(C):

for d=2, $\Gamma_a = (i\sigma_1, i\sigma_2)$

for d=3, $\Gamma_a = (i\sigma_1, i\sigma_2, i\sigma_3)$

for d=4, $\Gamma_a^{(4)} = (\gamma_E^1, \gamma_E^2, \gamma_E^3, \gamma_E^0)$

for d=5, $\Gamma_a^{(5)} = (\gamma_E^1, \gamma_E^2, \gamma_E^3, \gamma_E^0)$, $\Gamma_5^{(5)} = \gamma_E^5$,

for d=6, $\Gamma_a^{(6)} = -\sigma_2 \otimes \Gamma_a^{(5)}$ $a = 1,2,3,4,5$; $\Gamma_6^{(6)} = \sigma_1 \otimes I$

Vector rotations

$x' = \Lambda x$, Lorentz $\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right)$

6 elementary rotations $\omega_{\mu\nu}$,

Kronecker matrices $M_{\mu\nu}$

$(M_{kl})_{mn} = \eta_{km}\delta_{\ln} - \eta_{kn}\delta_{1m}$

$\Lambda = \exp(-i(\vec{\varphi}\,\vec{J} + \vec{v}\,\vec{K}))$

rotation vector $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$,

boost vector $\vec{v} = (v_1, v_2, v_3)$

rotation &boost $(J_i), (K_i)$ Lie algebra

$\left[K_i, K_j\right] = -\varepsilon_{ijk} J_k$, $\left[J_i, K_j\right] = \varepsilon_{ijk} K_k$

rotation operator $\Phi = (\vec{\varphi}\,\vec{J} + \vec{v}\,\vec{K})$

Spinor rotations

$\Psi' = S\Psi$, $\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$,

$S = exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)$, spin tensor generators $\Sigma^{\mu\nu}$

$\Sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^{\mu},\gamma^{\nu}\right] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \tilde{\sigma}^{\mu\nu} \end{pmatrix}$

with rotation vector $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$,

boost vector $\vec{v} = (v_1, v_2, v_3)$

$S = \begin{pmatrix} A_L & 0 \\ 0 & A_R \end{pmatrix}$ with the left- and right Weyl-spinor rotations

$A_L = exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right) = exp\left(-\frac{i}{2}(\vec{\varphi} - i\vec{v})\vec{\sigma}\right)$

$A_R = exp\left(-\frac{i}{2}\omega_{\mu\nu}\tilde{\sigma}^{\mu\nu}\right) = exp\left(-\frac{i}{2}(\vec{\varphi} + i\vec{v})\vec{\sigma}\right)$

rotation operator $\Phi_s = \frac{1}{2}(\vec{\varphi}\vec{\sigma} \pm i\vec{v}\vec{\sigma})$

0 Mathematical fundament of physics

Physical theories and their mathematical foundations
Theoretical physics can be roughly subdivided into four main branches
- General Relativity
- Classical physics
- Quantum physics
- Statistical mechanics & thermodynamics

The mathematical disciplines, on which all of physics and mathematics relies, are
- Mathematical logic & set theory

Those two provide the axiomatic fundament for other areas of mathematics, including those, on which the branches of physics are based.
The four branches of physics share a common mathematical fundament in three areas
- Euclidean geometry
- Real and complex functional analysis (including differentiation-integration calculus, and differential equations)
- the fundamental Lorentz rotation group and algebra SO(1,3)

In addition, quantum physics depends on three areas of algebra
- Lie algebras SU(n), which are the model for all four quantum gauge interactions:

AK-Quantum Gravity has the SU(2) symmetry
QED has the trivial abelian SU(1) symmetry
QCD color interaction has the SU(3) symmetry
QHCD hyper-color= ext. weak interaction has the SU(4) symmetry
- Clifford-Dirac spin algebras describe the spinor dynamics
- Heisenberg algebra defines the canonical momentum-position commutators, and the transition from classical to quantum regime

Statistical mechanics and thermodynamics is based on mathematical statistics, i.e. probability theory.

Physical theory and its mathematical and experimental formulation
According to Popper's philosophy, a scientific theory is valid, if its results are verified by experiments within the measurement error, and becomes wrong, if the verification fails in a single experiment.
Following this approach, we can describe a physical theory in the following way.
A physical theory is a consists of three entities
- model M={structure S, interaction R}

structure S=(composition,geometry)
interaction R =relation $R\left(p_1,\ldots,p_l;v_1,\ldots,v_l\right)$
p_i=parameters (phys. constants), v_i=observables (length,time,energy,pressure,temperature,momentum,force,....)
- interpretation Φ

map observables to phys. quantities
$\Phi:\left(v_1,\ldots,v_l\right)\rightarrow\left(w_1,\ldots,w_l\right)$
- set of experiments $X\left(w_1,\ldots,w_l\right)=\left\{\text{experiment}\,X_1,\text{experiment}\,X_2,\ldots\right\}$ in phys. quantities w_i

An experiment X_i has a formulation, a calculation and a measurement , with the following scheme:

experiment X_i
phys. quantities w_i
condition cond(w_i)=
geometry, range, init.vals, restrictions

calculation
num. error $\Delta_n(v_i)$
$v_i(\vec{x},t)$
orbit $\vec{x}_i(t)$

measurement
meas. error $\Delta_m(w_i)$
$w_i(\vec{x}_{m,i},t_m)$
orbit $\vec{x}_{m,i}(t_m)$

The calculation within the interaction R yields the values of observables $v_i(\vec{x},t)$ and their orbits $\vec{x}_i(t)$ with numerical error $\Delta_n(v_i)$ .. The measurement carried out within the experimental frame X_i on physical quantities w_i , yields their value over measured location and time $w_i(\vec{x}_{m,i},t_m)$ with experimental error $\Delta_m(w_i)$.
If the calculated and measured values agree within their respective error bounds, the theory is validated, otherwise the theory is falsified.

The structure is explained on several examples in flowcharts:
- falling object in constant gravitational field
- hadrons
- solid-fluid-gas phases

1 Symmetry in physics

[4]

1 Fundamental metric symmetry group

SO(1,3) Lorentzian rotation group

flat metric $\eta_{\mu\nu} = diag(+1,-1,-1,-1)$

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\, M^{\mu\nu}\right) = \exp(-i\,(\vec{\varphi}\,\vec{J} + \vec{v}\,\vec{K}))$$

$\vec{\varphi}$ =3 spatial rotation angles $\vec{v}$ =3 boost angles

$M^{\mu\nu}$ $=(\mu,\nu)$ coordinate swaps

$\vec{J}$ =3 elementary spatial rotations, $\vec{K}$ =3 elementary boosts

$$(M^{\mu\nu})^{\rho}_{\sigma} = -i\,(g^{\mu\rho}\delta^{\nu}_{\sigma} - g^{\nu\rho}\delta^{\mu}_{\sigma})$$

$$J^k = \frac{1}{2}\varepsilon^{lmk} M^{lm} \qquad K^k = M^{0k}$$

$$J^1 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad J^2 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad J^3 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K^1 = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K^2 = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K^3 = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

2 Lorentz-transformation behavior of vectors and spinors
vectors (bosons, spin s=1) , **spinors** (fermions, spin s=1/2)

rotation 4-vector X with Λ : $X' = \Lambda X$

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right) = \exp(-i(\vec{\phi}\,\vec{J} + \vec{v}\,\vec{K}))$$
,

where $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ = rotation angles, $\vec{v} = (v_1, v_2, v_3)$ = boost angles

rotation left-handed, right-handed Weyl-spinor Ψ with A_L , A_R : $\Psi' = A_L \Psi \quad \Psi' = A_R \Psi$

$$A_L = exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right) = exp\left(-\frac{i}{2}(\vec{\varphi} - i\vec{v})\vec{\sigma}\right)$$, left-handed Weyl-spinor

$$A_R = exp\left(-\frac{i}{2}\omega_{\mu\nu}\tilde{\sigma}^{\mu\nu}\right) = exp\left(-\frac{i}{2}(\vec{\varphi} + i\vec{v})\vec{\sigma}\right)$$, right-handed Weyl-spinor

rotation Dirac bi-spinor Ψ with S : $\Psi' = S\Psi$

$$S = exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right) = \begin{pmatrix} A_L & 0 \\ 0 & A_R \end{pmatrix} = \begin{pmatrix} exp\left(-\frac{i}{2}(\vec{\varphi} - i\vec{v})\vec{\sigma}\right) & 0 \\ 0 & exp\left(-\frac{i}{2}(\vec{\varphi} + i\vec{v})\vec{\sigma}\right) \end{pmatrix}$$
,

where Pauli 3-vector $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$

$$\Sigma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & \\ & \tilde{\sigma}^{\mu\nu} \end{pmatrix} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$, generators Clifford-algebra of Dirac-matrices $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,1(4,4)$

Pauli spin-tensor $\sigma^{\mu\nu} = \frac{i}{4}[\sigma^\mu, \tilde{\sigma}^\nu]$, $\tilde{\sigma}^{\mu\nu} = \frac{i}{4}[\tilde{\sigma}^\mu, \sigma^\nu]$, $\sigma^\mu = \frac{1}{2}(1, \vec{\sigma})$, $\tilde{\sigma}^\mu = \frac{1}{2}(1, -\vec{\sigma})$

Relation to quaternions: spinor=quaternion-vector

complex numbers $C = \left\{x_0 + i x_1 \mid x_l \in R, i = \sqrt{-1}\right\}$

quaternion numbers $H = \left\{x_0 + i_q x_1 + j_q x_2 + k_q x_3 \mid x_l \in R\right\}$

$$x_0 + i_q x_1 + j_q x_2 + k_q x_3$$

$$i_q^{\;2} = j_q^{\;2} = k_q^{\;2} = -1$$

$$i_q j_q = -j_q i_q = k_q \quad j_q k_q = -k_q j_q = i_q \quad k_q i_q = -i_q k_q = j_q$$

using the correspondence to Pauli matrices

$1 \to \sigma_0, i_q \to i\sigma_1, j_q \to i\sigma_2, k_q \to i\sigma_3$

Correspondence vector-spinor

3-vector $\vec{x} = (x_1, x_2, x_3)$	2-spinor $u = \vec{x}\bullet\vec{\sigma} = (u_1, u_2) = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$
	conjugate spinor $u^\dagger = (u_2{}^*, -u_1{}^*)$
orthogonal rotation $\vec{x}\,' = R\vec{x}$	transformation $u'_a = U_{ab} u_b$
$R^t R = I$, I =identity matrix	$U \in SU(2)$, $U^{*t} U = I$
$R(\vec{\varphi}) = \exp(-i\,\vec{\varphi}\bullet\vec{\tau})$	$U(\vec{\varphi}) = \exp\left(-\frac{i}{2}\vec{\varphi}\bullet\vec{\sigma}\right)$
scalar product $\vec{x}\bullet\vec{y} = x^i y_i = \sum_{i=1}^{3} x_i y_i$	scalar product $u\bullet v = u^i \varepsilon_i{}^j v_j = u_1 v_2 - u_2 v_1$
invariant $\vec{x}\bullet\vec{x} = \lvert\vec{x}\rvert^2 = \sum_{i=1}^{3} x_i{}^2$	invariant $u^\dagger \bullet u = u^\dagger{}_1 u_2 - u^\dagger{}_2 v_1 = u_2{}^* u_2 + u_1{}^* u_1 = 2\lvert\vec{x}\rvert^2$
$R(2\pi) = R(0) = I$	$U(2\pi) = -U(0) = -I$

where $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ = rotation angles, $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ elementary rotations ,

$$\tau_1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \tau_2 = i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \tau_3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\text{3-vector } \vec{x} = (x_1, x_2, x_3)} \xrightarrow{\boxed{\text{Pauli } \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)}} \boxed{\text{2-spinor } u = \vec{x}\bullet\vec{\sigma}}$$

Relation to quaternions: spinor=quaternion-vector

complex numbers $C = \{x_0 + i\,x_1 \mid x_l \in R, i = \sqrt{-1}\}$

quaternion numbers $H = \{x_0 + i_q x_1 + j_q x_2 + k_q x_3 \mid x_l \in R\}$

$$x_0 + i_q x_1 + j_q x_2 + k_q x_3$$

$$i_q{}^2 = j_q{}^2 = k_q{}^2 = -1$$

$$i_q j_q = -j_q i_q = k_q \quad j_q k_q = -k_q j_q = i_q \quad k_q i_q = -i_q k_q = j_q$$

Pauli matrices

$\sigma_1 = -i\,k_q \quad \sigma_2 = -i\,j_q \quad \sigma_3 = -i\,i_q$

Lorentz-vector $X = (x^\mu) = (x_0 = ct, x_1, x_2, x_3)$

$x_l \in R$

Lorentz rotation $X' = \Lambda X$

Spinor $\Psi = (\psi^\mu) = (\psi_0, \psi_1, \psi_2, \psi_3)$

$\psi_l \in H$, $\vec{\varphi}_q \in H$, $\vec{v}_q \in H$

spinor transformation $\Psi' = S\Psi$

$$S = \begin{pmatrix} exp\left(-\frac{i}{2}(\vec{\varphi}_q - i\vec{v}_q)\right) & 0 \\ 0 & exp\left(-\frac{i}{2}(\vec{\varphi}_q + i\vec{v}_q)\right) \end{pmatrix}$$

3 Fundamental duality wave-particle and fundamental equations
Classical regime
- **wave**

non-sharp location $\vec{x}(t)$, indeterminacy $\Delta x = \lambda$, velocity $v = c$, massless $m = 0$

electromagnetic: vector A_μ

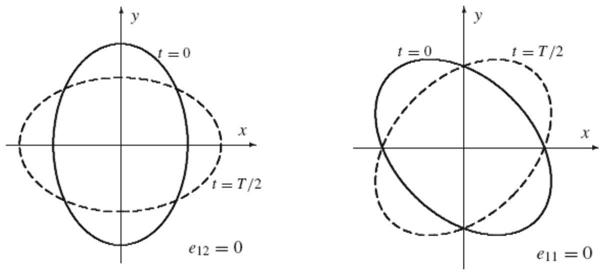
3 independent components, Poynting vectors of energy flux $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

basic mode: dipole $l=1$
plane wave
$A_\mu = e_\mu \, exp(\, i\, k(\, x^3 - ct\,))$
one linear polarization mode

gravitational: tensor $A_{\mu\nu}$
basic mode: quadrupole $l=2$
plane wave
$A_{\mu\nu} = e_{\mu\nu} \, exp(\, i\, k(\, x^3 - ct\,))$ where $k^\mu = (k, 0, 0, -k)$ is the wave vector and the amplitude tensor has the form

$$e_{\mu\nu} = \begin{pmatrix} e_{00} & 0 & 0 & 0 \\ 0 & e_{11} & e_{12} & e_{13} \\ 0 & e_{12} & 0 & e_{23} \\ 0 & e_{13} & e_{23} & e_{33} \end{pmatrix}$$ with 6 independent components

two fundamental polarizations $e_{12}=0$ and $e_{11}=0$



- **particle**

sharp location $\vec{x}(t)$, indeterminacy $\Delta x = 0$, velocity $v < c$, mass $m > 0$

Quantum regime
- **field massless vector/tensor**

spin integer $s = 1, 2$, velocity $v = c$, massless $m = 0$
spinors anticommute $\left\{\gamma^\mu, \gamma^\nu\right\} = 2\eta^{\mu\nu}$
vector rotations are Lorentz rotations Λ in SO(1,3) of the Minkowski metric
$\Lambda = \exp(-i\,(\vec{\varphi}\,\vec{J} + \vec{v}\,\vec{K}))$, with rotation vector $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ and boost vector $\vec{v} = (v_1, v_2, v_3)$, and 4x4 rotation and boost generators $(J_i), (K_i)$

field bosons
vectors commute $\left[x^\mu, x^\nu\right] = 0$
graviton tensor $A_{\mu\nu}$, spin $s = 2$, interaction gauge field SU(2)$_{\text{ext}}$ 4 generators 4x4 $\varepsilon^a{}_{\mu\nu}$
electromagnetic photon, vector A_μ, spin $s = 1$, interaction gauge field SU(1)
color 8 gluons, vector $A^a{}_\mu$, spin $s = 1$, interaction gauge field SU(3),
8 generators 3x3 $\tau_a = \lambda_a / 2$, λ_a 3x3 Gell-Mann matrices
extended-weak hypercolor 15 hc-bosons, vector $A^a{}_\mu$, spin $s = 1$, interaction gauge field SU(4),
15 generators 4x4 $\tau_a = \lambda_a / 2$, λ_a 4x4 Gell-Mann matrices

field equation

field equation: s=2 graviton 32 AK-equations $AK(A_\mu{}^\nu, E^{\mu\nu}, \Lambda)$

$F_{\mu\nu}{}^\kappa = \partial_\mu A_\nu{}^\kappa - \partial_\nu A_\mu{}^\kappa + \varepsilon^\kappa{}_{\kappa_1\kappa_2} A_\mu{}^{\kappa_1} A_\nu{}^{\kappa_2}$

$H_{(\mu,\nu)}{}^\kappa \equiv F_{\mu\nu}{}^\kappa + \frac{\Lambda}{3}\varepsilon_{\mu\nu\rho}E^{\rho\kappa} = 0$

$G^\mu \equiv \partial_\nu E^{\nu\mu} + \varepsilon^\mu{}_{\kappa\lambda}A_\nu{}^\kappa E^{\nu\lambda} = 0$

$I_\mu \equiv E^\kappa{}_\nu F_{\mu\kappa}{}^\nu = 0$

for small amplitude linearized Einstein equations

$\partial^\kappa\partial_\kappa A_{\mu\nu} + \partial_\mu\partial_\nu A^\kappa{}_\kappa - \partial_\nu\partial_\kappa A^\kappa{}_\mu - \partial_\mu\partial_\kappa A^\kappa{}_\nu = 0$

field equation: s=1

electromagnetic Maxwell field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

$\partial_\mu F^{\mu\nu} = j^\nu = -e\,\bar{\psi}\gamma^\nu\psi$

color. hypercolor field tensor $F^a{}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$,

field equation $\frac{1}{2}\partial_\nu\left(F^{a\mu\nu} - F^{a\nu\mu}\right) + \frac{g}{2}\left(f^{kac}A^c_\nu F^{k\mu\nu} + f^{kba}A^b_\nu F^{k\nu\mu}\right) = j^a{}_\nu$

- **particle massive spinor** $s = 1/2,\ 3/2$ **, massive vector/scalar** $s = 1, 0$

velocity $v < c$, massive $m > 0$

spinors anticommute $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

spinor rotations S are 4x4 spin matrices in spinor group Spin(1,3)

$$S = \begin{pmatrix} A_L & 0 \\ 0 & A_R \end{pmatrix} = \begin{pmatrix} exp\left(-\frac{i}{2}(\vec{\varphi} - i\vec{v}\,)\vec{\sigma}\right) & 0 \\ 0 & exp\left(-\frac{i}{2}(\vec{\varphi} + i\vec{v}\,)\vec{\sigma}\right) \end{pmatrix},$$

with rotation vector $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ and boost vector $\vec{v} = (v_1, v_2, v_3)$, and Pauli 3-vector $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$

because of the equivalence {1, i, j, k} $\to \{I, i\sigma_3, i\sigma_2, i\sigma_1\}$ between Pauli matrices and quaternions:

spinors are vectors of quaternions, Dirac matrices are 2x2 matrices of quaternions

spinor fermion: spin half-integer $s = n + \frac{1}{2}$,

vector boson: spin integer $s = 1$, scalar boson: $s = 0$

particles

basic SM particles Dirac spinors $s = 1/2$

3 generations leptons $(e, \nu_e)(\mu, \nu_\mu)(\tau, \nu_\tau)$, 3 generations quarks $(u, d)(c, s)(t, b)$

field equation

spinor $s = 1/2$ eom Dirac $\left(i\hbar\gamma^\mu D_\mu - mc\right)\Psi = 0$, covariant derivative $D \equiv \partial - ig\sum_a A^a\tau_a$, with generators τ_a

spinor $s = 3/2$ eom Rarita–Schwinger $\left(\hbar\varepsilon^{\mu\kappa\rho\nu}\gamma_5\gamma_\kappa D_\rho + \frac{1}{2}mc\left[\gamma^\mu, \gamma^\nu\right]\right)\psi_\nu = 0$

vector boson $s = 1$ eom Proca $\left(\hbar^2 D^\mu D_\mu + m^2c^2\right)A^\nu = 0$, $D_\mu A^\mu = 0$

scalar boson $s = 0$ eom Klein-Gordon $\left(\hbar^2 D^\mu D_\mu + m^2c^2\right)\varphi = 0$

4 Fundamental symmetry operators

charge conjugation $C = q \rightarrow -q$ quantum $C\psi = i\gamma^2\psi^*$

parity=spatial reversal $P = \vec{x} \rightarrow -\vec{x}$ quantum $P = i\gamma^0$

time reversal $T = t \rightarrow -t$ quantum $T = i\gamma^1\gamma^3$

the product $CPT = -i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^5$ is a Lorentz-invariant and CPT is conserved under all interactions

C, P **maximally violated** under weak interactions

CP **weakly violated** under weak interactions (kaon decay rel. deviation ~10^{-6})

Breaking of time-symmetry: arrow of time in classical and quantum regime

In physics. the two classical interactions gravity and electromagnetic, and the color(strong)-interaction are time-symmetric, only the weak interaction is not (CP-violation in weak kaon decay).

But the arrow of time is always present in statistical phenomena governed by thermodynamics (classical and quantum), because the entropy is non-decreasing (second law of thermodynamics).

In the quantum regime, time is locally reversible on micro-scale because of the fundamental uncertainty relations.

Spontaneous breaking of time-symmetry in quantum and in classical regime

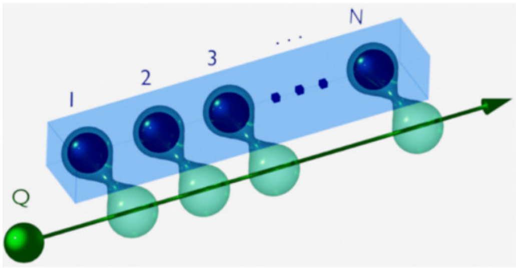
In the classical and quantum regime, there is a spontaneous breaking of time-symmetry classical in gravitational (and by analogy in classical electrodynamic and in quantum gravitational and quantum electrodynamic) systems, so the preferred time direction is taken (the one preferred by CP-violation, i.e. positive). This is an explanation for the observed Boltzmann-effect of **low initial entropy**.

Both classical and quantum **dispersing** dynamical systems are **ergodic** (every state is reached), **entropic** (obey the Second Law), **time-irreversible** (arrow of time).

Emergence of constructor-based irreversibility in quantum systems

Quantum machine works by qubit B moving along qubit-row, interacting into entanglement and out of it.

State transformation B: B1->B2 is possible, but inverse B: B2->B1 has probability->0 with N(qubit)->∞ .



5 SU(n) Lie group and Yang-Mills theory

SU(n)={unitary nxn matrices $U \mid U^{*t} = U^{-1},\ det(U) = 1$ }

with generators $\tau_a = \lambda_a / 2$, $a = 1 \ldots n^2 - 1$, $\lambda_a =$ generalized Gell-Mann matrices

commutator relations $[\tau_a, \tau_b] = i\, f^{abc} \tau_c$,structure constants f^{abc}

generator matrices are traceless $Tr(\tau_a) = 0$, Hermitian, and obey the extra trace orthonormality relation

$$Tr(\tau_a \tau_b) = \frac{\delta_{ab}}{2}$$

corresponding Lie algebra sU(n)=span(τ_a)= $\sum_i \alpha_i \tau_i$ ={hermitian traceless nxn matrices $H \mid H^{*t} = H$, $Tr(U) = 0$ }

generators τ_a generate SU(n): SU(n)= $\left\{ \prod_i \left(exp(\tau_i) \right)^{\alpha_i} \right\} \leftrightarrow \left\{ exp\left(\sum_i \alpha_i \tau_i \right) \right\}$

General Yang-Mills theory

gauge transformation $\Omega(x) = exp\left(-i \sum_a \theta_a(x) \tau_a \right)$ = generalized phase shift

covariant derivative defined as

$$D_\mu = \partial_\mu - i\, g\, A^a{}_\mu \tau_a$$

The Lagrangian is $L = \bar{\psi}\left(i\gamma^\mu D_\mu - m \right)\psi - \frac{1}{4} F^a{}_{\mu\nu} F_a^{\mu\nu}$

field tensor $F^a{}_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + g\, f^{abc} A^b{}_\mu A^c{}_\nu$

with the resulting eom = gauge Dirac equation

$$\left(i\hbar D_\mu \gamma^\mu - mc \right)\psi = 0$$

field equation $\frac{1}{2}\partial_\nu \left(F^{a\mu\nu} - F^{a\nu\mu} \right) + \frac{g}{2}\left(f^{kac} A^c_\nu F^{k\mu\nu} + f^{kba} A^b_\nu F^{k\nu\mu} \right) = j^a{}_\nu = g\, \bar{\psi}\gamma^\mu T^a \psi$

6 Interaction gauge symmetry groups

Electromagnetism

classical electrodynamics

Maxwell equations $\partial_\mu F^{\mu\nu} = j^\nu$

electric 3-vector-field $\vec{E}$

magnetic 3-pseudo-v-field $\vec{B}$, $\vec{B} = \vec{\nabla} \times \vec{A}$

four-vector potential $A_\mu = \left(\frac{\phi}{c}, \vec{A} \right)$

$E_i = -cF^{0i}$, $B_k = -\varepsilon_{kij} F^{ij}$ duality

quantum electrodynamic SU(1)

quantum electrodynamics

scale $r_B = \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2} = 5.29 * 10^{-11} m$,

E- scale $E_{em} = \frac{\hbar c}{r_B} = 3.7 keV$

$D_\mu = \partial_\mu + ieA_\mu$ SU(1)

Dirac eq. $\left(i\gamma^\mu D_\mu - m \right) \Psi = 0$

field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Maxwell equations $\partial_\mu F^{\mu\nu} = 0$

field eq. $\partial_\mu F^{\mu\nu} = j^\nu = -e\, \bar{\psi} \gamma^\nu \psi$

Lagrangian $L = \bar{\psi} \left(i\gamma^\mu D_\mu - m \right) \psi - \frac{1}{4} F^a{}_{\mu\nu} F_a{}^{\mu\nu}$

classical Coulomb potential $E_{el}(r) = \frac{e_0{}^2}{r}$

Lie group SU(1)
={real numbers}

Gravity

Ashtekar-Kodama gravitation classical

field carrier graviton $A_\mu{}^\nu$ S=2

32 AK-equation for graviton $A_\mu{}^\nu$ and tetrad $E^{\mu\nu}$

$$H_{(\mu,\nu)}{}^\kappa \equiv F_{\mu\nu}{}^\kappa + \frac{\Lambda}{3}\varepsilon_{\mu\nu\rho}E^{\rho\kappa} = 0$$

$$G^\mu \equiv \partial_\nu E^{\nu\mu} + \varepsilon^\mu{}_{\kappa\lambda}A_\nu{}^\kappa E^{\nu\lambda} = 0$$

$$I_\mu \equiv E^\kappa{}_\nu F_{\mu\kappa}{}^\nu = 0$$

with field tensor $F_{\mu\nu}{}^\kappa$

$$F_{\mu\nu}{}^\kappa = \partial_\mu A_\nu{}^\kappa - \partial_\nu A_\mu{}^\kappa + \varepsilon^\kappa{}_{\kappa_1\kappa_2}A_\mu{}^{\kappa_1}A_\nu{}^{\kappa_2}$$

boundary cond. at $x \to \infty$ for tetrad and external metric $g_{\mu\nu}$

$$E^{\mu\kappa}E^\nu{}_\kappa = g^{\mu\nu}/(-\det(g))^{3/4}$$

$\Lambda \to 0$

GR with Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

with $\kappa = \frac{8\pi G}{c^4}$

at the horizon $r = r_s$ $E^{\mu\nu} >> 1$ and $\Lambda|E| \sim 1$

singularity becomes a finite peak

AK-quantum-gravity

$r_{gr} = \sqrt{l_P\sqrt{\frac{1}{\Lambda}}} = 39\mu m$, E-scale $E_{gr} = \frac{\hbar c}{r_{gr}} = 5.04*10^{-3}eV$

covariant derivative $D_\mu = \partial_\mu - i\sqrt{\alpha_{gr}}\left(\tau_a \bullet A^a\right)_\mu$, $\tau^a = i\varepsilon_\nu{}^a{}_\lambda$

gauge group Lie(ε) ≅ $SU(2)$, $[\tau^a,\tau^b] = i\varepsilon_c{}^{ab}\tau^c$

Dirac eq. $\left(i\hbar\gamma^\mu D_\mu - mc\right)\Psi = 0$

field eq.= $AK(A_\mu{}^\nu, E^{\mu\nu}, \Lambda)$, graviton tensor $A_\mu{}^\nu$, tetrad $E^{\mu\nu}$

boundary cond. r→∞ $E\eta E^t = g_{ext}{}^{-1}/\left(-det\, g_{ext}\right)^{3/4}$

external metric g_{ext}

quantum metric $g = g_{Schw}\left(r_s = r_\Lambda\right)$ $g_{11} = 1/\left(1-\frac{r_\Lambda}{r}\right)$

approx. quantum grav. potential $E_{gr}(r) = \frac{r_\Lambda}{r}mc^2$, $r_\Lambda = 2.03*10^{-57}m$

grav. quantum noise $P_p = \frac{r_\Lambda}{r_p}m_p c^2\frac{c}{\lambda_0} \sim 10^{-27}eV/s$

(r_p, m_p particle radius, mass, λ_0 char. length)

SU(2) $\tau^a = i\varepsilon_\nu{}^a{}_\lambda$

$[\tau^a,\tau^b] = i\varepsilon_c{}^{ab}\tau^c$

Color interaction

color (strong) interaction SU(3)
field carrier gluon $A^a{}_\mu$ a=1..8 S=1
charge color=(r,g,b)
energy scale $E_{col} = 220MeV$, conf-scale

$$r_{col} = \frac{\hbar c}{E_{col}} = 0.89 * 10^{-15}\, m \approx r_{pr}$$

cov. derivative $D_\mu = \partial_\mu - igA^a{}_\mu T^a$

Dirac eq. $\left(i\ D_\mu \gamma^\mu - m\right)\psi = 0$

field eq.

$$\frac{1}{2}\partial_\nu\left(F^{a\mu\nu} - F^{a\nu\mu}\right) + \frac{g}{2}\left(f^{kac} A_\nu^c F^{k\mu\nu} + f^{kba} A_\nu^b F^{k\nu\mu}\right) = j^a{}_\nu = g\,\bar{\psi}\gamma^\mu T^a \psi$$

$\psi = \left(u_1, u_2, u_3\right)$, e.g. proton $p = \left(u, u, d\right)$

baryon $b : 8A_g$ meson $m : 2A_g$ v-meson $m : 4A_g$

field tensor $F^a{}_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$

Lagrangian $L = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}F^a{}_{\mu\nu}F_a^{\mu\nu}$

Yukawa-boson π^0, π^+, π^-

color Lie group SU3)
generators Gell-Mann matrices λ_a
a=1...8

π^0, π^+, π^- pions
low-energy strong force

Extended weak interaction

hyper-color interaction SU(4)
4 charges (L-, L+, R-, R+)
15 hc-bosons $A^a{}_\mu$ a=1...15, S=1
cov. derivative $D_\mu = \partial_\mu - ig A^a_\mu T_a$, $T_a = \lambda_a / 2$
Dirac eq. $\left(i\gamma^\mu D_\mu - m\right)\Psi = 0$
field eq.
$$\frac{1}{2}\partial_\nu\left(F^{a\mu\nu} - F^{a\nu\mu}\right) + \frac{g}{2}\left(f^{kac}A^c_\nu F^{k\mu\nu} + f^{kba}A^b_\nu F^{k\nu\mu}\right) = j^a{}_\nu = g\,\bar{\psi}\gamma^\mu T^a \psi$$
$\psi = \left(u_{L-}, u_{L+}, u_{R-}, u_{R+}\right)$, e.g. electron $e^- = \left(r_{L-}, 0, r_{R-}, 0\right)$
generations $f_1 : 1A_h$ $f_2 : 4A_h$ $f_3 : 15A_h$
field tensor $F^a{}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu A^c_\nu$
Lagrangian $L = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4} F^a{}_{\mu\nu} F_a^{\mu\nu}$
coupling Callan-Symanzik $g_{hc}(\mu) = 4\pi \sqrt{3 / \left(76 \sqrt{\left(\log(\frac{\mu}{\Lambda_{hc}})\right)^2 + c_{GE1}{}^2}\right)}$
E-scale $\Lambda_{hc} = 2m(Z_0) = 180 GeV$
$r_{hc} = \frac{\hbar c}{\Lambda_{hc}} = 1.08 * 10^{-18} m$

extended weak
hyper-color Lie group SU(4)
generators SU(4)-matrices λ_a
a=1...15

W, Z, H boson

symmetry breaking $SU(4) = SU(2)_L \otimes SU(1)_R \otimes SU(1)_{em}$
field Yukawa boson Z_0, W^+, W^-, H
interaction $L(W^\pm) \otimes Y(Z) \otimes em(A^\mu)$
doublet $SU(2)_L$ interaction $L(W^\pm) = W^\mu J_W{}^\mu$
current $J_W{}^\mu = \bar{\nu}_L \gamma^\mu e_L + \ldots$
singlet $SU(1)_R$ interaction $Y(Z) = Z^\mu J_Z{}^\mu$
current $J_Z{}^\mu = c_{eL}\bar{e}_L\gamma^\mu e_L + c_{eR}\bar{e}_R\gamma^\mu e_R + \ldots$
singlet $SU(1)_{em}$ interaction $em(A^\mu) = A^\mu J_A{}^\mu$
em-current $J_A{}^\mu = q(e)\,\bar{e}\,\gamma^\mu e + \ldots$
SU(1) right-handed derivative
$$D_\mu R = \left(\partial_\mu + i\, g' B_\mu\right) R$$
SU(2) left-handed derivative
$$D_\mu L = \left(\partial_\mu + \frac{i}{2} g' B_\mu - \frac{i}{2} g \sigma_i W^i_\mu\right) L$$

Pauli weak interaction
Lie group SU(2)
generators Pauli-matrices $i\sigma_k$
k=1...3

7 General Relativity

The fundamental symmetry of GR is “general covariance”, i.e. its fundamental equations (Einstein equations) are independent of arbitrary (analytic) coordinate transformations.
This is the central requirement for differential geometry on a differential manifold for a metric element
$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu$

with connection via Christoffel symbols $\Gamma^\kappa_{\lambda\mu} = \frac{g^{\kappa\nu}}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right)$

where *Riemann tensor* determines the *curvature* of the geometry

$$R^\rho{}_{\mu\lambda\nu} = \frac{\partial \Gamma^\rho{}_{\mu\lambda}}{\partial x^\nu} - \frac{\partial \Gamma^\rho{}_{\mu\nu}}{\partial x^\lambda} + \Gamma^\sigma{}_{\mu\lambda}\Gamma^\rho{}_{\sigma\nu} - \Gamma^\sigma{}_{\mu\nu}\Gamma^\rho{}_{\sigma\lambda}$$

and where the resulting *geodesics equation* describes the shortest lines in the geometry and is the equation-of-movement in GR

$$\frac{d^2 x^\kappa}{d\tau^2} = -\Gamma^\kappa{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

So, in essence, GR is the differential geometry of the space-time metric, with addition of the *equivalence of geometric curvature* $K_{\mu\nu}$ *and physical energy density* $T_{\mu\nu}$

$$K_{\mu\nu} = \kappa T_{\mu\nu}$$

Assuming that the fundamental equations are linear, follow the fundamental *Einstein equations*

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$
,

with Einstein constant $\kappa = \frac{8\pi G}{c^4} = \frac{8\pi l_P{}^2}{\hbar c}$ and Planck length $l_P = \sqrt{\frac{\hbar G}{c^3}}$

The formula for the Einstein constant results from the Newtonian gravity being the low-energy limit of the Einstein equations.
Einstein equations are derived by minimization of *Einstein-Hilbert action*

$S = \int \left(\frac{R - 2\Lambda}{2\kappa} + L_M\right)\sqrt{-g}\, d^4x$, where $g=det(g_{\mu\nu})$, R is the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ derived from the Ricci tensor $R_{\mu\nu} = R^\kappa{}_{\mu\kappa\nu}$ and the Riemann tensor , $\Gamma^\lambda{}_{\mu\nu}$ are the Christofffel symbols, and Λ is the cosmological constant.
The minimal action principle yields the Einstein equations by variation of $g_{\mu\nu}$ as the Euler–Lagrange equation, starting with the full action, where the Lagrangian L_M accounts for the energy-momentum tensor of the matter,

$$T_{\mu\nu} = L_M g_{\mu\nu} - 2\frac{\delta L_M}{\delta g^{\mu\nu}}$$

2 Mathematical models of cognition

[5]

1 General cognition

Natural science, with physics at its center, is essentially a *description of the reality*. This description is formulated in the language of quantitative mathematics for the quantitative sciences like physics, it is formulated in natural language and in images for qualitative sciences like biology.

An example for the *quantitative-mathematical* description are the Maxwell equations, which are the basis for classical and quantum electrodynamics.

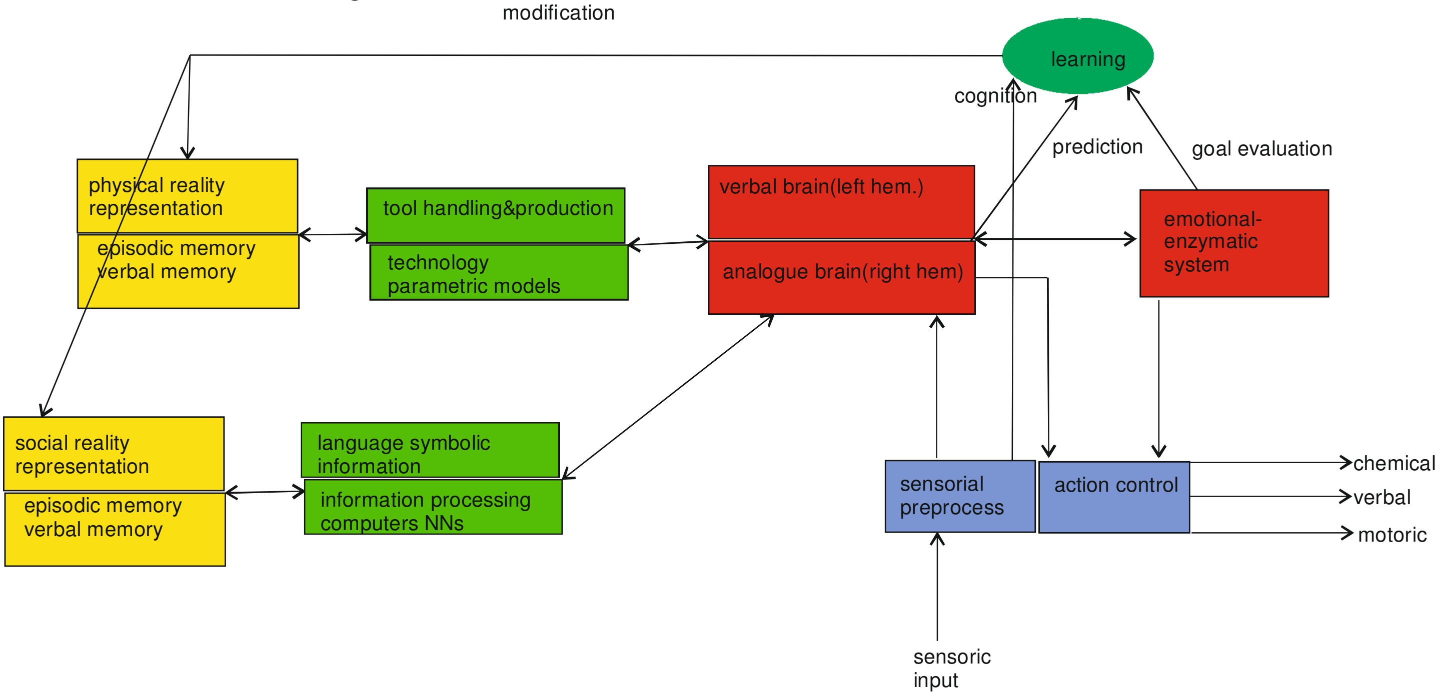
An example for the *qualitative-verbal* description are genotypes and behavioral schemes of a species, or evolution-taxonomy trees in biology.

Humans, like all biological beings have a *representation of reality*, in order to act appropriately within this reality, i.e. live and survive. This representation works in simple organisms through biochemical feed-back loops, in complex organisms through a nervous system with sensors, motoric actuators, memory, learning, executive control. In humans, it works through a complex dual-hemispherical brain with consciousness and linguistic communication.

This representation of reality is at the center of human cognition, i.e. acquirement of knowledge. The original form of it is the qualitative-verbal knowledge like in biology or history. The advanced form is the quantitative-mathematical knowledge, which began with the Greek geometry in antiquity and continued with the birth of classical physics founded by Galilei and Newton in the 17th century.

Acquirement of knowledge is an algorithmic process, and we humans have two ways to achieve it: the *analytic abstract quantitative-symbolical* process controlled basically by the left brain hemisphere and the *analog intuitive non-verbal* process controlled by the right brain hemisphere (which is also responsible for "discoveries, i.e. the so-called eureka-experiences).

Schematic of a human being



The schematic of the learning process in a human being is shown above.

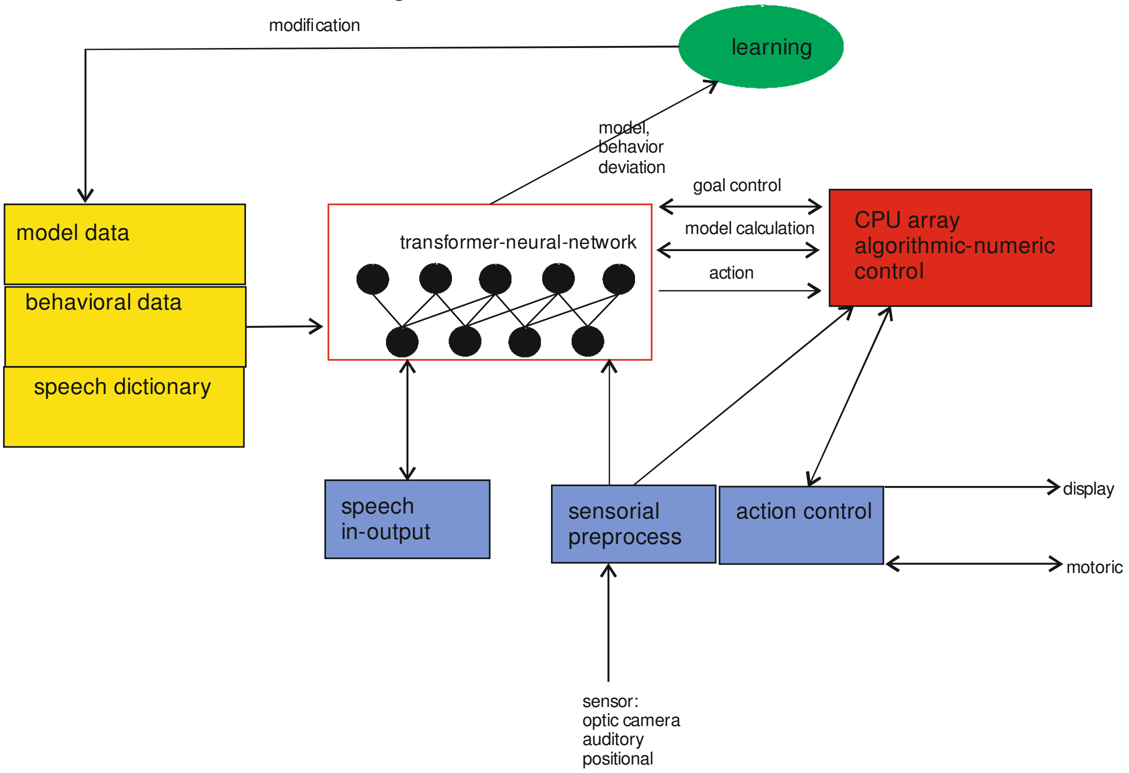
The significant brain centers are shown in red: the verbal-processing left cortex hemisphere, the pattern-processing right cortex hemisphere, the emotion-processing and chemical-controlling brainstem/ midbrain. The storage centers in yellow are the verbal-symbolic memory (reality representation), and the episodic memory. In green are shown the two mediating processes : *tool management* for interaction with the physical environment and *language* for interaction with the social environment.

The cortex processes the sensorial information, and generates an *action response and a prediction*.

In the learning process, the incoming sensorial input and the prediction are compared, evaluated by the emotional system, stored in episodic memory, and potentially used for *modification of the episodic and the verbal memory*.

The two mediating processes are a specifically human accomplishment, which distinguishes us from all other biological beings. Their evolution ran in parallel, and the language and motoric centers are closely related and situated and at short distance in the cortex.
They evolved in two levels: basic level1, which developed over ca 2 million years, and the machine level 2, which came with the industrial and the computer revolution.
• tool management
level1: manual production of tools and weapons in stone, wood, leather, metals, ceramics;
motoric ability to handle tools as an extension of the biological motoric apparatus
level2: mechanic and electronic machines as motoric extension;
electronic, optic, acoustic devices as sensorial extension
• language management
level1: symbolic language with complex grammar and generic semantics evolved from 2 word-sentences [96] into a powerful medium of communication, organization, and information storage via scripture
level2: natural language is transformed into programming language for computers, which is a logical-analytic-calculation extension of the brain;
humanoid autonomous robots equipped with transformer-type neural networks process and produce natural language as humans and are an extension of human behavior and thinking as a whole.

Schematics autonomous intelligent robot



An intelligent robot processes data with two methods
• pattern control, processed by transformer neural network hardware (TNN)
• algorithmic-numeric control, processed by CPU array (CPUA)
The internal-external world representation is stored in model & behavioral data.
TNN takes input from model & behavioral data, speech dictionary, sensorial preprocessor, model numerical calculation (from CPUA), and generates a prediction in physical external data (model data) and action (behavioral data). CPUA performs model numerical calculation, controls the goal model and behavioral data .
Finally, the deviation in goal and prediction data generates a modification of the world representation (learning process).
Speech input (commands and information) and speech output (information) is managed by TNN using the speech dictionary. In speech-learning mode, the speech dictionary is also modified.
Example: a humanoid robot drives a vehicle to a given address.
Sensorial data are actual location and velocity, surrounding traffic data.
Applicable model data are city map, traffic rules.
Applicable behavioral data are car movement control parameters: velocity, direction, position, distances, acceleration.

Applicable goal data are destination address, velocity and time goals, traffic rule boundary conditions, security conditions.
TNN generates in real time a position-direction-velocity (action) prediction, CPUA performs numerical calculation and goal control.

2 Philosophical approaches to cognition: analytic, pragmatic, hermeneutical

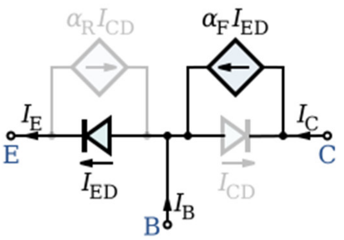
In respect to cognition and reality, there are **three fundamental philosophical approaches**: the analytic, the pragmatic, and the hermeneutical approach.

In the **analytic** approach, the reality is *precisely describable* by a mathematical model of a definite type (in classical and quantum physics: partial differential equations of second degree as equation-of-motion derived from minimization of action= integral of Lagrangian, plus symmetry described by gauge Lee groups and Minkowski metric with Lorentz group, plus statistics with some basic distributions (Gauss, exponential), plus structural parametric models (standard model for particle physics, general-relativistic lambda-CDM model for the universe)).
The validity of the model is verified by experiment with a finite precision. This approach holds that reality is a specific parametric representation of a parametrized mathematical structure. Famous proponents of this approach were, among others, Einstein and Heisenberg.
In this approach, reality is a fixed mathematic structure, which in principle can be deduced from fundamental principles, like Einstein deduced General Relativity from the principle of general covariance.
The model of reality is improved by extending the model in steps until it reaches the “real” interpretation
Example

ideal gas eos $p\,V = n_{mol}\;R\,T \;\rightarrow$ v.d.-Waals $\left(p + a\dfrac{n_{mol}^2}{V^2}\right)\left(V - n_{mol} b\right) = n_{mol}\;R\,T$

Our human approach to this way of knowledge acquisition is : the *analytic abstract quantitative-symbolical* process controlled basically by the left brain hemisphere. The underlying mathematical model machine is the *von-Neumann computer* or in a more abstract generalized form the *universal Turing machine*.

In the **adaptative** approach, the reality is *approximated* by a mathematical model, which is verified by experiment within the current measurement error. The model must be continuously refined and corrected according to the current experimental results. The experimentally verified (within the current measurement error) mathematical model is the valid knowledge of reality (Popper 1934), but it is valid only until falsified, there is no precisely accurate description. The most famous proponents of this approach are Popper from the philosophical point of view, and Bohr from the physical point of view.
In this approach, reality has a structure, not necessarily quantitative, which can be “learned” from experience, like mapping an unknown area.
The model of reality is improved by introducing and adapting new parameters and is always an approximation
Example
Ebers-Moll model of the bipolar transistor

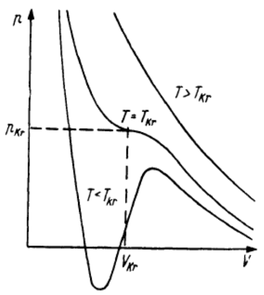


$$I_E = I_{ES}\left(e^{\frac{V_{BE}}{V_T}} - 1\right)$$
$$I_C = \alpha_F I_E$$
$$I_B = (1-\alpha_F)\, I_E$$

The underlying mathematical model is a *neural network*, which describes well learning and cognition, in biology (brain), and in artificial intelligence (transformer neural networks like GPT).
Our human approach to this way of knowledge acquisition is and the *analog intuitive non-verbal* process controlled by the right brain hemisphere. The fitting mathematical model machine is the *spiking binary neural network*, as we know from the modern measurement techniques of brain functionality.

In the **hermeneutic** approach, the mathematical description is only one possibility of cognition, and it cannot describe reality completely. Famous proponents are Bergson (with his concept of time as permanent movement, and his concept of life energy (“elan vital”)) and Heidegger with his concept of life as an evolving and non-reducible entity in time (“Sein und Zeit”).

Roughly speaking, in the hermeneutic “naturalistic” Bergson-Heidegger approach , building blocks of reality in our perception are seen as a collection of patterns (images) connected to notions, and events are seen as series of such patterns in time.
In terms of human brain functionality, this corresponds to the processing of impressions and responding to them in the senso-motoric cortex areas and the midbrain.
A possible fitting representation of this approach are *cellular automata*. Stephen Wolfram has proved that many physical phenomena, especially in regard to complex collective dynamical systems, can be represented by corresponding cellular automata, the same applies to parts of mathematics, e.g. Turing machines and differential equations.
Cellular automata with *n* states are interacting systems of state automata. As such, they are well adapted to describe collective phenomena in physics (e.g. solids, gases, galaxies), in biology (bacterial communities, animal communities), in biological informatics (nervous system, brain). Cellular automata describe such systems much better than statistical physics (thermodynamics).
Cellular automata can handle series of patterns, i.e. behavior: an input of a time-series of sensorial conditions (patterns) is transformed into a time-series of mechanical conditions (movement) and chemical conditions (chemical reaction, e.g. enzyme or neurotransmitter production).
The series of patterns which represents reality is improved in approximation by refining the rules of the input series and output series.
Example in biology: a rabbit sees a series of images and sounds caused by a fox, recognizes it as such, and responds with an action pattern of immediate and fast running away.
Example
real gas v.d.Waals eos $\left(p + a\frac{n_{mol}^2}{V^2}\right)(V - n_{mol}b) = n_{mol}\, R\, T$ represented by the corresponding graph produced by a cellular automaton increases in resolution



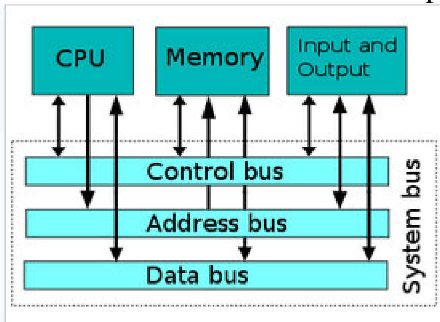
3 Mathematical models of cognition

There are three mathematical models capable of describing the process of cognition of reality:
-von-Neumann computers
-neural networks
-cellular automata

Von-Neumann computer (analytic approach)

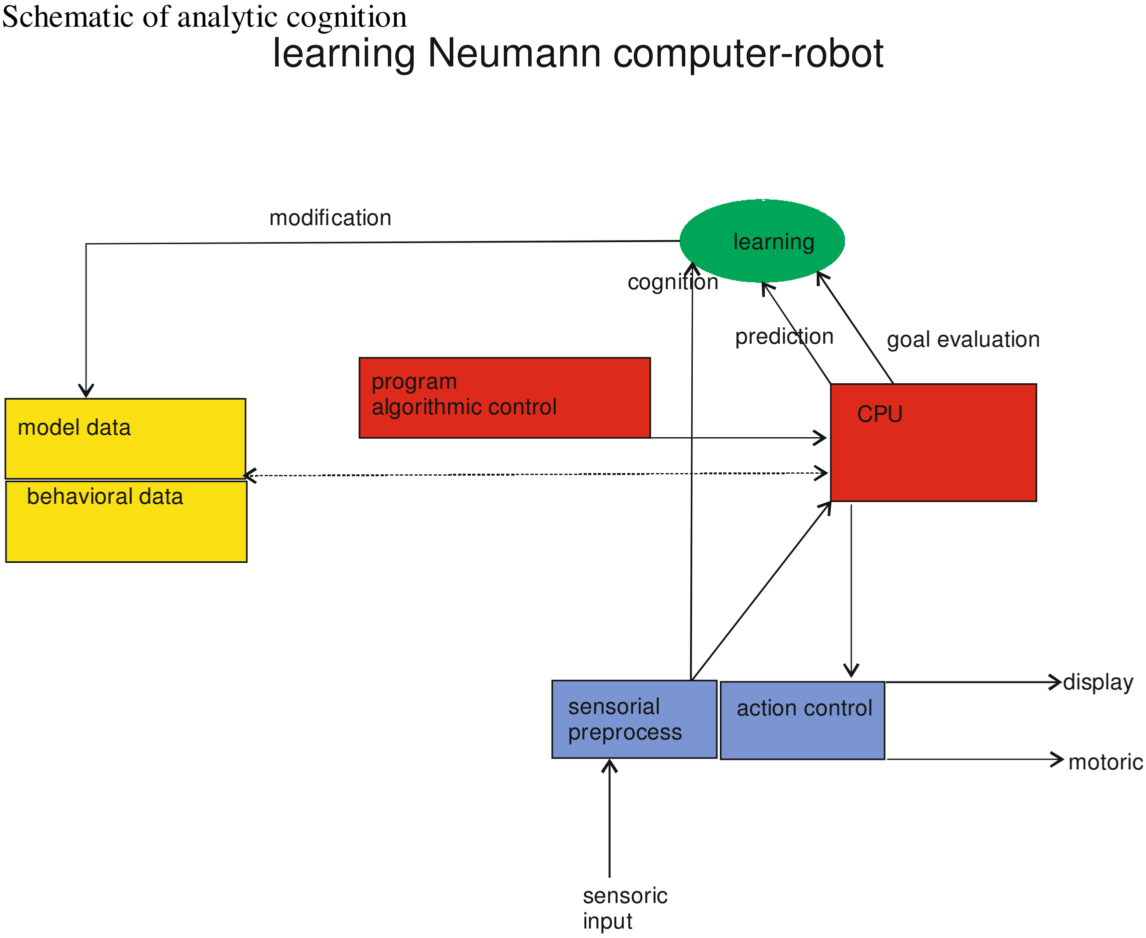
The abstract quantitative-symbolical verbal thinking with its logical deduction and numerical calculation, emerged probably from the complex sequential movement control located in the cerebellum and the senso-motoric cortex. The underlying mathematical model is the *von-Neumann computer* (specifically the ALU and program-data-memory). This model works well for rule-based deduction and numeric calculation, but is not well-adapted to "fuzzy" description, representation of psychic reality, pattern recognition and structure recognition, as required in AI.

Structure of a von-Neumann computer

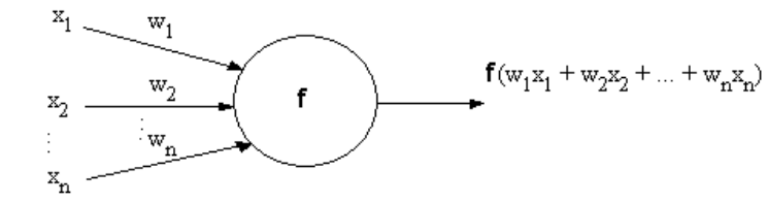


A von-Neumann-computer consists of CPU (processing unit), memory (containing opcode and data), input- and output-unit, they connected by control-bus (opcode), data-bus(data) and address-bus (memory address of current opcode).

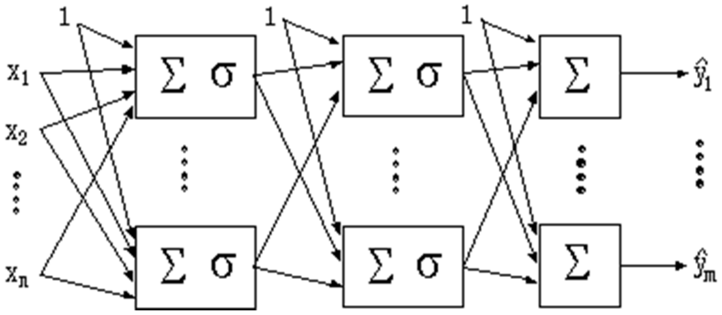
Schematic of analytic cognition



Neural network (pragmatic-adaptive approach)
We know that neural activity in brain happens on the basis of *spiking binary neural network*, where cognition works via pattern recognition, relational structuring, hierarchy forming, prediction by analogy, memory based on relational and spatial-temporal structure. This is the mathematical model, used and verified in artificial intelligence (AI), for our intuitive non-verbal thinking. Neural networks are capable of pattern memorizing recognition by enforced learning with weight adaptation, structure recognition by free learning, and of evolution by learning-induced structure adaptation. As practiced in AI, neural network work in two modes: learning mode with memorization by weight adaptation and recognition mode with similarity and analogy processing, exactly as high-level living organisms. A neural network is also of course well fitted for description of our psychic reality and the "fuzzy" functionality of natural language and its symbolic representation in the brain.



Structure of a neuron in a neural network

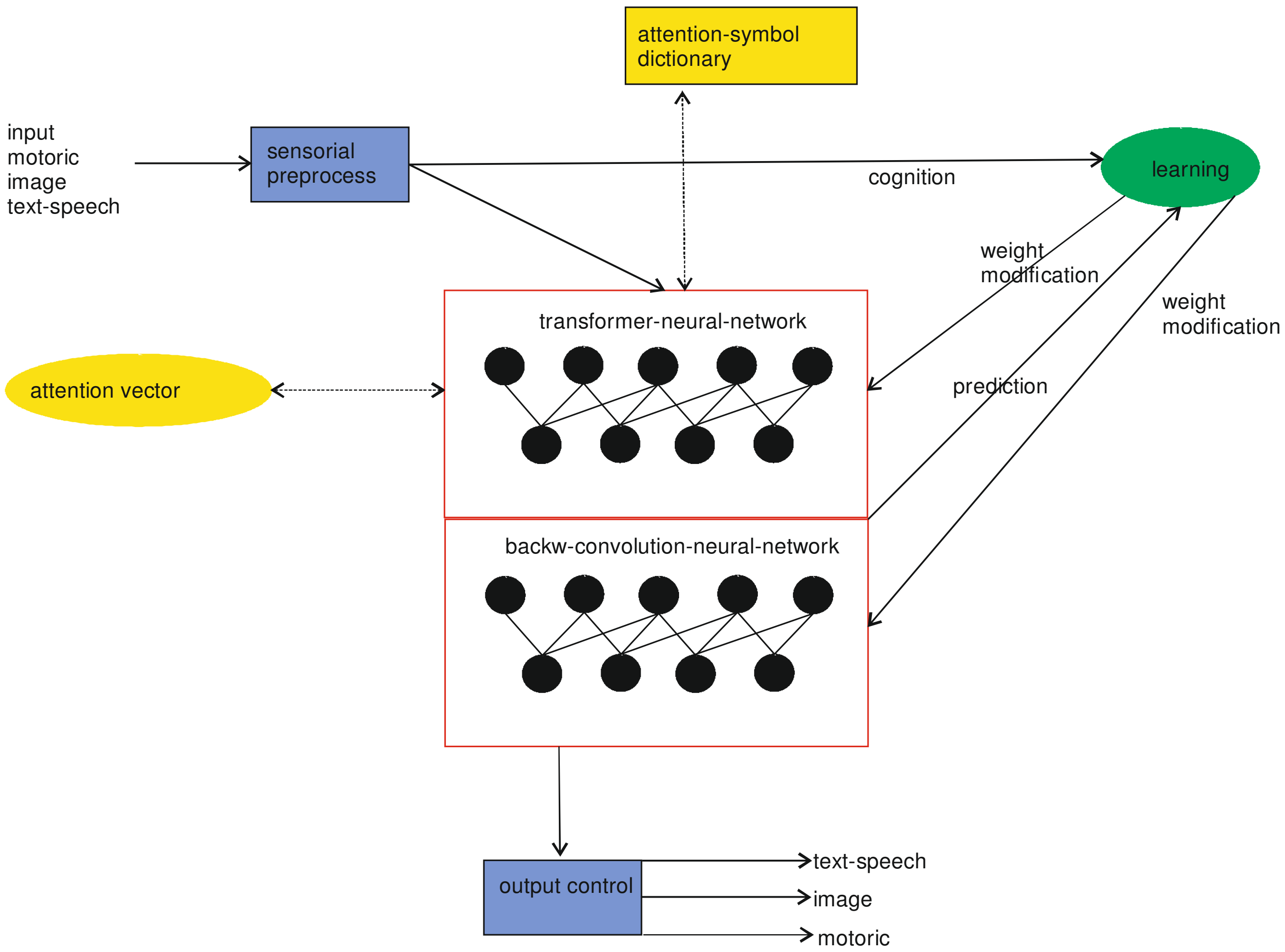


A multi-layer feedforward network with several hidden layers and one output

A neural network is a network of neurons with function f and input vector $x=(x_1, x_2,...,x_n)$ with weights $(w_1, w_2,...,w_n)$ and the output $y=f(w_1 x_1+ w_2 x_2.+...+ w_n x_n)$. The weights are adapted in such a way that the network of m neurons yields the output vector $Y=(y_1, y_2,...,y_m)$ with desired result.

Schematic of pragmatic-adaptive cognition

adaptive transformer neuralnetwork-robot

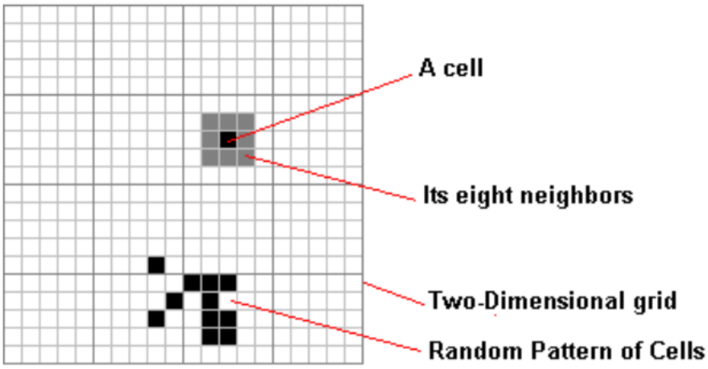


Cellular automaton (hermeneutic approach)

There is a third mathematical model for the description of reality, especially of deterministic and stochastic processes, collective phenomena and fractal structures. This is the *cellular automaton* governed by transition rules, known from Conway's "game-of-life" and from Stephen Wolfram's work. This model is also capable of performing algorithmic calculation, rule-based deduction, and evolution by adaptation of rules.

A cellular automaton is essentially a state machine based on a set of transition rules. But it generates also a process in time, which can be viewed as a series of images, and therefore resembles a film or a story from our human intuitive concept of reality.

A two-dimensional cellular automaton



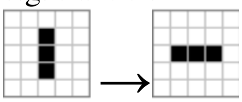
It is comprised of a two-dimensional grid of cells. Each cell can be in one of two states, on or off (dead or alive). Cells may transition from one state to the other, and become on or off, based on a set of rules.

Example: Conway's Game of Life

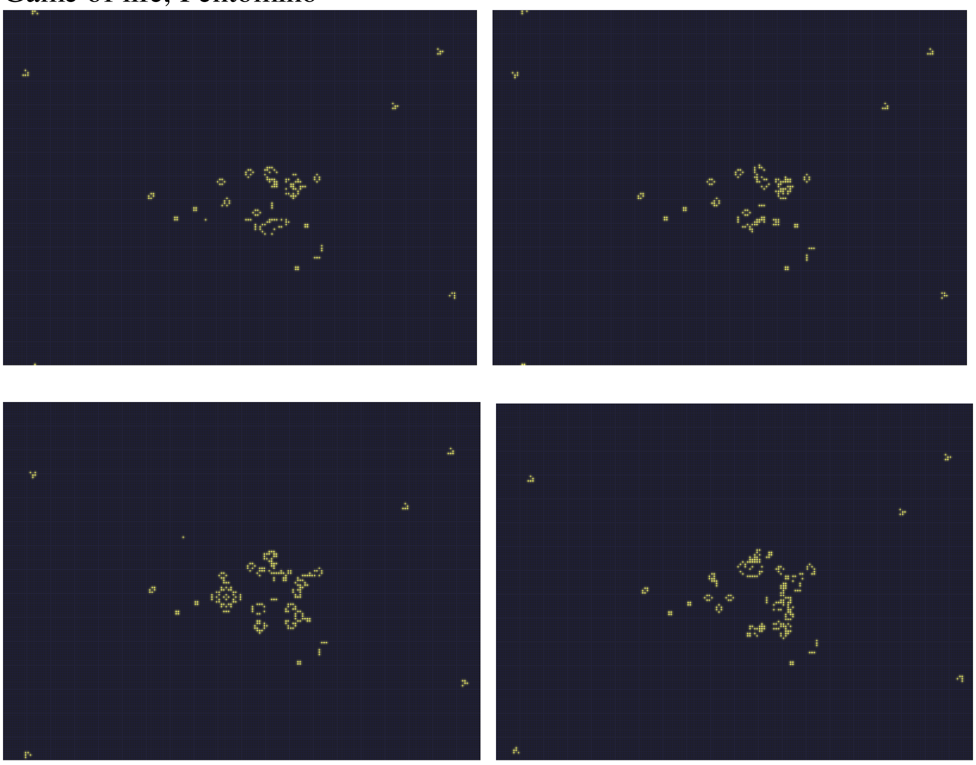
Rules of the Game of Life

Let N be the number of neighbors of a given cell. If N = 0 or 1, cell dies If N = 2, the cell maintains its current state (status quo)If N = 3, the cell becomes alive If N = 4,5,6,7 or 8, the cell dies.

e.g. transition

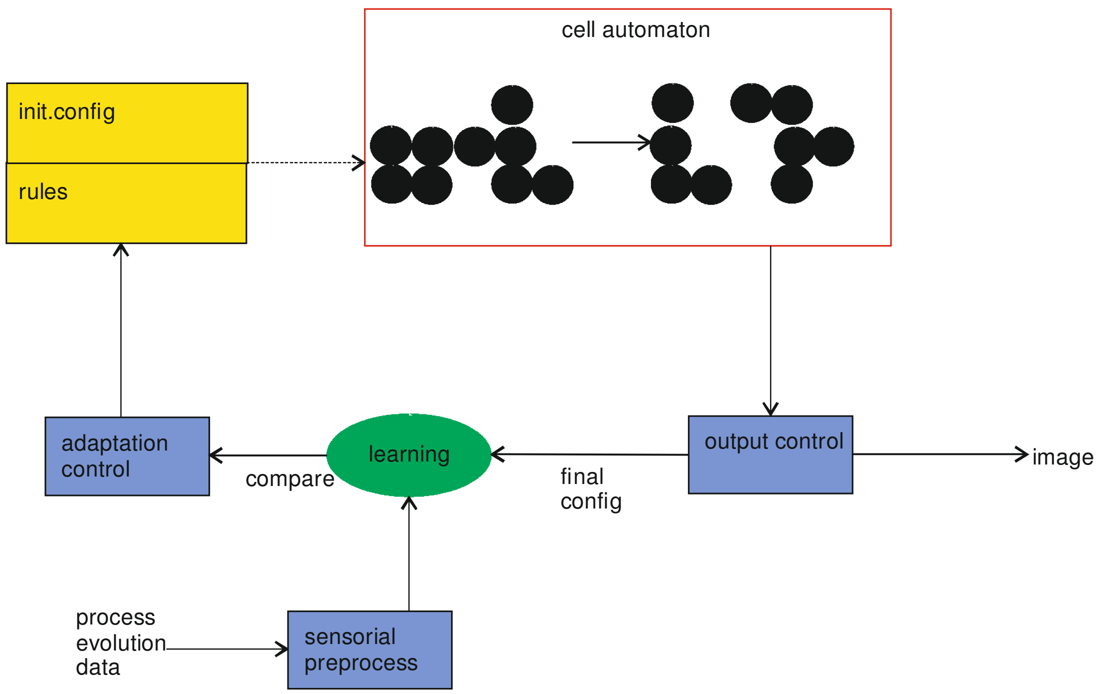


Game of life, Pentomino



Schematic of hermeneutic cognition

adaptive cell-automaton process simulator



3 Geometry in physics: Euclidean, hyperbolic and spherical geometry

1 Euclidean Geometry

In about 300 BCE, Euclid penned the Elements, the basic treatise on geometry for almost two thousand years. Euclid starts of the Elements by giving some 23 definitions. After giving the basic definitions he gives us five “postulates”. The postulates (or axioms) are the assumptions used to define what we now call Euclidean geometry.

The five axioms for Euclidean geometry are:
Any two points can be joined by a straight line. (This line is unique given that the points are distinct)
Any straight line segment can be extended indefinitely in a straight line.
Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
All right angles are congruent.
Through a point not on a given straight line, one and only one line can be drawn that never meets the given line.

The axioms are basic statements about lines, line segments, circles, angles and parallel lines. We need these statements to determine the nature of our geometry.

2 Hyperbolic Geometry

The five axioms for hyperbolic geometry are:
Any two points can be joined by a straight line.
Any straight line segment can be extended indefinitely in a straight line.
Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
All right angles are congruent.
Through a point not on a given straight line, infinitely many lines can be drawn that never meet the given line.

Lines: In hyperbolic geometry a geodesic line segment measures the shortest distance between two points. *Line segments:* Any finite piece of a geodesic.
Circles: Given any center C and any radius R we can draw a circle of radius R centered at C.
Angles: Right angles are congruent.
Infinitely many parallel lines: Given a line and a point not on the line, we can always draw infinitely many parallel lines through the point.
Sum of the angles in a triangle: On the sphere the sum of the angles in a triangle is always strictly less than 180 degrees.

3 Spherical Geometry

The five axioms for spherical geometry are:
Any two points can be joined by a straight line.
Any straight line segment can be extended indefinitely in a straight line.
Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
All right angles are congruent.
There are NO parallel lines.

Lines: The shortest distance between two points on a sphere always lies on a great circle, distance d is measured in radians, $d<=2\pi$.
Line segments: Line segments are arcs.
Circles: Given a center C and a radius $R<=\pi/2$ we can draw a circle of radius R centered at C
Angles: Right angles are congruent.
No parallel lines: Any two geodesics will intersect in exactly two points.
Sum of the angles in a triangle: On the sphere the sum of the angles in a triangle is always strictly greater than 180 degrees.

4 Hilbert's axioms Euclidean Geometry

[1]

Hilbert's axioms are a set of 20 assumptions proposed by David Hilbert in 1899 as the foundation for a modern treatment of Euclidean geometry. They are formulated in second-order logic, and represent Euclidean space geometry.

The axioms

Hilbert's axiom system is constructed with nine primitive notions: three primitive terms
point, straight line, plane,
and these six primitive relations:
Betweenness, a ternary relation linking points;
Containment, three binary relations, one linking points and straight lines, one linking points and planes, and one linking straight lines and planes;
Congruence, two binary relations, one linking line segments and one linking angles, each denoted by an infix ≅.
Note that line segments, angles, and triangles may each be defined in terms of points and straight lines, using the relations of betweenness and containment. All points, straight lines, and planes in the following axioms are distinct unless otherwise stated.
Axioms: Combination 7, Order 4, Parallel 1, Congruence 6, Continuity 2 -> 20 axioms

I. Combination

1. Two distinct points A and B always completely determine a straight line a. We write $AB = a$ or $BA = a$. Instead of "determine," we may also employ other forms of expression; for example, we may say "A lies upon a", "A is a point of a", "a goes through A and through B", "a joins A to B", etc. If A lies upon a and at the same time upon another straight line b, we make use also of the expression: "The straight lines a and b have the point A in common," etc.
2. Any two distinct points of a straight line completely determine that line; that is, if $AB = a$ and $AC = a$, where $B \neq C$, then also $BC = a$.
3. Three points A, B, C not situated in the same straight line always completely determine a plane α. We write $ABC = \alpha$. We employ also the expressions: "A, B, C, lie in α"; "A, B, C are points of α", etc.
4. Any three points A, B, C of a plane α, which do not lie in the same straight line, completely determine that plane.
5. If two points A, B of a straight line a lie in a plane α, then every point of a lies in α. In this case we say: "The straight line a lies in the plane α," etc.
6. If two planes α, β have a point A in common, then they have at least a second point B in common.
7. Upon every straight line there exist at least two points, in every plane at least three points not lying in the same straight line, and in space there exist at least four points not lying in a plane.

II. Order

1. If a point B is between points A and C, B is also between C and A, and there exists a line containing the points A,B,C.
2. If A and C are two points of a straight line, then there exists at least one point B lying between A and C and at least one point D so situated that C lies between A and D.
3. Of any three points situated on a straight line, there is always one and only one which lies between the other two.
4. Pasch's Axiom: Let A, B, C be three points not lying in the same straight line and let a be a straight line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the straight line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.

III. Parallels

1. In a plane α there can be drawn through any point A, lying outside of a straight line a, one and only one straight line which does not intersect the line a. This straight line is called the parallel to a through the given point A.

IV. Congruence

1. If A, B are two points on a straight line a, and if A' is a point upon the same or another straight line a', then, upon a given side of A' on the straight line a', we can always find one and only one point B' so that the segment AB (or BA) is congruent to the segment $A'B'$. We indicate this relation by writing $AB \cong A'B'$. Every segment is congruent to itself; that is, we always have $AB \cong AB$.

We can state the above axiom briefly by saying that every segment can be *laid off* upon a given side of a given point of a given straight line in one and only one way.

2. If a segment AB is congruent to the segment $A'B'$ and also to the segment $A''B''$, then the segment $A'B'$ is congruent to the segment $A''B''$; that is, if $AB \cong A'B'$ and $AB \cong A''B''$, then $A'B' \cong A''B''$
3. Let AB and BC be two segments of a straight line a which have no points in common aside from the point B, and, furthermore, let $A'B'$ and $B'C'$ be two segments of the same or of another straight line a' having, likewise, no point other than B' in common. Then, if $AB \cong A'B'$ and $BC \cong B'C'$, we have $AC \cong A'C'$.
4. Let an angle (h, k) be given in the plane α and let a straight line a′ be given in a plane α′. Suppose also that, in the plane α′, a definite side of the straight line a' be assigned. Denote by h' a half-ray of the straight line a' emanating from a point O' of this line. Then in the plane α′ there is one and only one half-ray k' such that the angle (h, k), or (k, h), is congruent to the angle (h', k') and at the same time all interior points of the angle (h', k') lie upon the given side of a'. We express this relation by means of the notation $\angle(h, k) \cong (h', k')$
   Every angle is congruent to itself; that is, $\angle(h, k) \cong (h, k)$
   or
   $\angle(h, k) \cong (k, h)$
5. If the angle (h, k) is congruent to the angle (h', k') and to the angle (h'', k''), then the angle (h', k') is congruent to the angle (h'', k''); that is to say, if $\angle(h, k) \cong (h', k')$ and $\angle(h, k) \cong (h'', k'')$, then $\angle(h', k') \cong (h'', k'')$.
6. If, in the two triangles ABC and A′B′C′ the congruences AB ≅ A′B′, AC ≅ A′C′, ∠BAC ≅ ∠B′A′C′ hold, then the congruences ∠ABC ≅ ∠A′B′C′ and ∠ACB ≅ ∠A′C′B′ also hold.

V. Continuity

1. Axiom of Archimedes. Let A_1 be any point upon a straight line between the arbitrarily chosen points A and B. Take the points $A_2, A_3, A_4, \ldots$ so that A_1 lies between A and A_2, A_2 between A_1 and A_3, A_3 between A_2 and A_4 etc. Moreover, let the segments $AA_1, A_1A_2, A_2A_3, A_3A_4, \ldots$ be equal to one another. Then, among this series of points, there always exists a certain point A_n such that B lies between A and A_n.
2. *Axiom of completeness*. To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.

Application

These axioms axiomatize Euclidean solid geometry. Removing four axioms mentioning "plane" in an essential way, namely I.3–6, omitting the last clause of I.7, and modifying III.1 to omit mention of planes, yields an axiomatization of Euclidean plane geometry.
Hilbert's axioms are second-order-logic, unlike Tarski's axioms, do not constitute a first-order theory because the axioms V.1–2 cannot be expressed in first-order logic.

The first four groups of Hilbert's axioms for plane geometry are bi-interpretable with Tarski's axioms minus continuity.

H-Euclidean geometry is consistent.
H-Euclidean geometry is categorical: all models are isomorphic .
H-Euclidean geometry is decidable: There exists an algorithm that decides for every wff whether is it is provable or disprovable from the axioms. This follows from the decidability of Tarski’s formulation, as Hilbert’s axioms are an extension of Tarski’s.

5 Tarski's axioms

[8]

The axiom system presented in 1926 is due to Alfred Tarski. It is formulated in first-order logic (whereas Hilbert's axioms are in second-order logic), and describes the Euclidean plane geometry (called in the following T-Euclidean geometry).

Primitive relations:

- *Betweenness*, Bxyz denotes that the point y is between the points x and z
- *Congruence*, Cwxyz denotes $wx \equiv yz$

Axioms

Congruence 3, Betweenness 2, Continuity 1, LUdimension 2, Parallel 3, Segment 2 → 13 axioms

Congruence axioms

$xy \equiv yx$ reflexivity

$xy \equiv zz \rightarrow x=y$ identity

$(xy \equiv zu \wedge xy \equiv vw) \rightarrow zu \equiv vw$ transitivity

Betweenness axioms

$Bxyx \rightarrow x=y$ identity=only point on *xx* is *x*

$(Bxuz \wedge Byvz) \rightarrow \exists a(Buay \wedge Bvax)$ axiom of Pasch=lines intersect in a point

Continuity axiom

$$\exists a \forall x \forall y \left(\left(\varphi(x) \wedge \psi(y) \right) \rightarrow Baxy \right) \rightarrow \exists b \forall x \forall y \left(\left(\varphi(x) \wedge \psi(y) \right) \rightarrow Bxby \right)$$

φ and ψ divide the ray into two halves , then there is point *b* dividing those two halves

Lower dimension axiom

$\exists a \exists b \exists c[\neg Babc \wedge \neg Bbca \wedge \neg Bcab]$

There are three non-collinear points.

Upper dimension axiom

$$(xu \equiv xv) \wedge (yu \equiv yv) \wedge (zu \equiv zv) \wedge (y \neq v) \rightarrow (Bxyz \wedge Byzx \wedge Bzxy)$$

Three points equidistant from two distinct points form a line

Parallel postulate

(A) $(Bxyw \wedge xy \equiv yw) \wedge (Bxuv \wedge xu \equiv uv) \wedge (Byuz \wedge yu \equiv uz) \rightarrow yz \equiv vw$

Triangle angles summing to two right angles.

(B) $(Bxyz \vee Byzx \vee Bzxy) \wedge \exists a (xa \equiv ya \wedge xa \equiv za)$

Triangle vertices lie on a circle

(C) $(Bxuv \wedge Byuz \wedge x \neq u) \rightarrow \exists a \exists b (Bxya \wedge Bxzb \wedge Bavb)$

For a point *v* in an angle, there is a line segment including *v*, with an endpoint on each side of the angle.

Five segments congruence

$$(x \neq y \wedge Bxyz \wedge Bx'y'z' \wedge xy \equiv x'y' \wedge yz \equiv y'z' \wedge yu \equiv y'u') \rightarrow zu \equiv z'u'$$

If angles uxz and u'x'z' are congruent and the two pairs of incident sides are congruent ($xu \equiv x'u'$ and $xz \equiv x'z'$), then the remaining pair of sides is also congruent ($uz \equiv u'z'$).

Segment Construction

$$\exists z (Bxyz \wedge yz \equiv ab)$$

T-Euclidean geometry is consistent.

T-Euclidean geometry is complete.

T-Euclidean geometry is decidable: There exists an algorithm that decides for every wff whether is it is provable or disprovable from the axioms.

T-Euclidean geometry is non-categorical (there are non-isomorphic models).

6 Axiomatics

Tarski's axioms (T-Euclidean, 2d)
Primitive relations:
- Betweenness, Bxyz
- *Congruence*, Cwxyz

Axioms (first-order logic) 10+3
Congruence 3, Betweenness 2, Continuity 1, LUdimension 2, (Parallel 3), Segment 2

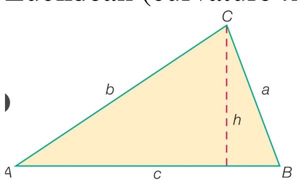
T-Euclidean geometry is decidable:

Euclid's axioms
1 "To draw a straight line from any point to any point."
2 "To produce [extend] a finite straight line continuously in a straight line."
3 "To describe a circle with any centre and distance [radius]."

Hilbert's axioms (H-Euclidean, 3d)
Primitive relations:
- *Betweenness*, $Bx_1x_2x_3$
- Containment Cxl, Cxp,Clp
- *Congruence*, Kx_1x_2 $K\alpha_1\alpha_2$

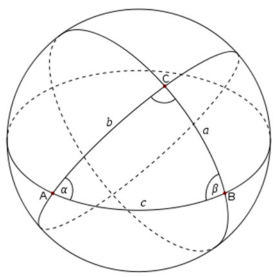
Elements: point x, line l, plane p, angle α
Axioms (second-order logic) 19+1
Combination 7, Order 4, (Parallel 1), Congruence 6, Continuity 2

Euclidean (curvature κ=0)



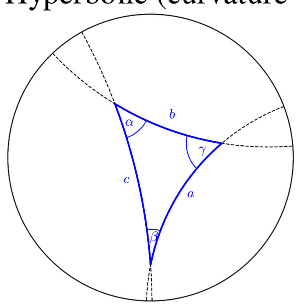
one parallel

Spherical (curvature κ>0)



no parallel

Hyperbolic (curvature κ<0)



more than one parallel

7 Properties of Euclidean, hyperbolic, and spherical geometries.
[2]

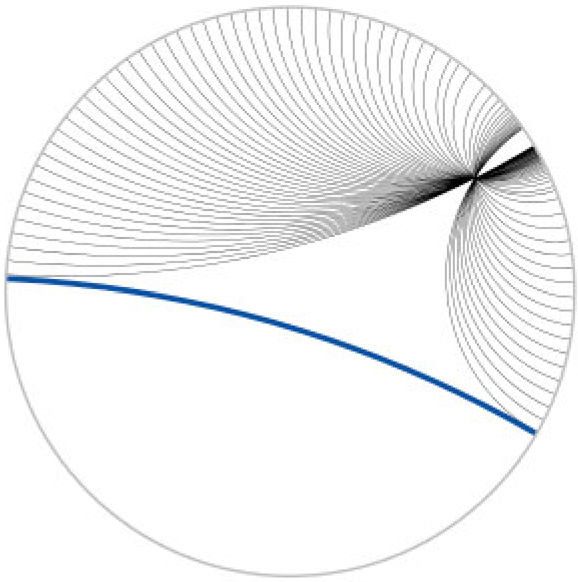
Euclidean	Hyperbolic	Spherical
Two distinct lines intersect in one point.	Two distinct lines intersect in one point.	Two distinct lines intersect in two points.
The sum of the measures of the angles of a triangle is 180.	The sum of the measures of the angles of a triangle is less than 180.	The sum of the measures of the angles of a triangle is more than 180.
Similar triangles that are not congruent exist.	Similar triangles are congruent.	Similar triangles are congruent.
Triangles are finite	Infinite triangles exist	Triangles are finite
Rectangles exist.	No quadrilateral is a rectangle.	No quadrilateral is a rectangle.
A line does not have finite length and is unbounded.	A line does not have finite length.	A line has finite length and is unbounded.
Two parallel lines are equidistant.	No two parallel lines are equidistant.	Parallel lines do not exist.
Two distinct lines do not enclose a finite area.	Two distinct lines do not enclose a finite area.	Two distinct lines enclose a finite area.
The area of a triangle is equal to half the product of the base and height.	The area of a triangle is proportional to its defect.	The area of a triangle is proportional to its excess.
A unique line perpendicular to a given line through a point not on the line.	A unique line perpendicular to a given line through a point not on the line.	All lines perpendicular to a given line intersect at two antipodal points.
A line separates a plane.	A line separates a plane.	A line separates a plane.

8 Poincare disk model

[7]

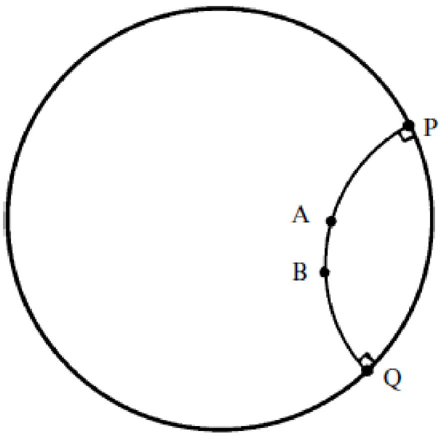
The Poincaré hyperbolic disk is a two-dimensional space having hyperbolic geometry defined as the disk $\{x \in R^2 \mid |x| < 1\}$ with hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{\left(1 - x^2 - y^2\right)}$$



Poincaré disk with hyperbolic parallel lines

Hyperbolic **straight lines** consist of all arcs of Euclidean circles contained within the disk that are orthogonal to the boundary of the disk, plus all diameters of the disk.

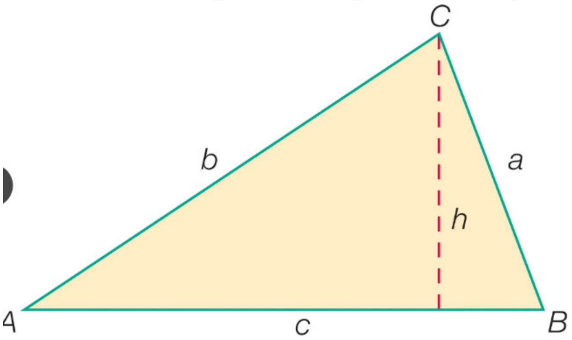


The distance between A and B is

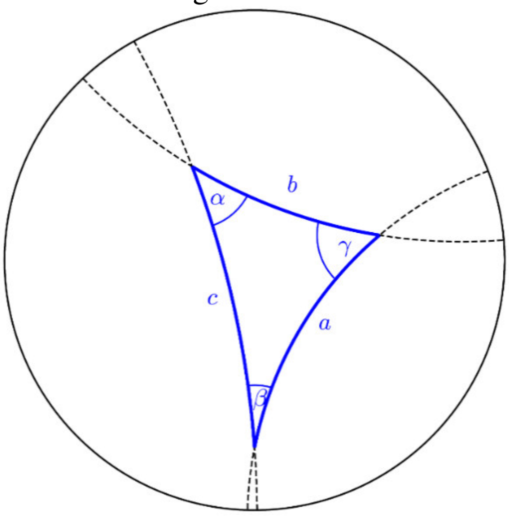
$$d(A,B) = \log\left(\frac{|QA|}{|QB|}\frac{|PB|}{|PA|}\right)$$, where $|QA| > |PA|$, $|PB| > |QB|$

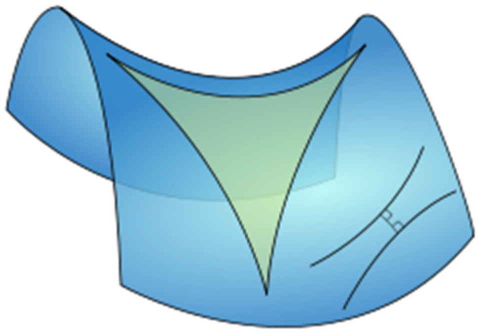
$$d(A,B) = arcosh\left(1 + \frac{2|AB|^2 r^2}{\left(r^2 - |OA|^2\right)\left(r^2 - |OB|^2\right)}\right)$$, r=radius of Poincare-disc

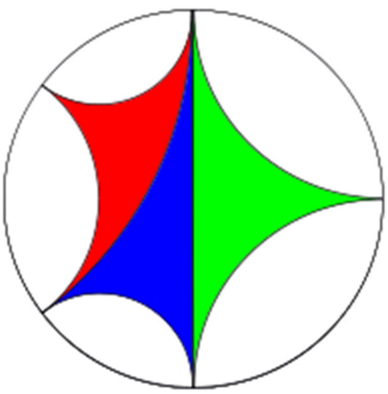
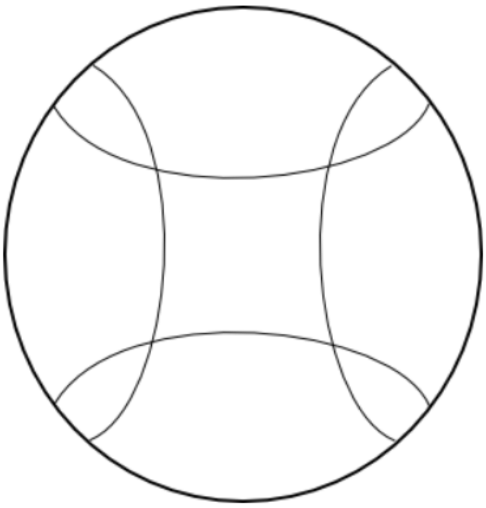
9 Euclidean, hyperbolic, spherical trigonometry



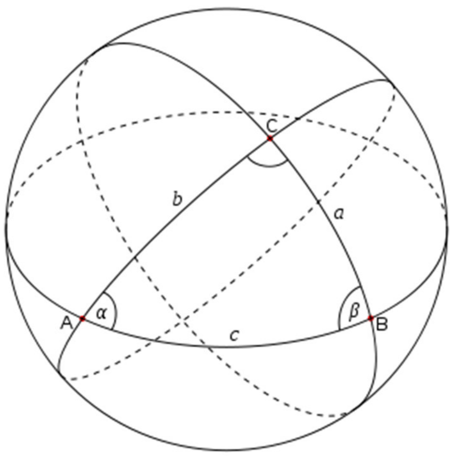
Euclidean triangle




Hyperbolic triangle



Hyperbolic infinite triangle

Hyperbolic square $A=\alpha$, $B=\beta$, $C=\gamma$



Spherical triangle $A=\alpha$, $B=\beta$, $C=\gamma$

Planar Triangle
[3]
Consider a triangle ABC with (interior) angles A;B;C and sides a; b; c (with a opposite to A, etc.).
s is the semiperimeter, σ semiangle, Δ area

$$s := \frac{a+b+c}{2},$$
$$\sigma := \frac{A+B+C}{2}.$$

The law of sines

E : $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

S : $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c},$$

H : $$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}$$

The law of cosines.

$$a^2 = b^2 + c^2 - 2bc\cos A,$$
$$\cos a = \cos b\, \cos c + \sin b\, \sin c\, \cos A,$$
$$\cosh a = \cosh b\, \cosh c - \sinh b\, \sinh c\, \cos A.$$

The second law of cosines.

E : $$\cos A = -\cos B\, \cos C + \sin B\, \sin C = -\cos(B+C),$$

S : $$\cos A = -\cos B\, \cos C + \sin B\, \sin C\, \cos a,$$

H : $$\cos A = -\cos B\, \cos C + \sin B\, \sin C\, \cosh a,$$

Angle sum and area

E : $$2\sigma := A + B + C = \pi,$$

S : $$2\sigma := A + B + C = \pi + \Delta,$$

H : $$2\sigma := A + B + C = \pi - \Delta.$$

Area

E : $$\Delta = \frac{bc\sin A}{2},$$

S : $$\sin\frac{\Delta}{2} = \frac{\sin b\, \sin c\, \sin A}{4\cos\frac{a}{2}\, \cos\frac{b}{2}\, \cos\frac{c}{2}},$$

H : $$\sin\frac{\Delta}{2} = \frac{\sinh b\, \sinh c\, \sin A}{4\cosh\frac{a}{2}\, \cosh\frac{b}{2}\, \cosh\frac{c}{2}}$$

Heron’s formula

E : $$\Delta^2 = s(s-a)(s-b)(s-c),$$

S : $$2\sin^2\frac{\Delta}{2} = 1 - \cos\Delta = \frac{4\sin s\, \sin(s-a)\, \sin(s-b)\, \sin(s-c)}{(1+\cos a)(1+\cos b)(1+\cos c)},$$

H : $$2\sin^2\frac{\Delta}{2} = 1 - \cos\Delta = \frac{4\sinh s\, \sinh(s-a)\, \sinh(s-b)\, \sinh(s-c)}{(1+\cosh a)(1+\cosh b)(1+\cosh c)}$$

triangle inequality

E, S, H : $$a < b + c, \qquad b < a + c, \qquad c < a + b.$$

S : $$B + C < \pi + A, \qquad A + C < \pi + B, \qquad A + B < \pi + C.$$

E, S, H :
$$\left.\begin{array}{l} a < b \\ a = b \\ a > b \end{array}\right\} \iff \left\{\begin{array}{l} A < B, \\ A = B, \\ A > B. \end{array}\right.$$

area inequality

S : $$\Delta < 2\pi,$$

H : $$\Delta < \pi.$$

E: Δ unbounded

Right triangle C =π/2

Pythagorean theorem

E : $$c^2 = a^2 + b^2,$$

S : $$\cos c = \cos a \, \cos b,$$

H : $$\cosh c = \cosh a \, \cosh b.$$

Law of sines

E : $$\sin A = \frac{a}{c},$$

S : $$\sin A = \frac{\sin a}{\sin c},$$

H : $$\sin A = \frac{\sinh a}{\sinh c}.$$

Law of cosines

E : $$\cos A = \frac{b}{c},$$

S : $$\cos A = \cos a \, \sin B = \frac{\cos a \, \sin b}{\sin c} = \frac{\tan b}{\tan c},$$

H : $$\cos A = \cosh a \, \sin B = \frac{\cosh a \, \sinh b}{\sinh c} = \frac{\tanh b}{\tanh c}$$

Area

E : $$\Delta = \frac{ab}{2},$$

S : $$\sin \Delta = \frac{\sin a \, \sin b}{1 + \cos c},$$

H : $$\sin \Delta = \frac{\sinh a \, \sinh b}{1 + \cosh c}.$$

Curves of constant curvature κ: line or circle with radius *r*

E : $\kappa = 1/r,$ circle

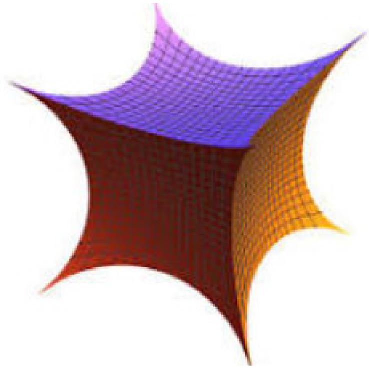
S : $\kappa = \cot r.$ circle

H : $$\kappa = \begin{cases} \coth r, & \text{(circle)} \\ 1, & \text{(horocycle)} \\ \tanh d. & \text{(hypercycle)} \end{cases}$$

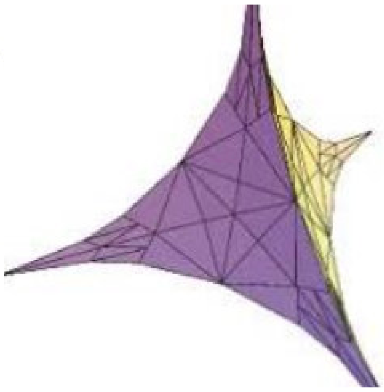
a *horocycle* is orthogonal to a pencil of parallel lines;

a *hypercycle* is the set of all points with a given distance d > 0 to some line *l* ,and lying on the same side of *l*

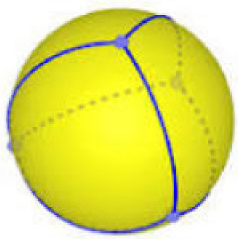
10 3D-4D graphics



hyperbolic cube

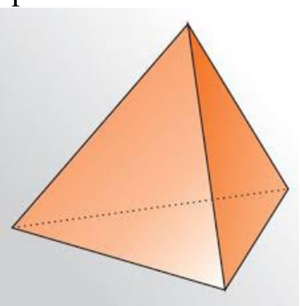


hyperbolic tetrahedron

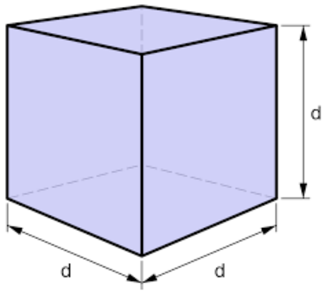


spherical tetrahedron

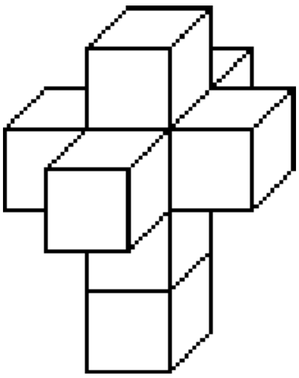
spherical cube



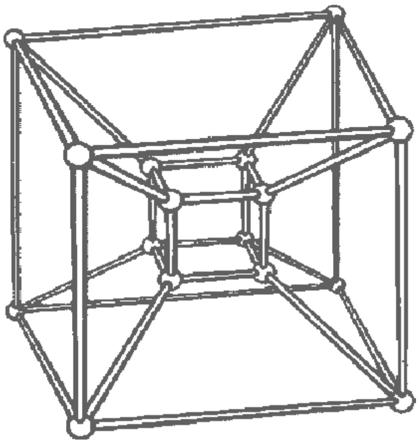
Euclidean tetrahedron



Euclidean cube



Euclidean hypercube 3D-unwrapping



Euclidean hypercube projection

4 Logic in physics: first-order-logic

1 First-order logic

[9] [10]

[10] First-order logic—also called predicate logic, is a collection of formal systems used in mathematics. First-order logic uses quantified variables over non-logical objects, and allows the use of sentences that contain variables.

The connectives of the first-order logic are

$\wedge \vee \neg \rightarrow \forall \exists$

where the two quantors $\forall$ "for all" and $\exists$ "there is" are included. They contain *quantified variables* in the form $\forall x$ and $\exists x$.

An interpretation *I(P)* of a formula *P* is defined on a structure (model).

The structure consists of a domain *D* and an interpretation function I mapping non-logical symbols to predicates, functions, and constants.

Interpretation

For a set of sentences *K* , and sentence *P* the rules for *I(P)* are:

If P is a constant predicate in K, then I(P) is a truth value, T or F

If Q is a unary predicate in K, then I(Q) is the set of members of the domain that the predicate Q holds of.

If R is a two-argument predicate in K, then I(R) is the set of pairs of members of the domain that the relation R holds between.

$M,s \vDash \phi$ for M *satisfies* ϕ *under the assignment* s.

If P is a constant predicate in K, then $M,s \vDash P$ if and only if I(P)=T .

If *S* is an n-argument predicate K and $t_1,\ldots,t_n$ are terms, then $M,s \vDash St_1\ldots t_n$ if and only if the n-tuple $\langle D_{M,s}(t_1),\ldots,D_{M,s}(t_n)\rangle$ is in I(S)

Satisfaction rules:

$M,s \vDash \neg\theta \Leftrightarrow$ not $M,s \vDash \theta$

$M,s \vDash (\theta \wedge \psi) \Leftrightarrow$ $M,s \vDash \theta$ and $M,s \vDash \psi$

$M,s \vDash (\theta \vee \psi) \Leftrightarrow$ $M,s \vDash \theta$ or $M,s \vDash \psi$

$M,s \vDash (\theta \rightarrow \psi) \Leftrightarrow$ not $M,s \vDash \theta$ or $M,s \vDash \psi$

$M,s \vDash \forall v\theta \Leftrightarrow$ for every s' $M,s' \vDash \theta$, where $s'(x) = s(x)\ \forall x \neq v$

$M,s \vDash \exists v\theta \Leftrightarrow$ for some s' $M,s' \vDash \theta$, where $s'(x) = s(x)\ \forall x \neq v$

2 Zermelo-Frankel set theory

[11]

Zermelo-Frankel set theory (ZFC) is an axiomatic formulation of set theory, which eliminates contradictions arising in the naïve set theory, e.g. Russel's antinomy. Basically, it eliminates predicate-all-sets like "set of all sets".

ZFC is a countable-infinite axiomatization, involving predicate variables (first-order logic).

The axioms (including axiom of choice VII) are as follows [1]:

(I) (Axioms of extensionality) Two sets A and B are equal if (and only if) they contain the same members; i.e. A = B~(A ⊆ B and B⊆ A).

(II) (Axiom of subsets) Given a set A and a predicate P(x) meaningful for the members of A (i.e., for each x either P(x) is true or P(x) is false), there exists the set containing exactly those members of A for which P(x) is true.

(III) (Axiom of pairing) If a and b are different objects, there exists the set {a, b} containing exactly a and b .

(IV) (Axiom of union) Given a set S of sets, there exists the set ∪S containing just the members of the members of S .

(V) (Axiom of infinity) There exists at least one infinite set: the set {0, 1,2,...} of the natural numbers.

(VI) (Axiom of the power set) Given a set A, there exists the set 2^A whose members are all the subsets of A.

(VII) (Axiom of choice) Given a disjointed set S of nonempty sets, there exists a set C which has as its members one and only one element from each member of S.

3 Calculus of deduction

[12] [14] [13]

Gentzen's tree notation

[14]

Example: "if it's raining then it's cloudly; it is raining; therefore it's cloudy"

Representing an example as a list of propositions, we would have:

1) P→Q

2) P

∴ Q

In Gentzen's notation, this would be written like this:

$$\frac{P \rightarrow Q, P}{Q}$$

The inference line represents syntactic consequence, which is also symbolized with ⊢.

So the above can also be written in one line as $P \rightarrow Q, P \vdash Q$

Syntactic consequence is contrasted with semantic consequence, which is symbolized with ⊨.

Deduction rules of semantic consequence [14]

Name	Sequent	Description
Modus Ponens		If p then q; p; therefore q
Modus Tollens		If p then q; not q; therefore not p
Hypothetical Syllogism		If p then q; if q then r; therefore, if p then r
Disjunctive Syllogism		Either p or q, or both; not p; therefore, q
Constructive Dilemma		If p then q; and if r then s; but p or r; therefore q or s
Destructive Dilemma		If p then q; and if r then s; but not q or not s; therefore not p or not r
Bidirectional Dilemma		If p then q; and if r then s; but p or not s; therefore q or not r
Simplification		p and q are true; therefore p is true
Conjunction		p and q are true separately; therefore they are true conjointly
Addition		p is true; therefore the disjunction (p or q) is true
Composition		If p then q; and if p then r; therefore if p is true then q and r are true
De Morgan's Theorem (1)	⫤⊨	The negation of (p and q) is equiv. to (not p or not q)
De Morgan's Theorem (2)	⫤⊨	The negation of (p or q) is equiv. to (not p and not q)
Commutation (1)	⫤⊨	(p or q) is equiv. to (q or p)
Commutation (2)	⫤⊨	(p and q) is equiv. to (q and p)
Commutation (3)	⫤⊨	(p iff q) is equiv. to (q iff p)
Association (1)	⫤⊨	p or (q or r) is equiv. to (p or q) or r
Association (2)	⫤⊨	p and (q and r) is equiv. to (p and q) and r
Distribution (1)	⫤⊨	p and (q or r) is equiv. to (p and q) or (p and r)
Distribution (2)	⫤⊨	p or (q and r) is equiv. to (p or q) and (p or r)
Double Negation	⫤⊨	p is equivalent to the negation of not p

Transposition	$\dashv\vDash$	If p then q is equiv. to if not q then not p
Material Implication	$\dashv\vDash$	If p then q is equiv. to not p or q
Material Equivalence (1)	$\dashv\vDash$	(p iff q) is equiv. to (if p is true then q is true) and (if q is true then p is true)
Material Equivalence (2)	$\dashv\vDash$	(p iff q) is equiv. to either (p and q are true) or (both p and q are false)
Material Equivalence (3)	$\dashv\vDash$	(p iff q) is equiv to., both (p or not q is true) and (not p or q is true)
Exportation		from (if p and q are true then r is true) we can prove (if q is true then r is true, if p is true)
Importation	$\dashv\vDash$	If p then (if q then r) is equivalent to if p and q then r
Idempotence of disjunction	$\dashv\vDash$	p is true is equiv. to p is true or p is true
Idempotence of conjunction	$\dashv\vDash$	p is true is equiv. to p is true and p is true
Tertium non datur (Law of Excluded Middle)		p or not p is true
Law of Non-Contradiction		p and not p is false, is a true statement
Explosion		p and not p; therefore q

Deduction rules of first-order syntactic consequence [13]

Γ is a set of sentences, $\phi\,\psi\,\theta$ are sentences , $\Gamma|-\phi$ syntactic consequence ϕ from Γ

(As) $\phi\in\Gamma \quad\Rightarrow\quad \Gamma|-\phi$ (assumption)

($\wedge$I) $\Gamma_1|-\theta$ and $\Gamma_2|-\psi \quad\Rightarrow\quad \Gamma_1,\Gamma_2|-(\theta\wedge\psi)$ (AND introduction)

($\wedge$E) $\Gamma|-(\theta\wedge\psi) \quad\Rightarrow\quad \Gamma|-\theta$, $\Gamma|-(\theta\wedge\psi) \quad\Rightarrow\quad \Gamma|-\psi$ (AND elimination)

($\vee$I) $\Gamma|-\theta \quad\Rightarrow\quad \Gamma|-(\theta\vee\psi)$, $\Gamma|-\psi \quad\Rightarrow\quad \Gamma|-(\theta\vee\psi)$ (OR introduction)

($\vee$E) $\Gamma_1|-(\theta\vee\psi)$, Γ_2 , $\theta|-\phi$ and Γ_3 , $\psi|-\phi \quad\Rightarrow\quad \Gamma_1,\Gamma_2,\Gamma_3|-\phi$ (OR elimination)

($\rightarrow$I) Γ , $\theta|-\psi \quad\Rightarrow\quad \Gamma|-(\theta\rightarrow\psi)$ (implication introduction)

($\rightarrow$E) $\Gamma_1|-\theta\rightarrow\psi$ and $\Gamma_2|-\theta \quad\Rightarrow\quad \Gamma_1,\Gamma_2|-\psi$ (implication elimination)

($\neg$I) $\Gamma_1,\theta|-\psi$ and $\Gamma_2,\theta|-\neg\psi \quad\Rightarrow\quad \Gamma_1,\Gamma_2|-\neg\theta$ (negation introduction)

(DNE) $\Gamma|-\neg\neg\theta \quad\Rightarrow\quad \Gamma|-\theta$ (double negation elimination)

5 Functional analysis in physics

1 Real analysis
Real analysis [17] studies real numbers R, focusing on continuity, limits, and integration on a line, while complex analysis explores functions over complex numbers c= a + b i, which form a 2D plane.
Real analysis is the branch of mathematical analysis that is concerned with properties of real numbers and real-valued functions. Classical real analysis provides the rigorous foundations of calculus, specifically convergent series, limits, continuity, differentiability, smoothness, and integrability.
Central to the subject is the completeness of the real numbers, which distinguishes the real numbers from the rational numbers and **underlies many basic results about limits and continuous functions.**
Basics
Limit Definition. Let f be a real-valued function defined on $E\subset R$. We say that f(x) tends to L as x approaches x0, or that the limit of f(x) as x approaches x0 is L if, for any $\varepsilon>0$, there exists $\delta>0$ such that for all $x\in E$, $0<|x-x0|<\delta$ implies that $|f(x)-L|<\varepsilon$. We write this symbolically as $f(x)\to L \quad x\to x_0$
Definition. Let (an) be a real-valued sequence. We say that (an) is a Cauchy sequence if, for any $\varepsilon>0$, there exists a natural number N such that $m,n\geq N$ implies that $|a_m-a_n|<\varepsilon$.
It can be shown that a real-valued sequence is Cauchy if and only if it is convergent.
Uniform and pointwise convergence
Roughly speaking, pointwise convergence of functions f_n to a limiting function $f:E\to R$, denoted $f_n\to f$, simply means that given any $x\in E$, $f_n(x)\to f(x)$ as $n\to\infty$.
In contrast, uniform convergence is a stronger type of convergence, in the sense that a uniformly convergent sequence of functions also converges pointwise, but not conversely.
Uniform convergence requires members of the family of functions, f_n, to fall within some error $\varepsilon>0$ of f for every value of $x\in E$, whenever $n\geq N$, for some integer N.
Compactness
Definition. A set $E\subset R$ is compact if it is closed and bounded.
Definition. A set E in a metric space is compact if every sequence in E has a convergent subsequence.
Continuity
Definition. If $I\subset R$ is a non-degenerate interval, we say that $f:I\to R$ is continuous at $p\in I$ if
$$\lim_{x\to p} f(x)=f(p)$$
We say that f is a continuous map if f is continuous at every $p\in I$.
Definition. If X is a subset of the real numbers, we say a function $f:X\to R$ is uniformly continuous on X if, for any $\varepsilon>0$, there exists a $\delta>0$ such that for all $x,y\in X$, $|x-y|<\delta$ implies that $|f(x)-f(y)|<\varepsilon$.
Definition. Let $I\subset R$ be an interval on the real line. A function $f:I\to R$ is said to be absolutely continuous on I if for every positive number ε, there is a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$ of I satisfies
$$\sum_{k=1}^{n}(y_k-x_k)<\delta$$
then
$$\sum_{k=1}^{n}\left|f(y_k)-f(x_k)\right|<\varepsilon$$
Differentiability
A function f: $R\to R$ is differentiable at a if the limit
$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$ exists
Series convergence
$$\sum_{k=1}^{n}\frac{1}{2^k}=1$$
$$\sum_{k=1}^{n}\frac{1}{k}=\infty$$

$$\sum_{k=1}^{n}\frac{(-1)^k}{k}=\log 2$$

Taylor series

The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series

$$f(x)=\sum_{i=0}^{k}\frac{f^i(a)}{i!}(x-a)^i$$

Fourier series

The Fourier series of a complex-valued P-periodic function *f(x)* , integrable over the interval [0,P] on the real line, is defined as a trigonometric series of the form

$$f(x)=\sum_{k=-\infty}^{\infty}c_k\exp(i\ 2\pi k\,x/P)$$

$$c_k=\frac{1}{P}\int_0^P f(x)\exp(-i\ 2\pi k\,x/P)\,dx$$

real form

$$f_N(x)=a_0+\sum_{k=1}^{N}\left(a_k\cos(\ 2\pi k\,x/P)+b_k\sin(\ 2\pi k\,x/P)\right)$$

$$a_0=\frac{1}{P}\int_0^P f(x)\,dx\quad , \quad a_k=\frac{2}{P}\int_0^P f(x)\cos(\ 2\pi k\,x/P)\,dx\quad , \quad b_k=\frac{2}{P}\int_0^P f(x)\sin(\ 2\pi k\,x/P)\,dx$$

Integration

We say that the Riemann integral of f on [a,b] is S if for any ε>0 there exists δ>0 such that, for partition $\Delta_i=(x_{i+1}-x_i)$ $x_i\in[a,b]$ with mesh ‖Δi‖<δ , we have

$$\left|S-\sum_{k=1}^{n}f(x_i)\Delta_i\right|<\varepsilon$$

i.e. $\int_a^b f(x)\,dx=\lim_{n\to\infty}\sum_{i=1}^{n-1}f(x_i)(x_{i+1}-x_i)$ for $x_i\in[a,b]$

Important results

Bolzano-Weierstraß

The theorem states that each infinite bounded sequence in R_n has a convergent subsequence, i.e. R_n is locally compact.

Heine–Borel theorem

For a subset S of Euclidean space R_n , the following two statements are equivalent:

- S is compact, that is, every open cover of S has a finite subcover
- S is closed and bounded.

L'Hôpital's rule

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

where f′ and g′ are the derivatives of f and g .

Mean value theorem

Let f:[a,b]→R be a continuous function on the closed interval [a,b] , and differentiable on the open interval (a,b) , where a<b .

Then there exists $c\in[a,b]$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

Taylor's theorem

Let k≥1 be an integer and let the function f:R→R be k times differentiable at the point a∈R .

Then there exists a function h_k:R→R such that

$$f(x)=\sum_{i=0}^{k}\frac{f^i(a)}{i!}(x-a)^i+h_k(x)$$

and remainder $h_k(x)=\frac{f^{k+1}(\xi)}{(k+1)!}(x-a)^{k+1}$, $\xi\in[a,x]$

$\lim_{x\to a} h_k(x)=0$

Weierstrass approximation theorem

Let $f:[a,b]\to R$ be a continuous function.

If ε > 0 is given, then there exists a polynomial function p_n such that $|f(x)-p_n(x)|<\varepsilon$ for all $x\in[a,b]$

$$p_n(x)=\sum_{i=0}^{n} f\left(a+\frac{i}{n}(b-a)\right)B_{i,n}^{a,b}(x)$$

Here, the approximating polynomials are the Bernstein polynomials

$$B_{i,n}^{a,b}(x)=\frac{1}{(b-a)^n}\binom{n}{i}(x-a)^i(x-b)^{n-i}$$

with the error

$$\|f-p_n\|\le\frac{\pi}{2}\frac{1}{(n+1)^k}\|f^{(k)}\|$$

Fundamental theorem of calculus

Let *f* be a continuous real-valued function defined on a closed interval [a, b].

Let $F=I_a f$ be the function defined, for all x in [a, b], by

$$(I_a f)(x)=\int_a^x f(t)\,dt$$

Then F is uniformly continuous on [a, b] and differentiable on the open interval (a, b), and

$$(I_a f)'(x)=f(x)$$

for all *x* in *(a, b)* so *F* is an antiderivative of *f*.

Newton–Leibniz theorem

Let *f* be a real-valued function on a closed interval *[a,b]* and *F* a continuous function on *[a,b]* which is an antiderivative (inverse-derivative) of *f* in *(a,b)* : $F'(x)=f(x)$

If *f* is Riemann integrable on *[a,b]* then $F(b)-F(a)=\int_a^b f(t)\,dt$

Dini theorem

If *X* is a compact topological space, and $(f_n)_{n\in N}$ is a monotonically increasing sequence (meaning $f_n(x)\le f_{n+1}(x)$ for all $n\in N$ and $x\in X$) of continuous real-valued functions on *X* which converges pointwise to a continuous function f:X→R , then the convergence is uniform.

Banach fixed-point theorem

Let (X,d) be a non-empty complete metric space with a contraction mapping *f: X→X*

$d(f(x),f(y))\le q\,d(x,y)$, q<1 , $x,y\in X$

Then *f* admits a unique fixed point $x_*\in X$, meaning $f(x_*)=x_*$

Furthermore, x_* can be found as follows:

start with $x_0\in X$ and iterate

$x_{n+1}=f(x_n)$, then $x_n\to x_*$

Hahn-Banach theorem: functional with given values

Let $(x_i)_{i\in I}$ be vectors in a real or complex normed space X and let $(c_i)_{i\in I}$ be scalars also indexed by I≠∅.

There exists a continuous linear functional f on *X* such that $f(x_i)=c_i$ for all $i\in I$

if and only if there exists a K>0 such that for any choice of scalars $(s_i)_{i\in I}$ where all but finitely many s_i are 0, the following holds:

$\left|\sum_{i\in I} s_i c_i\right|<K\left|\sum_{i\in I} s_i x_i\right|$ (linear form is low-bounded)

Stokes theorem
Let Σ be a smooth oriented surface in R_3, parametrized by $\Sigma(u,v)$,
with boundary $\partial\Sigma\equiv\Gamma$, parametrized by $\Gamma(t)$.
If a vector field $F(x,y,z)$ has continuous first-order partial derivatives in Σ,
then
$$\iint_{\Sigma}(\nabla\times F)\cdot d\Sigma=\oint_{\partial\Sigma}F\cdot d\Gamma$$
Newton-Raphson approximation method
The Newton-Raphson method is a solution procedure for a zero α of a differentiable function f
$$x_{n+1}=x_n-f(x_n)/f'(x_n)$$
the error at n steps $\varepsilon_n=|\alpha-x_n|$, $\xi_n\in[x_n,\alpha]$
is $|\varepsilon_{n+1}|=\varepsilon_n{}^2\dfrac{|f''(\xi_n)|}{2|f'(\xi_n)|}$, so the convergence is quadratic.

2 Complex analysis

[18]
Complex analysis, traditionally known as the **theory of functions of a complex variable**, is the branch of mathematical analysis that investigates functions of complex numbers.
Complex analysis features superior rigidity in comparison with real analysis—differentiability implies infinite differentiability (analyticity)—unlike real analysis where derivatives may not be continuous.

Complex functions that are differentiable at every point of an open subset Ω of the complex plane are said to be holomorphic on Ω. In the context of complex analysis, the derivative of f at z_0 is defined to be
$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$
Important results
Great Picard's Theorem
If an analytic function f has an essential singularity at a point w, then on any punctured neighborhood of w, $f(z)$ takes on all possible complex values, with at most a single exception, infinitely often.
Laurent series
The Laurent series for a complex function $f(z)$ about a point c is given by
$$f(z)=\sum_{k=-\infty}^{\infty}a_k(z-c)^k$$
where a_n and c are constants, with a_n defined by a line integral that generalizes Cauchy's integral formula:
$$a_k=\frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)}{(z-c)^{k+1}}dz$$
The path of integration γ is counterclockwise around a curve enclosing c and lying in an annulus A in which f (z) is holomorphic (=analytic).
Liouville's theorem
A bounded function that is holomorphic in the entire complex plane must be constant.
Riemann mapping theorem
If U is a non-empty simply connected open subset of the complex number plane **C** which is not all of **C**, then there exists a biholomorphic mapping f from U onto the open unit disk
$$D=\{z\in C\,|\,|z|<1\}$$
The map f is essentially unique: if z_0 is an element of U and φ is an arbitrary angle, then there exists precisely one f as above such that $f(z_0)=0$ and such that the argument of the derivative of f at the point z_0 is equal to φ.
Runge's approximation theorem : approximation by rational functions
Denoting by C the set of complex numbers, let K be a closed subset of $C\cup\{\infty\}$ and let f be a function which is holomorphic on an open set containing K. If A is a set containing at least one complex number from every connected component of $C\cup\{\infty\}\backslash K$, then there exists a sequence (r_n) of rational functions which converges uniformly to f on K and such that all the poles of the functions (r_n) are in A.

3 Quaternionic analysis

[27]

Basics

Quaternionic analysis is an extension of complex analysis to the setting of quaternions, which retains the central concepts of analyticity, differentiation, and integration, but is profoundly more intricate due to the noncommutative nature of quaternions .
Quaternionic analysis extends complex analytic concepts into four-dimensional quaternionic space, replacing the Cauchy-Riemann equations with the Fueter equation. It reconciles analyticity, harmonicity, and integral structures within a noncommutative setting,
forming the foundation for further explorations in both pure and applied mathematics.

Quaternions

A quaternion q∈H can be expressed as:
$q = q_0 + q_1 i + q_2 j + q_3 k$, or in vector representation $q = (q_0, q_1, q_2, q_3)$
where $x_0, x_1, x_2, x_3 \in R$ and the imaginary units satisfy:
$i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$
Quaternions form a non-commutative division algebra over R, and a non-commutative field,
with multiplication

$$pq = (p_0, p_1, p_2, p_3)(q_0, q_1, q_2, q_3)$$
$$= \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

It is also possible to present the quaternion multiplication in vector form
$$pq = (p_0, \vec{p})(q_0, \vec{q}) = (p_0 q_0 - \vec{p} \cdot \vec{q}, p_0 \cdot \vec{q} + q_0 \cdot \vec{p} + \vec{p} \times \vec{q})$$
For a quaternion in vector form $q = q_0 + \vec{q} = q_0 + q_1 i + q_2 j + q_3 k$
the conjugate is . $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$, $\overline{q} = q_0 - \vec{q}$
The conjugate has the properties
$$\overline{\overline{q}} = q , \ \overline{(p+q)} = \overline{p} + \overline{q} , \ \overline{(pq)} = \overline{q}\,\overline{p} , \ q\overline{q} = \overline{q}q$$
The norm of quaternions is $N(q) = |q| = \sqrt{q\overline{q}} = \sqrt{q_0^{\ 2} + q_1^{\ 2} + q_2^{\ 2} + q_3^{\ 2}}$ with the properties
$N(\overline{q}) = N(q)$, $N(pq) = N(p)N(q)$, $N(rq) = |r| N(q)$, $r \in R$
The inverse of $q \neq 0$ is $q^{-1} = \dfrac{\overline{q}}{|q|^2}$.
There is a isomorphic mapping from quaternion to Pauli-matrices
$(1, i, j, k) \to (1, i\sigma_x, i\sigma_y, i\sigma_z)$

Quaternions and rotation

A unit quaternion has the vector form $q = \cos\theta + \hat{u}\sin\theta$, which is a 3D-rotation by angle 2θ around the axis $\hat{u}$, the rotated vector is
$R(\hat{v}) = q\hat{v}\overline{q}$, where the vector $v = (v_1, v_2, v_3)$ with the quaternion vector $\hat{v} = v_1 i + v_2 j + v_3 k$.
The polynomial ring H[x] has polynomials without zeros in H, so H is not algebraically closed.
$p(x)=ix+xi-j$ has no root in H, because $ix+xi$ always lies in the plane spanned by $\{1,i\}$.
H is topologically complete (all Cauchy sequences converge in H).

Quaternionic functions

rotation by a versor u ($|u| = 1$): $f(q) = uqu^{-1}$
multiplicative inverse $f(q) = q^{-1}$
affine transform $f(q) = aq + b$

linear fractional transformations $f(q) = uqv$ with versors *u, v*

quaternion conjugation $f(q) = \bar{q} = -(q + iqi + jqj + kqk)/2$

extension of a complex map $f(z) = u(x, y) + iv(x, y)$, *z=x+iy*

generalized extension

$f(q) = u(x, y) + c\, v(x, y)$, *u* even, *v* odd, extension to a quaternion variable $q = x + y\,c$, $c^2 = -1$

for conjugate $\bar{q} = x - y\bar{c}$, we obtain $f_5(\bar{q}) = f_5(q)$

Differentiability of quaternion functions

A function $f : H \to H$, $f(q)$, $q = t + ix + jy + kz$, $t, x, y, z \in R$ is real-differentiable, if derivatives

$\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ exist.

A function $f : H \to H$ is quaternion-left-differentiable at q if the limit

$\lim_{h \to 0} \left(h^{-1} \left(f(q+h) - f(q) \right) \right)$ exists,

$f : H \to H$ is quaternion-right-differentiable at q if the limit

$\lim_{h \to 0} \left(\left(f(q+h) - f(q) \right) h^{-1} \right)$ exists.

Example *f(q)=q*, $\lim_{h \to 0} \left(h^{-1} \left(f(q+h) - f(q) \right) \right) = \lim_{h \to 0} \left(\left(f(q+h) - f(q) \right) h^{-1} \right) = f'(q) = 1$,

so *f* is left-differentiable and right-differentiable

Example *f(q)=i q+1*,

left-derivative $\lim_{h \to 0} \left(h^{-1} \left(f(q+h) - f(q) \right) \right) = \lim_{h \to 0} h^{-1} i h$

for *h=it* , t real, $\lim_{h \to 0} h^{-1} i h = i$,

for *h=jt* , t real, $\lim_{h \to 0} h^{-1} i h = -i$,

so the left-derivative limit does not exist.

right-derivative $\lim_{h \to 0} \left(h^{-1} \left(f(q+h) - f(q) \right) \right) = \lim_{h \to 0} i h h^{-1} = i$, so *f* is right-differentiable

The quaternion-differentiability is too strong to be a useful property, according to the theorem by Fueter.

Theorem If a function $f : H \to H$ is left-differentiable on a connected open set $U \subset H$, then it is linear $f(q) = aq + b$ for all $q \in U$

Regularity

A quaternionic function *f* is left-regular if

$$\frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 0$$

and right-regular if

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = 0$$

The left-derivative operator is defined by

$$\bar{\partial}_l = \frac{1}{2}\left(\frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \right)$$

the right-derivative operator is defined by

$$\bar{\partial}_r = \frac{1}{2}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k \right)$$

Using derivative operators, quaternionic function *f* is left-regular if

$\bar{\partial}_l f = 0$, function *f* is right-regular if $\bar{\partial}_r f = 0$

Example: $f(q)=q=t+ix+jy+kz$, $\bar{\partial}_l f=-1/2$, f is not left-regular

Example: $f(q)=q=t+ix-jy+kz$, $\bar{\partial}_l f=0$, $\bar{\partial}_r f=0$, so f is left-regular and right-regular

Example: $f(q)=q=jx+iy$, $\bar{\partial}_l f=0$, so f is left-regular

But $\bar{\partial}_l(if)$ $\bar{\partial}_l(if)=-j/2$, so if is not left-regular.

Properties of regular functions

- The sum of two left-regular functions is left-regular
- If f is left-regular and a is a quaternion then fa is left-regular, but af may not be left-regular.

Regular and harmonic functions

For quaternionic function $f:H\to H$, we define the following derivative operators

$$\partial_l f=\frac{1}{2}\left(\frac{\partial f}{\partial t}-i\frac{\partial f}{\partial x}-j\frac{\partial f}{\partial y}-k\frac{\partial f}{\partial z}\right)$$

$$\bar{\partial}_l f=\frac{1}{2}\left(\frac{\partial f}{\partial t}+i\frac{\partial f}{\partial x}+j\frac{\partial f}{\partial y}+k\frac{\partial f}{\partial z}\right)$$

$$\partial_r f=\frac{1}{2}\left(\frac{\partial f}{\partial t}-\frac{\partial f}{\partial x}i-\frac{\partial f}{\partial y}j-\frac{\partial f}{\partial z}k\right)$$

$$\bar{\partial}_r f=\frac{1}{2}\left(\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k\right)$$

and the Laplace operator

$$\Delta f=\left(\frac{\partial^2 f}{\partial t^2}+\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)$$

Theorem For every 2x real-differentiable function $f:H\to H$, we have $\Delta f=4\partial_l\bar{\partial}_l f=4\bar{\partial}_l\partial_l f$

Corollary

- if f is left-regular, then it is harmonic
- If f is harmonic, then $\partial_l f$ is left-regular.

Quaternionic integrals

Let $f:H\to H$ be a real-differentiable quaternionic function, and

γ:[0,1]→H a piecewise-smooth path.

The left line integral is defined as:

$$\int_\gamma f(q)dq=\int_0^1 f(\gamma(t))\gamma'(t)\,dt$$

Here, the differential γ′(t) dt multiplies on the right, and the integral is order-sensitive due to non-commutativity

Example

$\gamma(t)=t(1+i+j+k)$, $t\in[0,1]$, $f(q)=q$

compute $\int_\gamma q\,dq=\int_0^1 t(1+i+j+k)(1+i+j+k)\,dt=\frac{1}{2}(1+i+j+k)$

Quaternionic Cauchy-Type Integral

Theorem (Cauchy–Fueter integral formula)

If f is regular on every point of a (positively oriented) 4-parallelepiped Ω and q_0 is a point within Ω,

$$f(q_0)=\frac{1}{2\pi^2}\int_{\partial\Omega}\frac{(q-q_0)^{-1}}{|q-q_0|^2}Dq\,f(q)$$

, which generalizes the complex integral $\oint_{\partial\Omega}\frac{z^n}{z}dz=0$, n>1

A practical approach is to integrate component functions:

$$f(q)=f_0+f_1 i+f_2 j+f_3 z\ ,\ q=x_0+x_1 i+x_2 j+x_3 k$$

$$\int_\gamma f(q)dq=\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} f_\mu dx_\nu\left(i^\mu i^\nu\right)$$

which is a sum of 16 integrals.

Example

$$\int_0^1 t(1+i+j+k)\,dt=\int_0^1 t\,dt+i\int_0^1 t\,dt+j\int_0^1 t\,dt+k\int_0^1 t\,dt=\frac{1}{2}(1+i+j+k)$$

Power series expansion

THEOREM (Laurent series). Suppose *f* is regular in an open set *U* except possibly at $q_0 \in U$.
Then there is a neighbourhood *N* of q_0 such that if $\mathbf{q} \in \mathbf{N}$, $f(q)$ can be represented by a series

$$f(q)=\sum_{n=0}^{\infty}\sum_{\nu\in\sigma_n}\left(P_\nu(q-q_0)a_\nu+G_\nu(q-q_0)b_\nu\right)$$

which converges uniformly in any hollow ball $\{q; r\le|q-q_0|\le R\}$ inside *N*

The coefficients aν and bν are given by

$$a_\nu=\frac{1}{2\pi^2}\int_{\partial U}G_\nu(q-q_0)Dq\,f(q)$$

$$b_\nu=\frac{1}{2\pi^2}\int_{\partial U}P_\nu(q-q_0)Dq\,f(q)$$

Here

$$G_\nu(q)=\frac{q^{-1}}{|q|^2}$$

Let ν be an unordered set of n integers $\{i_1,\ldots,i_n\}$ with $1 \le i_r \le 3$;
there are (n + 1)(n + 2)/2 such sets ν; we will denote the set of all of them by σ_n.

$$\partial_\nu=\frac{\partial^n}{\partial x_{i_1}\ldots\partial x_{i_n}}$$

$G_\nu(q)$ is a rational function of q : $G_\nu(q)=\partial_\nu G(q)$

$P_\nu(q)$ is a polynomial in q: $P_\nu(q)=\frac{1}{n!}\sum_{\pi(\nu)=(i_1,\ldots,i_n)}\left(te_{i_1}-x_{i_1}\right)\ldots\left(te_{i_n}-x_{i_n}\right)$, summed over all different orderings $\pi(\nu)$

4 P-adic number analysis

Galois fields

A Galois field, often denoted as F_q, is a finite commutative field containing a finite number of elements, where ($q = p^n$) is a power of a prime number *p*.
It is a set where addition, subtraction, multiplication, and division are defined and satisfy basic algebraic rules.
Galois fields are not ordered, i.e. there is no relation “<” .

Example:
F5, simple Galois field with elements {0,1,...,4} [2]
Solution of x^2+1==0
$\sqrt{-1}=2$, $\sqrt{-1}=3$
Solution of x^2-1==0
$\sqrt{1}=1$, $\sqrt{1}=4$
Solution of x^2-2==0: no solution

Multiplication table F_5

	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4

2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Addition table F_5

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

P-adic numbers

P-adic numbers Q_p [15] are an extension of the field of rationals Q such that congruences modulo powers of a fixed prime *p* are related to proximity in the p-adic metric.

Any rational number $x = \frac{a}{b} \in Q$ can be written uniquely in the form $p^n \frac{u}{v}$ where $p \nmid u \quad p \nmid v$ (*u* and *v* not divisible by *p*), where *p* is a prime number, *r* and *s* are integers not divisible by *p* .

The corresponding p-adic representation of *x* is then $x = \sum_{k=n}^{\infty} a_k p^k$, where $a_k \in \{0,1,...,p-1\}$, and $a_n \neq 0$.

The integer *n* is the order of the p-adic number (index of the lowest non-zero coefficient), *n* can be negative. The coefficient a_k obey the arithmetic mod(p), i.e. they are elements of the underlying Galois field F_p , $a_k \in F_p$.

Then we define the *p*-adic norm of *x* by $|x|_p = p^{-n}$,

also, $|0|_p = 0$

For natural numbers $x \in N$ we have the finite sum representation $x = \sum_{k=n}^{m} a_k p^k$ with basis *p* analogous to the usual decimal representation.

For negative integers $x \in Z$, $x = -|x|$ we have the representation $x = \sum_{k=n}^{m} \tilde{a}_k p^k$, where the coefficients are the negative coefficients mod(p) $\tilde{a}_k + a_k = 0 \bmod (p)$ of the positive number representation $|x| = \sum_{k=n}^{m} a_k p^k$.

The p-adic integers Z_p obey multiplication and addition mod(p) with carry, and they have no zero-dividers , i.e. they allow multiplicative inverse, and can be extended to a field Q_p .

Rationals 1/q , $q \neq p^k$, k>0, are p-adic integers $q^{-1}{}_p \in Z_p$

Rationals 1/q , $q = p^k$, k>0, are proper p-adic fractures e.g. (p=5) $5^{-1}{}_5 = .1_5$

Arbitrary rationals $\frac{u}{v}$ (*u* and *v* not divisible by *p*) are formed by multiplication mod(p) $u*(1/v) \bmod (p)$

Valid identities are

$$-1 = \sum_{k=0}^{\infty} (p-1) p^k$$, e.g. (p=5) $...4444444_5$

$$\frac{1}{1-p} = \sum_{k=0}^{\infty} p^k$$

p-adics are written as

- fractions, from right to left, where the point signifies the order (and is skipped, when at beginning, i.e. when order v=0), e.g. p=5

$1/5 = .1_5$

$1/11 = ...4233042330423304233 1_5$

- lists, from left to right, where the order is the first element of the list, e.g. p=5

1/5= {-1, {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

1/11= {0, {1, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4}}

The cardinality of Z_p (and of Q_p) is the cardinality of continuum like R, whereas Q is countable (cardinality of N).

The p-adics Q_p were first introduced by Hensel (1897) in a paper which was concerned with the development of algebraic numbers in power series, p-adic numbers were then generalized to valuations by Kűrschák in 1913.

Rationals as p-adic numbers

The p-adic valuation (=norm) $|x|_p$ on Q gives rise to the p-adic metric $d(x,y)=|x-y|_p$ which in turn gives rise to the p-adic topology.

It can be shown that the rationals Q, together with the p-adic metric, do not form a complete metric space.

The completion of this space is the set of complete p-adic numbers $\bar{Q}_p = Comp\left(Q_p, |x-y|_p\right)$.

The real numbers R are the completion of the rationals Q with respect to the usual absolute value $|x|$,

$R=\bar{Q}$ $R=Comp\left(Q,|x-y|\right)$ is the set of all convergent series in Q in the metric $d(x,y)=|x-y|$

A rational number with p-adic absolute value 1 has a purely periodic p-adic expansion if and only if it lies in the real interval [−1,0) .

A genuine fractional rational number with p-adic absolute value 1, outside the real interval [−1,0) has a fixed initial expansion followed by a purely periodic p-adic expansion.

p-adic faculty *n!*

The p-adic faculty *n!* (without p-powers) has p-adic valuation

$$|n!|_p = p^{-(n-A_p(n))/(p-1)}$$

where the p-adic expansion of n is $n = a_0 + a_1 p^1 + \ldots + a_L p^L$

and $A_p(n) = a_0 + a_1 + \ldots a_L$

For sufficiently large n , $|n!|_p \le p^{-n/(2p-2)}$

p-adic metric

The p-adic metric [15] is a non-Archimedean distance function where two numbers are close if their difference is highly divisible by a prime p.

Any rational number $x=\frac{a}{b}\in Q$ can be written uniquely in the form $p^n \frac{u}{v}$ where $p\nmid u$ $p\nmid v$ (*u* and *v* not divisible by *p*).

The p-adic norm is then $|x|_p = p^{-n}$, the metric is $d(x,y)=|x-y|_p$

The metric is non-archimedean (ultrametric), it obeys the strong triangle inequality $|x-y|\le \max\left(|x|,|y|\right)$ instead of the simple (archimedean) triangle inequality $|x-y|\le|x|+|y|$.

p-adic half-metric

The p-adic metric yields a distance function sufficient for convergence criteria, but it is little sensitive near zero, therefore not well-suited for numerical calculations.

A better metric function is $f_{rat}\left(x_p,n,L\right)$ used for recovering rational numbers from the p-adic numbers.

$\left|x_p\right|_{hp} = \left|f_{rat}\left(x_p,n,L\right)\right|$, where period start position *n=1* and period length *L=N-1, N*=precision.

$\left|x_p\right|_{hp}$ is a half-norm, because it is not symmetric $\left|x_p\right|_{hp} \ne \left|-x_p\right|_{hp}$, and neither is the distance $\left|x_p - y_p\right|_{hp}$, but it satisfies the triangle relation $\left|x_p - y_p\right|_{hp} \le \left|x_p\right|_{hp} + \left|y_p\right|_{hp}$.

Examples [16]

p-adic numbers p=5, 10 or 20 digits

p-adic fraction form (right to left), below: list form (left to right)

$1=1_5$

{0, {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

finite

$-1=4444444444444444444_5$

{0, {4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4}}

period 4

$-1/5=444444444444444444.4_5$

{-1, {4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4}}

period 4, order v=-1, norm $|\ |_5 = 1/5$, point at position 2

$1/5=.1_5$

{-1, {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

here order v=-1, norm $|\ |_5 = 1/5$, point at position 2

$11= 21_5$

{0, {1, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

finite

$1/11=42330423304233042331_5$

{0, {1, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4}}

fixed=1 , period 04233

$3/11=23304233042330423304 3_5$

{0, {3, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3}}

fixed=3 , period 23304

$-3/11= 4233042330423304232311402114021140211402_5$

{0, {2, 0, 4, 1, 1, 2, 0, 4, 1, 1, 2, 0, 4, 1, 1, 2, 0, 4, 1, 1}}

period 11402

$12/11=20423304233042330423 32_5$

{0, {2, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4, 0, 3, 3, 2, 4}}

fixed=2 , period 04233

Non-archimedean (ultra-metric) distance

p-adic distance is $d_p(x,y)=|x-y|_p$ [15]

A non-Archimedean distance, or ultrametric, is a distance metric d(x,y) that satisfies a strict form of the triangle inequality:

$|x-y| \le \max(|x|,|y|)$, whereas Euclidean distance obeys the simple triangle inequality $|x-y| \le |x|+|y|$.

Unlike Euclidean distance, it implies distances do not "add up" or accumulate, often resulting in topologies that are totally disconnected, where all balls are clopen (both open and closed).

- Strong triangle inequality $|x-y| \le \max(|x|,|y|)$, weaker in general $|x-y| \le |x|+|y|$
- Strong multiplicativity $|x*y|=|x|*|y|$, weaker in general $|x*y| \le |x|*|y|$
- Isosceles Triangles: All triangles are isosceles, with the two larger sides equal.
- Ball Structure: Two balls $B_{x,r}=\{x \mid |x|<r\}$ either are disjoint or one is contained within the other.
- Center Property: Any point within a ball can be considered its center.

• Topology: The resulting topology is totally disconnected, where all balls are clopen (open and close)
• Smallness: Numbers that are large in the real sense (like 5^{100}) are small in the p-adic sense mod5, while small numbers (like $1/5^{100}$ mod5) are large.
• **Ostrowski's Theorem**: This theorem states that any non-trivial absolute value on the rational numbers Q is equivalent to either the usual real absolute value or a p-adic absolute value.

Mapping Qp->R
There is **no canonical embedding** (a natural, structure-preserving map) from the p-adic numbers Qp into the real numbers R.
Here is a breakdown of why this is the case based on number theory principles:

• **Different Completions:** Both Qp and R are completions of the field of rational numbers Q, but they are completed with respect to completely different, non-equivalent metrics (absolute values).

• **Topological Incompatibility:** R is a connected, archimedean field, while Qp is totally disconnected and non-archimedean. There is no continuous field homomorphism between them.

• **Algebraic Closure:** R is not algebraically closed (its algebraic closure is C), whereas Q_p is also not algebraically closed (its algebraic closure is Q_p^a).

Mapping Q->Qp

• **Definition:** The map is defined by $\iota_p : Q \to Q_p$, where $\iota_p(x) = x$

• **Completeness:** The p-adic field Qp is defined as the completion of Q with respect to the p-adic absolute value $| \ |_p$

Therefore, Q acts as a dense subfield of Qp

• **Dense Subset:** The image $\iota_p(Q)$ is dense in Qp, meaning any p-adic number can be approximated by a sequence of rational numbers using the p-adic norm.

• **Field Structure:** This mapping is a injective ring homomorphism, preserving both addition and multiplication.

• **p-adic expansion**
Any rational number $\frac{a}{b} \in Q$ can be written uniquely in the form $p^n \frac{u}{v}$ where $p \nmid u \quad p \nmid v$

The canonical embedding is then $\iota : p^n \frac{u}{v} \to \sum_{k=n}^{\infty} a_k p^k$, $Q \to Q_p$, with $a_k \in \{0, 1, \ldots p-1\}$

Extensions of Qp
[15]
• The elements of Qp are exactly those represented by series of the form $\sum_{r \in S} a_r p^r$, where S is a bounded-below subset of the integers and each a_r is a multiplicative lift (i.e., a "Teichmüller lift") of an element of F_p.
• The Teichmüller character is a homomorphism (lift) of multiplicative groups:
$\omega : F_p^x \to Z_p^x$, such that ω(a) is the unique (p−1)th root of unity in Zp which is congruent to a modulo p
• The elements of Q_{nrp} (the maximal unramified extension of Qp) are exactly represented by series of the form $\sum_{r \in S} a_r p^r$, where S is again a bounded-below subset of the integers, but the a_r can now be lifts of any element of F_q, where q is a fixed power of p.

- The elements of $\bar{\mathbb{Q}}_p$ and its completion C_p are represented by certain series of the form $\sum_{r\in S} a_r p^r$, where the a_r are again lifts of elements of $\bar{F}_p$, but now S can be a more general well-ordered subset of Q.

Theorem [Kedlaya] (Huang, Rayner, Ştefănescu). Let L be the set of generalized power series of the form $f=\sum_{i\in S} x_i t^i \ (x_i \in K)$, where the set S⊂Q (which depends on f) has the following properties: 1. Every nonempty subset of S has a least element (i.e. S is well-ordered). 2. There exists a natural number m such that every element of mS has denominator a power of p.
Then L is an algebraically closed field

- The elements of Ω_p, the *spherical completion* of C_p, are represented exactly by series of the form $\sum_{r\in S} a_r p^r$,

where the a_r are as before, but now S can be *any* well-ordered subset of Q (with no other restrictions).
And this is in some sense as far as we can go, since Ω_p is "maximally complete": we can't extend it any further without "adding geometry", in the sense that it's the unique largest field of characteristic zero with value group Q and residue field the algebraic closure of F_p.

Visualization

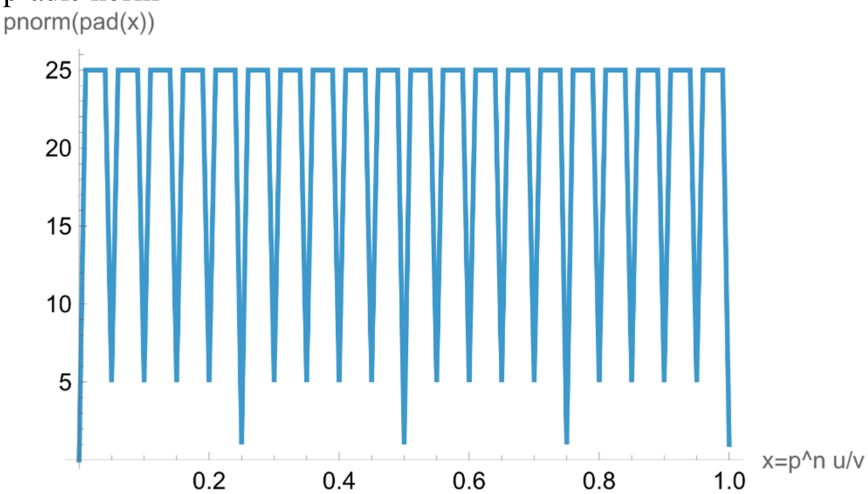
The canonical mapping [16] is $\iota: p^n \frac{u}{v} \to \sum_{k=n}^{\infty} a_k p^k$, $Q \to Q_p$, with $a_k \in \{0,1,...p-1\}$

The canonical p-adic scalar valuation is $v_c : \left(\sum_{k=n}^{\infty} a_k p^k\right)_p \to \left(p^{-n} \sum_{k=n}^{\infty} a_k p^{k-n}\right)$, $Q_p \to Q$

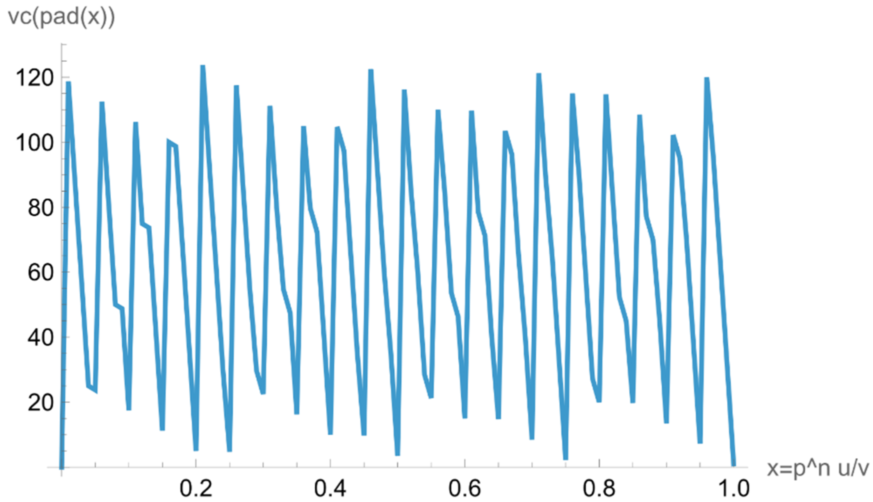
The canonical p-adic vector valuation length n_c is $v_v : \left(\sum_{k=n}^{\infty} a_k p^k\right)_p \to p^{-n}\left\{a_k p^{k-n}\right\}_{k=n}^{n+n_c-1}$, $Q_p \to Q^{n_c}$

Example 1 p-adic norm , prime base p_0=5 [16]
npts=100 number grid points
dint=1 interval [0,1]
p0=5 base prime p=5
Ndigits=100 number of elements in series expansion

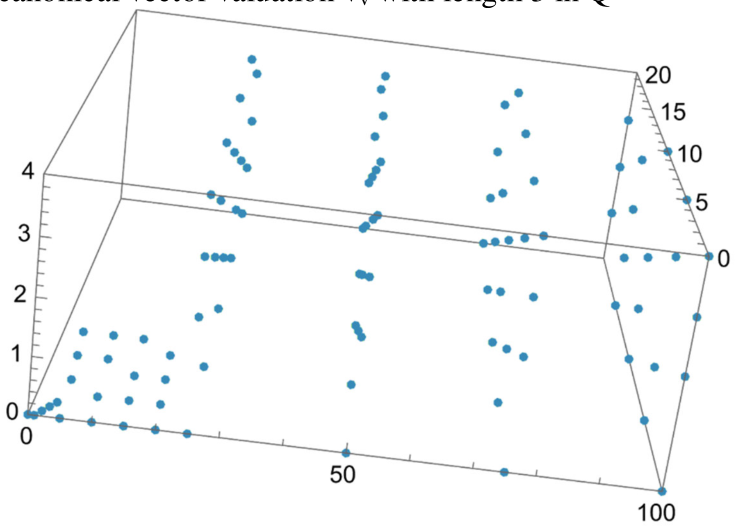
p-adic norm $|x|_p$



canonical scalar valuation v_c in Q



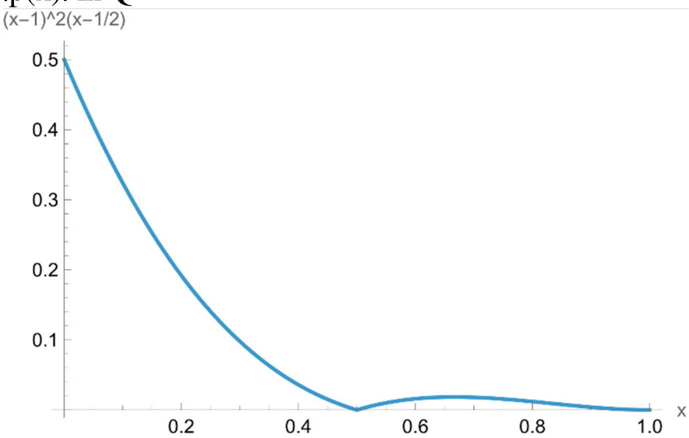
canonical vector valuation v_v with length 3 in Q^3



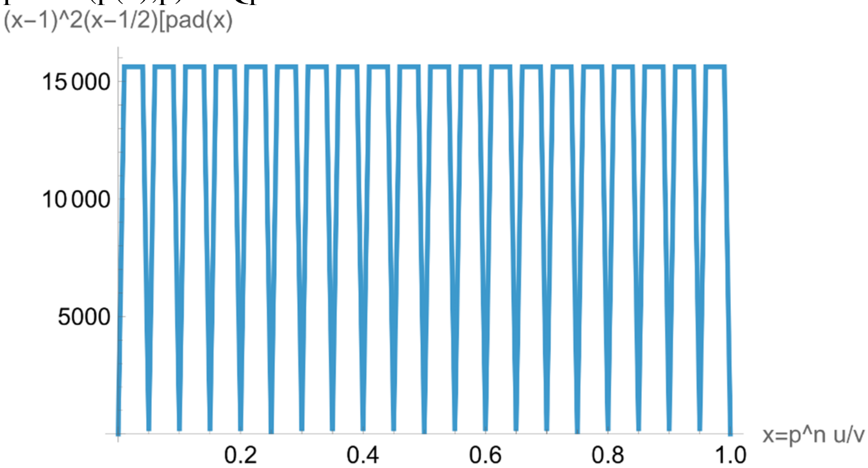
Example 2 polynomial p(x)=(x−1)^2 (x−12), prime base p_0=5 [16]

npts=100 number grid points
dint=1 interval [0,1]
p_0=5 base prime
Ndigits=100 number of elements in series expansion
pnorm=p-adic norm $|x|_p$
pval=p-adic scalar valuation v_c
vpval vector valuation v_v with length 3 in Q^3

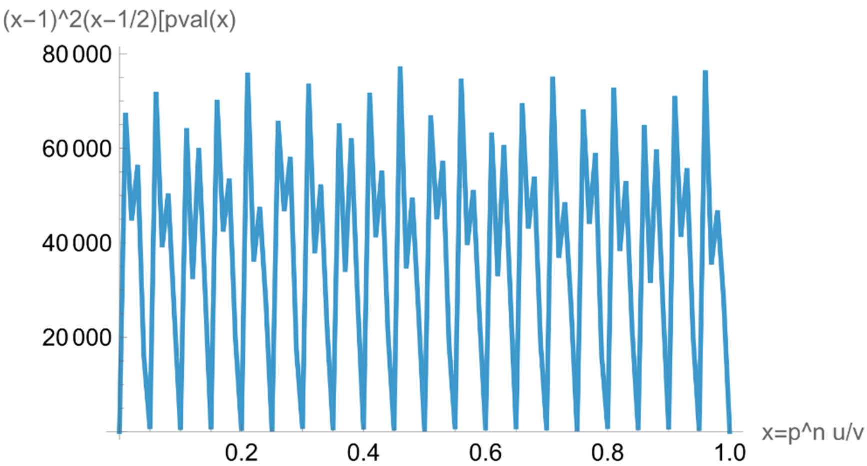
|p(x)| in Q



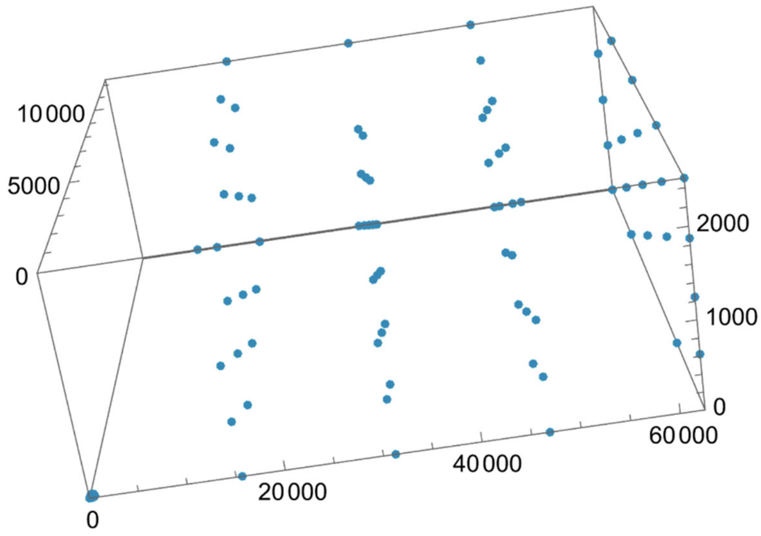
pnorm(p(x),p) in Qp



scalar valuation pval(p(x),p) in Qp



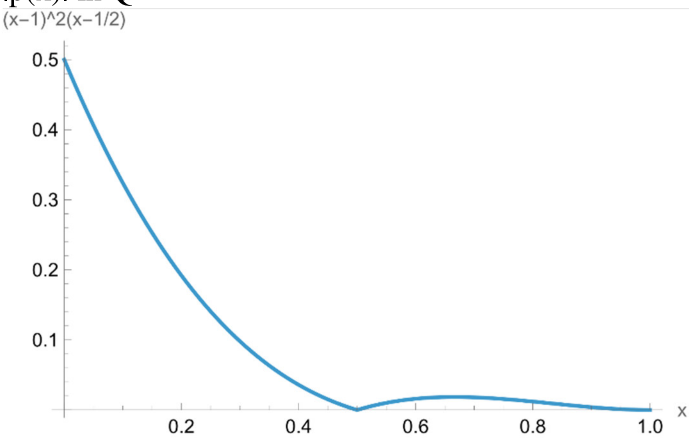
vpval vector valuation v_v with length 3 in Q^3


10 000
5000
0
0
20 000
40 000
60 000
2000
1000
0

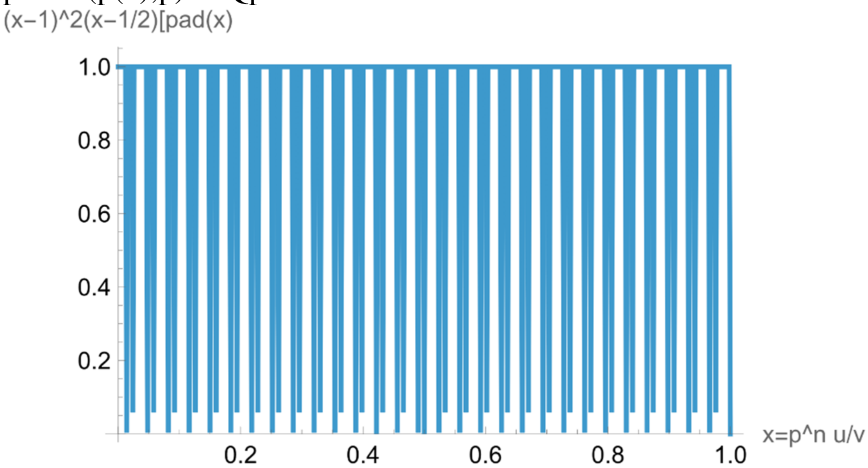
Example 3 polynomial p(x)=$(x-1)$^2 $(x-12)$, prime base p_0=17 [16]

npts=100 number grid points
dint=1 interval [0,1]
p_0=17 base prime
Ndigits=500 number of elements in series expansion
pnorm=p-adic norm $|x|_p$
pval=p-adic scalar valuation v_c
vpval vector valuation v_v with length 3 in Q^3

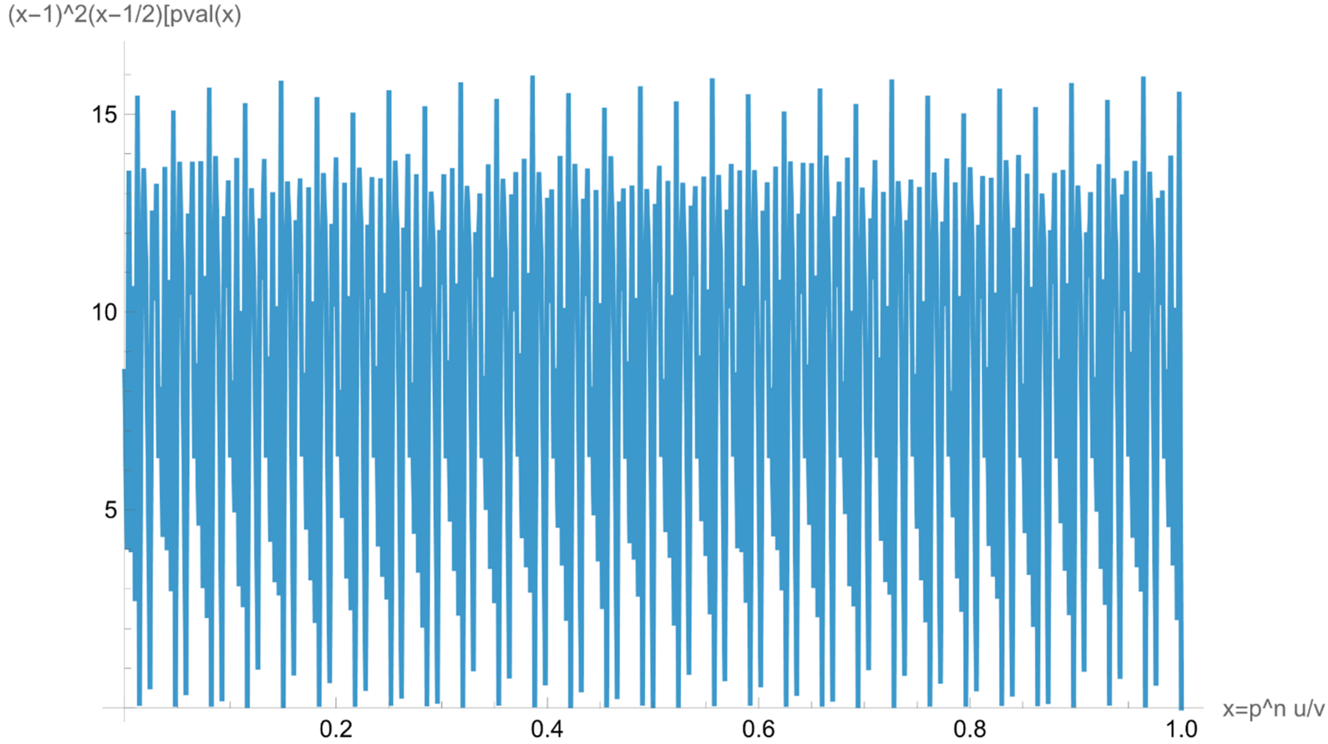
|p(x)| in Q



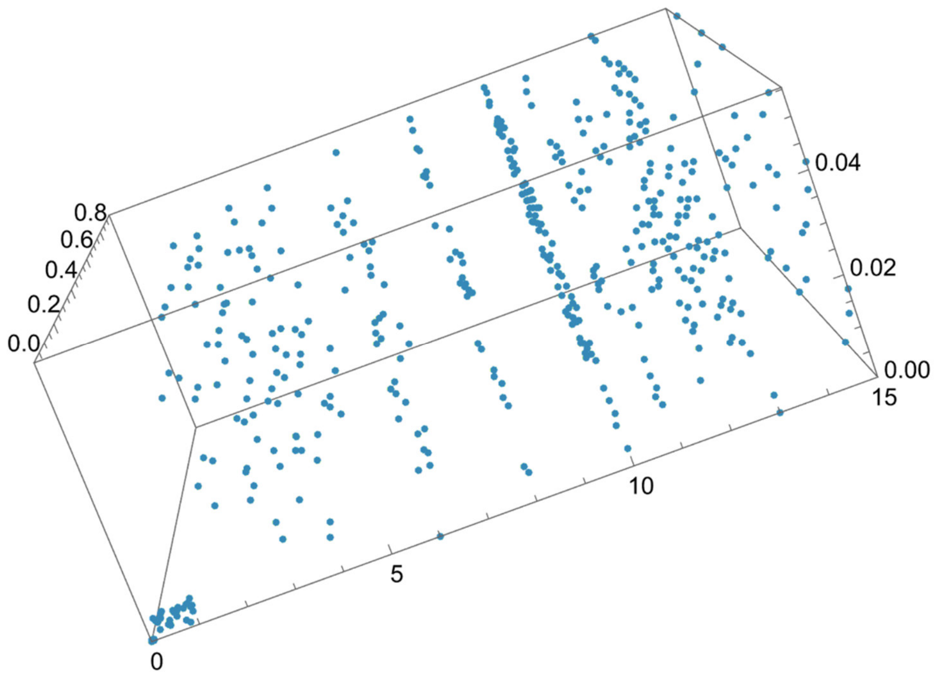
pnorm(p(x),p) in Qp



scalar valuation pval(p(x),p) in Qp



vpval vector valuation v_v with length 3 in Q^3



Hensel's principles

[15]

First principle: zeros of polynomials

Let $P(X, Y) \in Z[X, Y]$ be a polynomial with integral coefficients

Proposition The following properties are equivalent

- $P=0$ has a solution in Z_p
- for all $n>= 0$, $P=0$ has a solution in Z_p/p^nZ
- for all $n>= 0$, there are integers a_n b_n such that $P(a_n, b_n) = 0 \bmod (p^n)$

Second principle: Newton method for zeros

$P(X) \in Z[X]$, integer x, with $P(x)=0 \bmod (p)$, i.e. $P(a0)= p\, t$

then for some integer b
$P(a_0+a_1p)=P(a_0)+P'(a_0)a_1p+(a_1p)^2b$
we have the Newton approximation method
$$\hat{x}=a_0+a_1p=a_0-\frac{pt}{P'(a_0)}=x-P(a_0)/P'(a_0)$$
$a_1=-t/P'(a_0)\bmod(p)$
and $P(\hat{x})=P(a_0+a_1p)=0\bmod(p^2)$
Newton iteration $\hat{x}(x)$

Proposition Let P∈ Z_p[X] and x∈ Z_p be such that $P(x)=0\bmod(p^n)$
If $k=v(P'(x))<n/2$ then $\hat{x}(x)=N_p(x)=x-P(x)/P'(x)$ satisfies

- $P(\hat{x})=0\bmod(p^{n+1})$ improvement
- $\hat{x}=x\bmod(p^{n-k})$ a controlled loss
- $v(P'(\hat{x}))=v(P'(x))=k$ invitation to iteration

Example: Newton algorithm for roots of polynomial p(x)=x^2-2, i.e. $\sqrt{2}$ [2]
error in p-adic half-norm
start value $x_0=7/5$, x_{p0}= {-1, {2, 1, 0, 0, 0, 0, 0, 0, 0, 0}}
iter=1, in Q $x_1=99/70$ error $|p(x_1)|$=1/4900 , in Q_p x_{p1}= {-1, {1, 9, 0, 4, 2, 3, 11, 15, 4, 10}}
error $|p(x_{p1})|_{hp}$ =0.001397
iter=2, in Q $x_1=19601/13660$ error $|p(x_1)|$=5.2 10^{-9} , in Q_p x_{p1}=
{-1, {9, 4, 1, 10, 13, 12, 14, 0, 11, 16}} error $|p(x_{p1})|_{hp}=0.000778$

Reduction of zeros mod(p^n) to mod(p^{n-k})
Theorem Hensel's Lemma
Let P∈ Z_p[X] and x∈ Z_p satisfy $P(x)=0\bmod(p^n)$
If $k=v(P'(x))<n/2$, the there exist a unique root of P, ξ∈ Z_p such that
$\xi=x\bmod(p^{n-k})$ and $v(P'(\xi))=v(P'(x))=k$
ξ is the unique root satisfying the apriori weaker congruence
$\xi=x\bmod(p^{k+1})$

Mahler series
[15]
Theorem
Let be $f:Z_p\to C_p$ arbitrary map.
Then there is a unique sequence $(a_i)_{i\geq 0}\in C_p$ such that
$$f(x)=\sum_{x\geq i\geq 0}a_i\binom{x}{i}\qquad a_i=\nabla^i f(0)$$
$$\binom{x}{i}=\frac{x(x-1)...(x-i+1)}{i!}$$ binomial polynomials
forward difference operator $\nabla f(x)=f(x+1)-f(x)$

Mahler series $f(x)=\sum_{x\geq i\geq 0}\nabla^i f(0)\binom{x}{i}$

analogous to classical analysis Taylor series $f(x)=\sum_{i=0}^{k}\frac{f^i(a)}{i!}(x-a)^i$

Theorem 1

Let $f:Z_p\to C_p$ continuous, $a_k=\nabla^k f(0)$, then

$|a_k|_p\to 0$, $\sum_{x\geq k\geq 0}a_k\binom{x}{k}\to f(x)$ uniformly convergent, and $\|f\|\leq\sup_{k\geq 0}|a_k|_p$

Theorem 2

Let $f:Z_p\to C_p$ continuous, $a_k=\nabla^k f(0)$, then the following properties are equivalent

- $|a_k|_p\to 0$
- $\sum_{x\geq k\geq 0}a_k\binom{x}{k}\to f(x)$ uniformly convergent
- f(x) uniformly continuous
- $\|\nabla^k f\|\to 0$

Theorem 3 extension on Q_p

Let $f:Q_p\to C_p$ continuous, $a_k=\nabla^k f(0)$, then

$|a_k|_p\to 0$, $\sum_{x\geq k\geq 0}a_k\binom{x}{k}\to f(x)$ uniformly convergent, and $\|f\|\leq\sup_{k\geq 0}|a_k|_p$

Integer-valued polynomials on Z

Theorem : Module of polynomial functions $L=L(Z)\subset Q[x]$ integer-valued on N is generated by the binomial polynomials $\sum m_i\binom{x}{i}$

Corollary If a polynomial $P\in Q[x]$ with degree d>=0 takes integral values on d+1 consecutive integers, then takes integral values on all integers.

Periodic functions on Z

For a finite (Galois) field F_p

Proposition

For $i<p^t$, the functions $\binom{\cdot}{i}:Z\to F_p$, $x\mapsto\binom{x}{i}\bmod(p)$ are periodic of period $T=p^t$.

They form a base of space of T-periodic maps.

Theorem

Le M be a vector space over F_p , and $f:Z\to M$ periodic function of period $T=p^t$.

Then f can be uniquely written in the form

$$f(x)=\sum_{T\geq i\geq 0}a_i\binom{x}{i}$$

Van-der-Putt expansion of functions on Zp

Proposition

If $f=\sum_i a_i\psi_i\in F_j$ locally constant function f: $Z_p\to\overline{Q}_p$,

then $a_0=f(0)$, $a_n=f(n)-f(n_-)$ for $n>0$

and $\|f\|=\max_i|a_i|=\sup_i|a_i|$

Van-der-Putt theorem

Continuous function f: $Z_p \to \bar{Q}_p$,

$a_0 = f(0)$, $a_n = f(n) - f(n_-)$ for *n>0*

$n_- = n - n_{v-1}p^{v-1}$ obtained by deleting its top digit in base *p*

Then $|a_n| \to 0$, $\sum_n a_n$ converges uniformly to *f*, and $\|f\| = \max_i |a_i| = \sup_i |a_i|$

Differentiability

[15]

Definition For $f: X \to K$, X a ring, K a field , the sup-norm of is $\|f\|_s = \sup_{x\in X} |f(x)|$

Definition For $f: X \to K$, power series $f(x) = \sum_{k\geq 0} a_k x^k$, the Gauss norm of is $\|f\|_G = \sup_k |a_k|$

Definition We say that f is strictly differentiable at a point $a \in X$ and denote this property by $f \in S^1(a)$ if the difference quotients

$$\Phi f(x,y) = \frac{f(x) - f(y)}{x - y}$$

have a limit $f'(a) = \lim_{(x,y)\to(a,a)} \Phi f(x,y)$

Then in a neighborhood *V*: $|f(x) - f(y)| = |f'(a)||x - y|$ for $(x,y) \in V \times V$

Proposition

For $f: X \to K$, X a ring, K a field, the following properties are equivalent

- $f \in S^1(a)$ for all $a \in X$
- the function $\Phi f(x,y)$ defined on $X \times X - \Delta_X$ has a continuous extension on $X \times X$
- f is differentiable at every point $a \in X$, and there is a continuous function α on $X \times X$, vanishing on Δ_X

with $f(y) = f(x) + (y-x)f'(x) + (y-x)\alpha(x,y)$ for $x, y \in X$

Theorem

Let *f* be a continuous function on Z_p.

Then *f* is differentiable at *y* precisely when $(\nabla^k f)(y)/k \underset{k\to\infty}{\to} 0$

Then $f'(y) = \sum_{k\geq 1} (-1)^{k-1} (\nabla^k f)(y)/k$

Definition $f: X \to K$ is Lipschitz, when its differential quotient is uniformly bounded

$|f(x) - f(y)| = M\,|x - y|$ for $x, y \in X$

$|f(x) - f(y)| = M\,|x - y|$, $M \leq \|\Phi f\|_s$

binomial polynomials are Lipschitz : For $k \geq 1$ and $p^j \leq k \leq p^{j+1}$ we have

$$\left|\binom{x}{k} - \binom{y}{k}\right| = p^j\,|x - y|$$

Proposition

A continouous function $f(x) = \sum_{k\geq 0} c_k \binom{x}{k}$ $f \in C(Z_p)$ is Lipschitz precisely when $\{k|c_k|\}_{k\geq 0}$ is bounded

Corollary $f \in Lip(Z_p)$ and $f(x) = \sum_{k\geq 0} c_k \binom{x}{k} \in C(Z_p)$ Mahler series, then $\|\Phi f\|_s = \sup_{k\geq 1} \kappa_k |c_k|$

where $\kappa_k = [\log_p k]$

Theorem: Mahler series and strict differentiability

For a continuous function $f(x)=\sum_{k\geq 0} c_k \binom{x}{k}$ $f\in C(Z_p)$, we have

if $k|c_k|\to 0$ then $f\in S^1(Z_p)$

For a strictly differentiable function $f\in S^1(Z_p)$, then its indefinite sum is also strictly differentiable

$Sf(x)=\sum_{0\leq i\leq x} f(i)$, $Sf\in S^1(Z_p)$

Restricted power series and mean value
[15]

Definition power series $f(x)=\sum_{k\geq 0} a_k x^k$ with $a_k\to 0$ over field K is called restricted.

They form a normed algebra over K, with the Gauss norm $\|f\|_G=\sup_k |a_k|=\max_k |a_k|$

where $\|f\,g\|_G\leq \|f\|_G\|g\|_G$ for $|x|\leq 1$

Restricted ps are uniformly continuous

$|f(x+h)-f(x)|\leq |h|\|f\|_G$ if $|x|\leq 1$ and $|h|\leq 1$

and $|f(x+y)-f(x)-f(y)+f(0)|\leq \|f\|_G|x\,y|$ for $|x|\leq 1$ and $|y|\leq 1$

Theorem1 (analogy to calculus on R)

f(x) restricted ps, then f is a twice strictly differentiable function on the unit ball A of K : $f\in S^2(A)$.

The derivative of *f* is given by the restricted formal power series

$f'(x)=\sum_{k\geq 1} a_k k x^{k-1}$

Lemma n >=1 integer and $S_p(n)=S_p(n)=\sum_{k=1}^{n} k \bmod (p)$

Then the p-adic order of n! is $ord_p n!=\dfrac{n-S_p(n)}{p-1}$

Theorem2F1 (mean value)

f(x) restricted ps, $f(x)=\sum_{k\geq 0} a_k x^k$

and $|x|\leq 1$ and $|h|\leq r_p$, $r_p=\begin{cases}|p|^{1/(p-1)} & p>2\\ \sqrt{2} & p=2\end{cases}$

then $|f(x+h)-f(x)|\leq |h|\|f'\|_G$

Theorem2F2 (mean value)

$f\in C_p(X)$ ps that converges in the open unit ball $M_p = p\,Z_p$, and $\|f'(x)\|_{<1}<\infty$

with the restricted norm $\|f\|_{<1}=\sup_{|x|<1}|f(x)|$

then $|f(x+h)-f(x)|\leq |h|\|f'\|_{<1}$ for $|h|<r_p$, $r_p=\begin{cases}|p|^{1/(p-1)} & p>2\\ \sqrt{2} & p=2\end{cases}$

Theorem2F3 (mean value)

If $|x|\leq 1$ and $|h|\leq r_p$, $r_p=\begin{cases}|p|^{1/(p-1)} & p>2\\ \sqrt{2} & p=2\end{cases}$

then $|f(x+h)-f(x)-f'(x)h|\leq |h^2/2|\|f''\|_G$

Theorem3 fixed-point theorem

f(x) restricted ps in Q_p , with $\|f'\|_G < 1$, $B_{\le 1}$ closed unit ball, and $\inf_{x \in B_{\le 1}} |f(x) - x| \le r_p \quad r_p = \begin{cases} |p|^{1/(p-1)} & p > 2 \\ \sqrt{2} & p = 2 \end{cases}$

then f has a fixed point in $B_{\le 1}$: $\exists x_* \in B_{\le 1} : f(x_*) = x_*$

Volkenborn integral and summation
[15]
Definition Volkenborn integral of $f \in S^1(Z_p)$

$$\int_{Z_p} f(x)\,dx = \lim_{n\to\infty} \frac{1}{p^n} \sum_{j=0}^{p^n} f(j) = (Sf)'(0)$$

Proposition1

$$\int_{Z_p} f(x)\,dx \le p\|f\|_1$$

where
$$\|f\|_1 = \sup\left(|f(0)|, \left\| \frac{f(x) - f(y)}{x - y} \right\| \right)$$

If $\|f_n - f\|_1 \to 0$ then $\int_{Z_p} f_n(x)\,dx \to \int_{Z_p} f(x)\,dx$

Proposition2
For $f \in S^1(Z_p)$

$$\int_{Z_p} \nabla f(x)\,dx = f'(0)$$

Proposition3 integration of Mahler series
For Mahler series $f(x) = \sum_{k\ge 0} c_k \binom{x}{k}$

$$\int_{Z_p} f(x)\,dx = \sum_{k\ge 0} (-1)^k c_k / (k+1)$$

Proposition4
For shift $\tau_x f(t) = f(x+t)$ we have the following properties

- $\tau_x f(t) = f(x+t)$
- $S\tau_x f = \tau_x S f - f(0)$
- $S\nabla f = \nabla S f - f(0)$
- $\int_{Z_p} \tau_x f(t)\,dt = (Sf)'(x)$
- $S(f)'(x) = \int_{Z_p} f(x+t)\,dt - \int_{Z_p} f(t)\,dt$
- $(\nabla S)(x) = f(floor(x+1))$

Proposition5
For $F(x) = \int_{Z_p} f(x+t)\,dt$, $F \in S^1(Z_p)$ and $F'(x) = \int_{Z_p} f'(x+t)\,dt$

Proposition6
For involution $\sigma(x) = -(x+1)$, then $\int_{Z_p} f(\sigma(x))\,dx = \int_{Z_p} f(x)\,dx$

Proposition7

For power series $f(x)=\sum_{k\geq 0} a_k x^k$, $\int_{Z_p} f(x)\,dx=\sum_{k\geq 0} a_k b_k$

with Bernoulli numbers b_k $\frac{t}{e^t-1}=\sum_{k\geq 0} b_k \frac{t^k}{k!}$

Sums of powers

$$S\left(x^k\right)=b_k x+\sum_{2\leq j\leq k}\binom{k}{j-1} b_{k+1-j}\frac{x^{k+1}}{k+1}$$

$$S_k(n):=\sum_{i=1}^{n} i^k$$

$$S_1(n)=\frac{n(n-1)}{2}$$

$$S_2(n)=\frac{n(n-1)(2n-1)}{6}$$

$$S_3(n)=\frac{n^2(n-1)^2}{4}$$

$$S_k(p)=p\,b_k \bmod\left(p\,k\,Z_p\right)$$

$$S_k(2)=1 \bmod\left(k\,Z_2\right)$$

6 Lie and Clifford algebras in physics

1 Basics

Lie groups are mathematical objects that are simultaneously groups (allowing for operations like multiplication and inversion) and smooth manifolds (allowing for continuous, "smooth" transformations). Lie algebras represent the infinitesimal structure of Lie groups, serving as the "flat" tangent space at the group's identity.

- Lie Groups: Represent continuous global symmetries. Because they are curved manifolds, doing global calculations (like multiplying group elements) can be geometrically complex
- Lie Algebras: Represent infinitesimal symmetries. They form a flat vector space, meaning you can add elements together and scale them.
- The Correspondence: The exponential map connects the two. It translates tangent vectors (from the Lie algebra) into curved trajectories on the Lie group.

Lie Groups

A Lie group G is a set that possesses two compatible structures:

It is a **group**: It is closed under an associative multiplication operation, has an identity element, and every element has an inverse.

It is a **smooth manifold**: It locally resembles n-dimensional Euclidean space R^n, allowing for calculus (differentiation, integration) to be performed on the group.

Examples: The circle group (U(1)), the group of n × n invertible matrices GL(n, R), and the group of 3D rotations SO(3).

Lie Algebras

A Lie algebra g is a vector space equipped with a binary operation called the **Lie bracket**. The Lie bracket, denoted as [.,.] measures the failure of commutativity

The bracket must satisfy three primary rules:

Bilinearity: ([ax + by, z] = a[x,z] + b[y,z])

Antisymmetry: ([x,y] = -[y,x])

Jacobi Identity: ([x, [y,z]] + [y, [z,x]] + [z, [x,y]] = 0)

For matrices, the Lie bracket is the commutator: [A, B] = AB – BA

Structure constants

Structure constants (or structure coefficients) are the numerical values that uniquely define the algebraic multiplication in a Lie algebra. They are the coefficients that appear when the Lie bracket of any two basis vectors is expressed as a linear combination of the basis itself.

For a Lie algebra g equipped with a basis $\{e_i\}$, the Lie bracket $[e_i, e_j]$ of any two basis elements produces a new element within the algebra. Because of bilinearity, you only need to know the products of the basis elements to determine the multiplication for the entire algebra.

Adjoint representation

The mapping $ad_x(y)=[x,y]$ is called the adjoint of $x\in G$ of Lie algebra *G*.

Derivation

A derivation on a Lie algebra G is a linear map D:G→G that satisfies the Leibniz rule for all elements $x, y\in G$

$D([x,y])=[Dx,y]+[x,Dy]$

The collection of all such derivations forms a Lie algebra known as the derivation Lie algebra, denoted *Der(G)* . The key aspects of derivations are

- The Bracket Operation: The Lie bracket on Der(G) is the standard commutator operation, defined for any two derivations $D_1, D_2\in Der(G)$ as:

$[D_1,D_2]=D_1\circ D_2-D_2\circ D_1$

- Inner Derivations: For any element $x\in G$, the adjoint operator $ad_x(y)=[x,y]$ is a derivation. These are called inner derivations and they form an ideal inside *Der(G)* .

Killing form

The Killing form is the bilinear symmetric form $\kappa(x,y)=Tr(ad(x)\circ ad(y))$, with adjoint $ad_x(y)=[x,y]$, for basis vectors $\kappa_{ij} = c^{a}_{ib} c^{b}_{ja}$ expressed by the structure constants.

Key Theorems

- **Lie's Third Theorem:** Every finite-dimensional, real Lie algebra is the Lie algebra of some Lie group.
- **The Correspondence:** While two different Lie groups can share the exact same Lie algebra (for example, SU(2) and SO(3)), simply connected Lie groups have a one-to-one correspondence with their Lie algebras.

Applications in physics

- **Quantum Mechanics**: The angular momentum operators satisfy the commutation relations of the SU(2) Lie algebra.
- **Particle Physics**: The Standard Model relies heavily on the gauge groups U(1) × SU(2) × SU(3) (where SU(3) relates to quantum chromodynamics).

2 Lie groups

A **Lie group** is a mathematical object that is both a smooth manifold and a group, where the group operations of multiplication and inversion are differentiable. This structure allows mathematicians to study continuous symmetries using the tools of calculus and geometry, mapping complex curved spaces into linear structures called Lie algebras.

At its core, a Lie group sits beautifully at the intersection of algebra and geometry. It combines two fundamental mathematical concepts:
The Manifold: A space that locally resembles flat Euclidean space. For instance, a circle looks like a straight line when you zoom in.
The Group: A set of elements coupled with an operation (like rotation or translation) that obeys closure, associativity, has an identity element, and allows for inverses
For the two structures to be compatible, the multiplication map $(x, y) \mapsto x \cdot y$ and the inversion map $x \mapsto x^{-1}$ must be smooth (continuous) functions.
Lie groups can be highly non-linear and curved, making global calculations difficult. The primary breakthrough in Lie theory is the Lie algebra, which essentially acts as a "flat map" or a linear approximation of the curved group.
Tangent Space: The Lie algebra is defined as the tangent space of the Lie group at the identity element.
Exponential Map: The relationship between the Lie algebra and the Lie group is established via the exponential map, which projects tangent vectors (elements of the Lie algebra) directly onto the curved group (the Lie group).
The Lie Bracket: While the group has a multiplication operation, the Lie algebra possesses a special operation called the Lie bracket, which captures the non-commutativity and curvature of the group's operations.
Because Lie algebras are flat (vector spaces), it is often vastly simpler to perform algebraic and geometric computations within them.
Lie algebra $\mathfrak{g}$ is nilpotent if its lower central series terminates in the zero subalgebra. The *lower central* series is the sequence of subalgebras
$\mathfrak{g} \supseteq [\mathfrak{g},\mathfrak{g}] \supseteq [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] \supseteq \ldots$
Lie algebra $\mathfrak{g}$ is *solvable* if its derived series terminates in the zero subalgebra. The *derived Lie algebra* of the Lie algebra $\mathfrak{g}$ is the subalgebra of $\mathfrak{g}$, denoted
$[\mathfrak{g},\mathfrak{g}]$ that consists of all linear combinations of Lie brackets of pairs of elements of g. The *derived series* is the sequence of subalgebras
$\mathfrak{g} \supseteq [\mathfrak{g},\mathfrak{g}] \supseteq [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] \supseteq \ldots$
Lie algebra $\mathfrak{g}$ is semisimple if it is a direct sum of simple Lie algebras. A simple Lie algebra is a non-abelian Lie algebra without any non-zero proper ideals.
For a finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0, if nonzero, the following conditions are equivalent:
$\mathfrak{g}$ is semisimple;
the Killing form $\kappa(x, y) = tr(ad(x)ad(y))$ is non-degenerate;
$\mathfrak{g}$ has no non-zero abelian ideals;
$\mathfrak{g}$ has no non-zero solvable ideals;
the radical (maximal solvable ideal) of $\mathfrak{g}$ is zero.

3 Lie algebras

Lie Algebras

A Lie algebra g is a vector space equipped with a binary operation called the **Lie bracket**. The Lie bracket, denoted as [.,.] measures the failure of commutativity
The bracket must satisfy three primary rules:
Bilinearity: ([ax + by, z] = a[x,z] + b[y,z])
Antisymmetry: ([x,y] = -[y,x])
Jacobi Identity: ([x, [y,z]] + [y, [z,x]] + [z, [x,y]] = 0)
For matrices, the Lie bracket is the commutator: [A, B] = AB – BA

Structure constants
Structure constants (or structure coefficients) are the numerical values that uniquely define the algebraic multiplication in a Lie algebra. They are the coefficients that appear when the Lie bracket of any two basis vectors is expressed as a linear combination of the basis itself.
For a Lie algebra g equipped with a basis $\{e_i\}$, the Lie bracket $[e_i, e_j]$ of any two basis elements produces a new element within the algebra. Because of bilinearity, you only need to know the products of the basis elements to determine the multiplication for the entire algebra.
The basis vectors (also known as generators) satisfy the commutation relation:
$$\left[e_i, e_j\right] = c_{ij}^k e_k$$
where:
c_{ij}^k are the structure constants , and Einstein summation notation is implied (sum over k).
Structure constants inherently possess specific mathematical properties:
• Anti-symmetry: Since the Lie bracket is anti-symmetric, swapping the first two indices of the structure constants flips the sign:
$$c_{ij}^k = -c_{ji}^k$$
• The Jacobi Identity: The Jacobi identity into a crucial constraint on the structure constants:
$$c_{ij}^m c_{mk}^k + c_{jk}^m c_{mi}^i + c_{ki}^m c_{mj}^j = 0$$
• Killing Form: The structure constants are directly used to compute the Killing form, a symmetric bilinear form that helps classify whether a Lie algebra is semisimple or solvable:
$$\kappa_{ij} = c_{ib}^a c_{ja}^b$$
The properties antisymmetry and Jacobi identity are sufficient in order to define a Lie algebra.

Adjoint representation
The mapping $ad_x(y) = [x, y]$ is called the adjoint of $x \in G$ of Lie algebra *G*.
Derivation
A derivation on a Lie algebra G is a linear map D:G→G that satisfies the Leibniz rule for all elements $x, y \in G$
$$D([x, y]) = [Dx, y] + [x, Dy]$$
The collection of all such derivations forms a Lie algebra known as the derivation Lie algebra, denoted *Der(G)* .
The key aspects of derivations are
• The Bracket Operation: The Lie bracket on Der(G) is the standard commutator operation, defined for any two derivations $D_1, D_2 \in Der(G)$ as:
$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$$
• Inner Derivations: For any element $x \in G$, the adjoint operator $ad_x(y) = [x, y]$ is a derivation. These are called inner derivations and they form an ideal inside *Der(G)* .

Killing form
The Killing form is the bilinear symmetric form $\kappa(x, y) = Tr(ad(x) \circ ad(y))$, with adjoint $ad_x(y) = [x, y]$, for basis vectors $\kappa_{ij} = c_{ib}^a c_{ja}^b$ expressed by the structure constants.
The Killing form characterizes the Lie algebra, and has the properties:
1. Cartan's First Criterion (Detecting Solvability)
A finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is solvable if and only if the Killing form vanishes identically on the commutator subspace:
$\kappa(x, y) = 0$ for all $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$
2. Cartan's Second Criterion (Detecting Semisimplicity)
A finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is semisimple if and only if its Killing form is non-degenerate.
Non-degeneracy means that if
$\kappa(x, y) = 0$ for all $y \in \mathfrak{g}$, then $x = 0$, i..e. $\kappa_{ij} = \kappa(e_i, e_j)$ has full rank $\det(\kappa_{ij}) \neq 0$

3. Nilpotent Lie Algebras
If a Lie algebra $\mathfrak{g}$ is nilpotent, its adjoint operators ad(x) are nilpotent endomorphisms, meaning all their eigenvalues are zero. Consequently, the Killing form is identically zero.
4. Compact Real Forms and Definiteness
For real semisimple Lie algebras, the Killing form provides a characterization based on its signature (the number of strictly positive and negative eigenvalues).
A real semisimple Lie algebra is compact (the Lie algebra of a compact Lie group) if and only if the Killing form is strictly negative definite.
In this case, all eigenvalues of the quadratic matrix $\kappa_{ij} = \kappa(e_i, e_j)$ are strictly negative.
5 Associativity (Adjoint Invariance): The Killing form is invariant under the Lie bracket, meaning that for any $x, y, z \in \mathfrak{g}$, it respects cyclic permutation:
$\kappa([x, y], z) = \kappa(x, [y, z])$
6 Orthogonal Ideals: If $\mathfrak{i}$ is an ideal in $\mathfrak{g}$, its orthogonal complement with respect to the Killing form is also an ideal.
$x, y \in g$ are orthogonal resp. $\kappa_{ij} = \kappa(e_i, e_j)$, if their vectors x_b, y_b in the basis b={e_i} satisfy
$x_b \cdot \kappa(e_i, e_j) \cdot y_b = 0$

4 Fundamental theorems
The Fundamental Theorems of Lie Theory—often called Lie's Theorems—form the backbone of the relationship between Lie groups (continuous symmetries) and Lie algebras (their infinitesimal counterparts). They establish a direct dictionary between the calculus of smooth manifolds and linear algebra.
Lie's Three Theorems
The core of Lie theory revolves around three foundational theorems that connect a Lie group G with its Lie algebra $\mathfrak{g}$ (typically represented as the tangent space at the group's identity element).
Lie's First Theorem: This states that every finite-dimensional (local) differentiable manifold or group can be embedded into a local Lie group. It establishes that the tangent space at any point fundamentally characterizes the infinitesimal transformations of the group.
Lie's Second Theorem: This theorem establishes a direct link between the structure of a Lie group and its algebra. It states that two elements (x,y) correspond to the commutator product, meaning the structure of the algebra is entirely determined by its structure constants.
Lie's Third Theorem: This is the most profound result. It states that every finite-dimensional, real Lie algebra is the Lie algebra of some unique, connected, and simply connected Lie group.
The Homomorphisms and Subgroups Theorems
The Homomorphisms Theorem: If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, and the domain group G is simply connected, there exists a unique global Lie group homomorphism $\Phi : G \to H$ whose derivative at the identity maps to φ.
The Subgroups–Subalgebras Theorem: For any Lie group G and any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there is a unique connected Lie subgroup $H \subset G$ that has $\mathfrak{h}$ as its Lie algebra.
Engel's Theorem
This theorem applies to nilpotent Lie algebras. It states that a finite-dimensional Lie algebra $\mathfrak{g}$ is nilpotent if and only if every element in the adjoint representation is a nilpotent endomorphism. Consequently, there exists a common basis in which all matrices representing elements of $\mathfrak{g}$ are strictly upper triangular.
Lie's Theorem solvable algebras
This applies to solvable Lie algebras. It states that if $\mathfrak{g}$ is a solvable Lie algebra over an algebraically closed field of characteristic zero (like C), and $\rho : \mathfrak{g} \to End(V)$ is a finite-dimensional representation (on a matrix space V), then all elements of $\mathfrak{g}$ can be represented by upper triangular matrices simultaneously.
Cartan's Criterion
This provides a powerful test for the semisimplicity of a Lie algebra. It states that a finite-dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form (a symmetric bilinear form defined by the trace of the adjoint representations) is non-degenerate.
Levi's Decomposition Theorem

This is a cornerstone for the structure theory of all finite-dimensional Lie algebras. It asserts that any finite-dimensional Lie algebra $\mathfrak{g}$ can be decomposed as a semi-direct sum of its radical (the unique maximal solvable ideal) and a semisimple subalgebra (known as the Levi factor).

5 Roots, weights, Cartan subalgebra, Dynkin diagrams

Root system

[19]

Let E be a finite-dimensional Euclidean vector space, with the standard Euclidean inner product denoted by (·,·). A root system Φ in E is a finite set of non-zero vectors (called roots) that satisfy the following conditions:

1 The roots span E.

2 The only scalar multiples of a root α∈Φ that belong to Φ are α itself and −α.

3 For every root α∈Φ, the set Φ is closed under reflection through the hyperplane perpendicular to α.

4 (Integrality) If α and β are roots in Φ, then the projection of β onto the line through α is an integer or half-integer multiple of α.

Equivalent ways of writing conditions 3 and 4, respectively, are as follows:

3 For any two roots α,β∈Φ, the set Φ contains the projection

$$\sigma_\alpha(\beta)=\beta-2\alpha\frac{(\alpha,\beta)}{(\alpha,\alpha)}$$

4 For any two roots α,β∈Φ , the number (double projection β onto α)

$$\langle\beta,\alpha\rangle=\alpha\frac{(\alpha,\beta)}{(\alpha,\alpha)}$$

is an integer.

The **rank** of a root system Φ is the dimension of *E*.

Two root systems (E1, Φ1) and (E2, Φ2) are called isomorphic if there is an invertible linear transformation E1 → E2 which sends Φ1 to Φ2 such that for each pair of roots, the projection ⟨x,y⟩ is preserved

The group of isometries of E generated by reflections through hyperplanes associated to the roots of Φ is called the Weyl group of Φ. As it acts faithfully on the finite set Φ, the Weyl group is always finite.

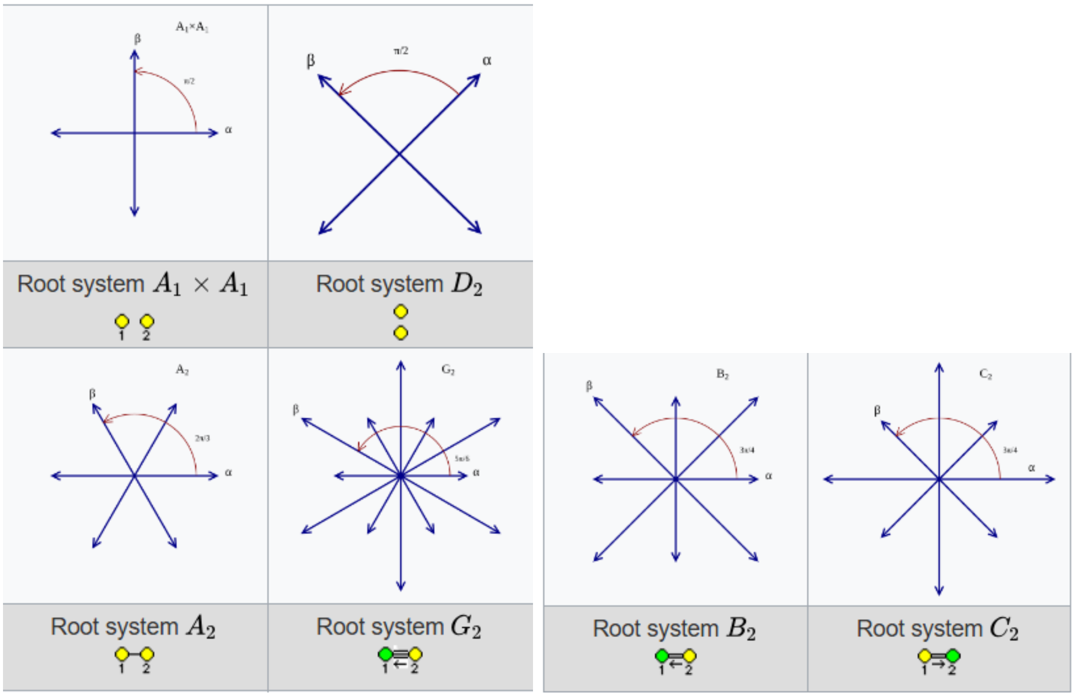
Root system rank 2

In rank 2 there are four possibilities, corresponding to $\sigma_\alpha(\beta)=\beta+n\alpha$, where n=0,1,2,3 .

The root system is not determined by the lattice that it generates: A1×A1 and B2 both generate a square lattice while A2 and G2 both generate a hexagonal lattice.

Whenever Φ is a root system in E, and S is a subspace of E spanned by Ψ = Φ ∩ S, then Ψ is a root system in S. Thus, the exhaustive list of four root systems of rank 2 shows the geometric possibilities for any two roots chosen from a root system of arbitrary rank.

In particular, two such roots must meet at an angle of 0, 30, 45, 60, 90, 120, 135, 150, or 180 degrees.



Rank 2 root systems

Lie Algebras roots and weights
[20]
Lie Algebra Root
The roots of a semisimple Lie algebra g are the Lie algebra weights occurring in its adjoint representation. The set of roots form the root system, and are completely determined by g. It is possible to choose a set of Lie algebra positive roots, every root α is either positive or -α is positive. The Lie algebra simple roots are the positive roots which cannot be written as a sum of positive roots.
The simple roots can be considered as a linearly independent finite subset of Euclidean space, and they generate the root lattice.
For example, in the special Lie algebra $sl_2(C)$ of two by two matrices with zero matrix trace, has a basis given by the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The adjoint representation is given by the brackets

$$ad(H)(X) = [H, X] = 2X$$

$$ad(H)(Y) = [H, Y] = -2Y$$

so there are two roots of $sl_2(C)$ given by α(H)=2 and - α(H)=-2. The Lie algebraic rank of $sl_2(C)$ is one, and it has one positive root α(H)=2.
Adjoint representation
The adjoint representation of a Lie group G is a way of representing the elements of the group as linear transformations of the group's Lie algebra, considered as a vector space.
For example, if G =GL(n,R), the Lie group of real n-by-n invertible matrices, then the adjoint representation is the group homomorphism that sends an invertible nxn matrix g to an endomorphism of the vector space of all linear transformations of R^n defined by:

$$x \mapsto g\,x\,g^{-1}$$

Roots in e_i basis
For a Lie algebra with dimension *d=2n* or *d=2n+1* (e.g. orthogonal Lie algebras so(2n) , so(2n+1), sp(2n)) we use an orthonormal basis of extended Kronecker matrices $i, j = 1 \ldots n$

$$E^1_{jk} = E_{jk} - E_{k+n,j+n}\,, \quad E^2_{jk} = E_{j,k+n} - E_{k,j+n}\,, \quad E^3_{jk} = E_{j+n,k} - E_{k+n,j}\,, \quad E^3_{jk} = E_{j+n,k} - E_{k+n,j}\,,$$

where the ordinary Kronecker matrices are $E_{ij} = \left(\delta_{ik}\delta_{jl}\right)_{kl}$.

For su(n+1), we use the basis $E^1_{jk} = E_{jk}$.

Cartan subalgebra is spanned by the diagonal operators $h = \langle E_{ii} \rangle$,

the remaining subspace is spanned by the non-diagonal operators $h_\perp = \langle E^1_{ij} \rangle \ i \neq j$

We make the ansatz for the Cartan subalgebra h : $H = \sum_{i=1}^{n} \lambda_i E_{ii}$ (n=dim(h)

we obtain the commutator relations

$[H, E^1_{jk}] = (\lambda_j - \lambda_k) E^1_{jk} \quad i \neq j$

$[H, E^2_{jk}] = (\lambda_j + \lambda_k) E^2_{jk} \quad i \leq j$

$[H, E^3_{jk}] = -(\lambda_j + \lambda_k) E^3_{jk} \quad i \leq j$

For the classical Lie algebras we obtain simple root vectors e_i in the form

$e_1 = E^1_{12}$, $e_2 = E^1_{23}$, ..., $e_n = f(E_{ij})$

with the commutators

$r_1 = [H, e_1] = (\lambda_1 - \lambda_2) e_1$

$r_2 = [H, e_2] = (\lambda_1 - \lambda_2) e_2$

...

$r_n = [H, e_n] = (c_{n-1}\lambda_{n-1} + c_n \lambda_n) e_n$

Here, the final term differs

for su(n+1) $r_n = (\lambda_n - \lambda_{n+1}) E_{n,n+1}$

for so(2n) $r_n = (\lambda_{n-1} + \lambda_n)(E_{2n-1,n} - E_{2n,n-1})$

for so(2n+1) $r_n = \lambda_n (E_{n,2n+1} - E_{2n+1,2n})$

for sp(2n) $[X_i, X_j] = c^k_{ij} X_k$

Roots in Weyl-Cartan representation

We start for the Lie algebra g with the generator set $g = \langle H_1, ..., H_k, Y_1, ..., Y_{n-k} \rangle$,

where the Cartan algebra h is spanned by the first k generators $h = \langle H_1, ..., H_k \rangle$.

The form ladder operators $L_{1\pm} = Y_1 \pm Y_2, ..., L_{(n-k-1)/2\pm} = Y_{n-k-1} \pm Y_{n-k}$ and let them span the remaining subspace $h_\perp = \langle L_{1\pm}, ..., L_{(n-k-1)/2\pm} \rangle$

The mapping $ad(H_i)(X) = [H_i, X] \ X \in h_\perp$ maps the subspace $h_\perp$ into itself, it is a *(n-k)x(n-k)* matrix $ad(H_i)$ on the vectors of $h_\perp$. The k common eigenvectors of these matrices, which belong to the eigenvalues $(\lambda_{p1}(H_1), ..., \lambda_{pk}(H_k))$, *p=1,..k*

are the k simple roots of the Lie algebra g.

Lie Algebra Weight

Consider a collection of diagonal matrices $H_1, ..., H_k$, which span a subspace h. Then the ith eigenvalue, i.e., the ith entry along the diagonal, is a linear functional on h, and is called a weight.

The general setting for weights occurs in a Lie algebra representation of a semisimple Lie algebra, in which case the Cartan subalgebra h is Abelian and can be put into diagonal form.

For example, consider the standard representation of the special linear Lie algebra sl3(C) on C^3.

Then $H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

span the Cartan subalgebra *h*.

There are three eigenvectors (standard orthonormal basis) e_i with $e_i[j] = \delta_{ij}$

There are three weights,

$\alpha = (1,-2,1)$,

which are the eigenvalues of $H_1 + H_2$,

corresponding to the decomposition of

$C^3 = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \langle e_3 \rangle$

into its eigenspaces.

Note that $\alpha_1 + \alpha_2 + \alpha_3 = 0$, because the matrices have zero matrix trace. The eigenvectors e_1, e_2, e_3 are called weight vectors, and the corresponding eigenspaces are called weight spaces.

In the important special case of the adjoint representation of a semisimple Lie algebra, the weights are called Lie algebra roots and the weight space is called the root space. The roots generate a discrete lattice, called the root lattice, in the dual vector space *h**. The set of all possible weights forms a weight lattice, which contains the root lattice. The Lie Algebra representations of g can be classified using the weight lattice.

Cartan matrix

A Cartan matrix A=(a_{ij}) is an nxn integer matrix that uniquely encodes the structure and root system of a semisimple Lie algebra. It captures the angular and length relationships between the simple roots.

For the simple roots (α_i, α_j) the matrix entries are defined as:

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

- Diagonal entries are always a_{ij}=2.
- Off-diagonal entries are non-positive integers $a_{ij} \le 0 \quad i \ne j$ and represent how roots reflect into one another.
- The product a_{ij} a_{ji} can only equal 0,1,2,3 .

The Cartan matrices of the types An, Bn, Cn, Dn are

- A_n(su(n+1))

$$\begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 \\ -1 & 2 & -1 & ... & 0 & 0 \\ 0 & -1 & -2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

- B_n(so(2n+1))

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & -2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

- C_n(sp(2n))

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & -2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{pmatrix}$$

- D_n(so(2n))

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & -2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 & 2 \end{pmatrix}$$

Five exceptional algebras:

- E6: A (6 times 6) matrix characterized by a "Y" shape on its Dynkin diagram with a central branch.
- E7: A (7 times 7) matrix corresponding to a 7-vertex diagram.
- E8: An (8 times 8) matrix associated with the largest simple exceptional Lie group.
- F4: A (4 times 4) matrix corresponding to the 24-dimensional exceptional Lie algebra.
- G2: The smallest exceptional Lie algebra (14-dimensional), whose Cartan matrix is $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

Simple roots and weights

In general, for a simple Lie algebra g, the relation between the simple roots α_i and the fundamental weights ω_j reads

α_i = j A_{ij} ω_j ,

where A is the Cartan matrix of g

Cartan Subalgebra

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over some field *K*.

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a Cartan subalgebra if it is nilpotent and equal to its normalizer, which is the set of those elements $x \in \mathfrak{g}$ such that $[x, \mathfrak{h}] \subset \mathfrak{h}$.

Lie algebra $\mathfrak{g}$ is nilpotent if its lower central series terminates in the zero subalgebra. The lower central series is the sequence of subalgebras

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supseteq \dots$$

Let $\mathfrak{g}$ be the Lie algebra of all endomorphisms *f* of vector space k^n (for some natural number n), with

$$[f, g] = f \circ g - g \circ f$$

Then the set of all endomorphisms *f* of k^n of the form

$$f(x_1, \dots, x_n) = (\lambda_1 x_1, \dots, \lambda_n x_n)$$

$$f\ (x_1,\ \dots,\ x_n) = (\lambda_1\ x_1,\ \dots,\ \lambda_n\ x_n)$$

is a Cartan subalgebra of $\mathfrak{g}$.

Properties of Cartan subalgebra

1. If K is infinite, then $\mathfrak{g}$ has Cartan subalgebras.
2. If the characteristic of K is equal to 0, then all Cartan subalgebras of $\mathfrak{g}$ have the same dimension.
3. If K is algebraically closed and its characteristic is equal to 0, then, given two Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ of $\mathfrak{g}$, there is an automorphism f of $\mathfrak{g}$ such that $f(\mathfrak{h})=\mathfrak{h}'$ f(h)=h'.
4. If $\mathfrak{g}$ is semisimple and K is an infinite field whose characteristic is equal to 0, then all Cartan subalgebras of $\mathfrak{g}$ are Abelian.
5. Over field C, a Cartan subalgebra is precisely equivalent to a maximal commutative (or abelian) subalgebra.
6. In general, if a maximal commutative subalgebra contains non-semisimple (nilpotent) elements, it fails to be a Cartan subalgebra.
7. Every Cartan subalgebra of a Lie algebra $\mathfrak{g}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$. However, a maximal nilpotent subalgebra of $\mathfrak{g}$ doesn't have to be a Cartan subalgebra.

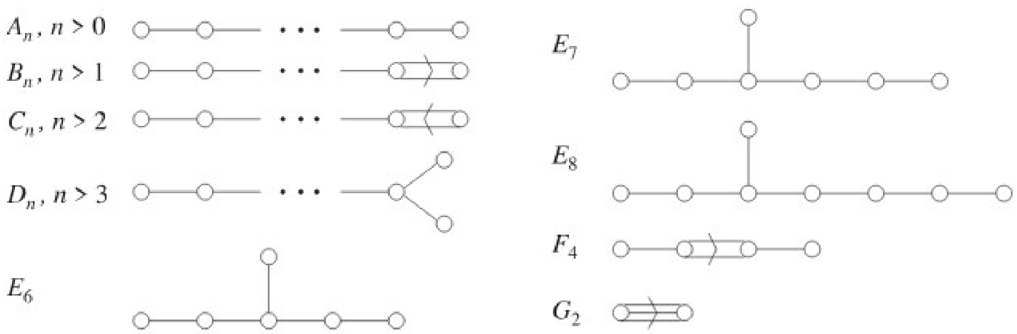
Semisimple Lie Algebra

A Lie algebra over a field of characteristic zero is called semisimple if its Killing form is nondegenerate. The following properties can be proved equivalent for a finite-dimensional algebra L over a field of characteristic 0:

1. L is semisimple.
2. L has no nonzero Abelian ideal.
3. L has zero ideal radical (the radical is the biggest solvable ideal).
4. Every representation of L is fully reducible, i.e., is a sum of irreducible representations.
5. L is a (finite) direct product of simple Lie algebras (Lie algebra is called simple if it is not Abelian and has no nonzero ideal smaller than L).

The radical of an ideal $\mathfrak{a}$ in a ring R is the ideal $r(\mathfrak{a})=\{x;\exists n>0: x^n\in\mathfrak{a}\}$

Dynkin Diagram



Every semisimple Lie algebra $\mathfrak{g}$ is classified by its Dynkin diagram. A Dynkin diagram is a graph with a few different kinds of possible edges. The connected components of the graph correspond to the irreducible subalgebras of $\mathfrak{g}$. So a simple Lie algebra's Dynkin diagram has only one component.

In fact, there are only certain possibilities for each component, corresponding to the classification of semi-simple Lie algebras.

The roots of a complex Lie algebra form a lattice of rank k in a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$, where k is the Lie algebra rank of $\mathfrak{g}$. Hence, the root lattice can be considered a lattice in R^k.

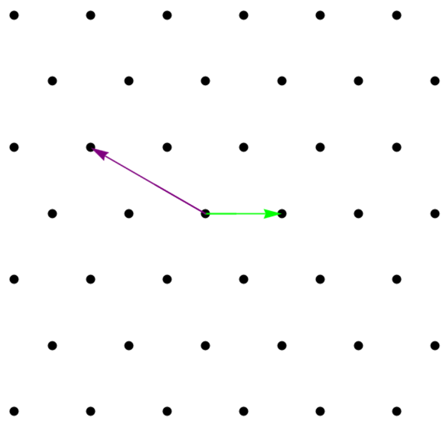
A vertex, or node, in the Dynkin diagram is drawn for each Lie algebra simple root, which corresponds to a generator of the root lattice. Between two nodes α and β, an edge is drawn if the simple roots are not perpendicular.

One line is drawn if the angle θ between them is $2\pi/3$, two lines if the angle is $3\pi/4$, and three lines are drawn if the angle is $5\pi/6$. There are no other possible angles between Lie algebra simple roots.

Alternatively, the number of lines N between the simple roots α and β is given by

$$N=A_{\alpha\beta}A_{\beta\alpha}=\frac{2\langle\alpha,\beta\rangle}{|\alpha|^2}\frac{2\langle\beta,\alpha\rangle}{|\beta|^2}=\cos^2\theta$$

where $A_{\alpha\beta}$ is an entry in the Cartan matrix. In a Dynkin diagram, an arrow is drawn from the longer root to the shorter root (when the angle θ is $3\pi/4$ or $5\pi/6$).



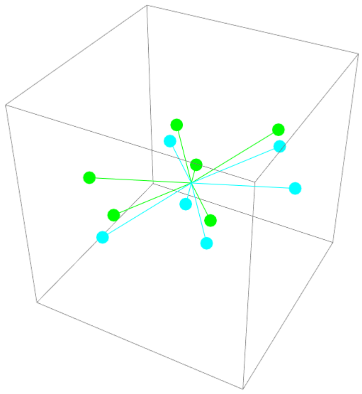
The picture above shows the two simple roots for G_2, at an angle of $5\pi/6$, in the root lattice. Therefore, the Dynkin diagram for G_2 has two nodes, with three lines between them.
Here are some properties of admissible Dynkin diagrams.
Recover a simple Lie algebra from its Dynkin diagram
The simplest way to recover a simple Lie algebra from its Dynkin diagram is to first reconstruct its Cartan matrix A_{ij}. The ith node and jth node are connected by $A_{ij}A_{ji}$ lines. Since $A_{ij}=0$ iff $A_{ji}=0$, and otherwise $A_{ij}\in\{-3,-2,-1\}$, it is easy to find A_{ij} and A_{ji}, up to order, from their product.
The arrow in the diagram indicates which is larger. For example, if node 1 and node 2 have two lines between them, from node 1 to node 2, then $A_{12}=-1$ and $A_{21}=-2$.

Weyl Group
Let *L* be a finite-dimensional split semisimple Lie algebra over a field *K* of field characteristic 0, *H* a splitting Cartan subalgebra, and Λ a weight of *H* in a representation of *L*. Then

$$\Lambda'=\Lambda S_\alpha=\lambda-\frac{2(\Lambda,\alpha)}{(\alpha,\alpha)}(\alpha)$$

$$\Lambda' = \Lambda S_\alpha = \lambda - \frac{2\,(\Lambda,\,\alpha)}{(\alpha,\,\alpha)}\,(\alpha)$$

is also a weight. Furthermore, the reflections S_α with root α, generate a group of linear transformations in H_0^* called the Weyl group W of *L* relative to *H*, where H^* is the algebraic conjugate space of *H* and H_0^* is the Q-space spanned by the roots.
The Weyl group acts on the roots of a semisimple Lie algebra, and it is a finite group.



Weyl Group acting on the roots of the Cartan matrix of the infinite family of semisimple lie algebras A_3

Calculation of the simple roots of a simple Lie algebra
To calculate the simple roots of a simple Lie algebra, you find a linearly independent subset of the algebra's roots that cannot be written as the sum of other positive roots. These simple roots act as a basis for the root system, meaning every other root can be expressed as a linear combination of these roots with non-negative integer coefficients.
The number of simple roots will always equal the rank of the Lie algebra (the dimension of the Cartan subalgebra).

For standard classical Lie algebras (e.g. A_i , B_i , C_i , D_i the simple roots are commonly defined using the standard orthonormal basis vectors e_i with factors λ_i

- An =su(n+1) The simple roots are $\alpha_i = \lambda_i - \lambda_{i+1}$ for $i=1,...,n$
- Bn =so(2n+1) The simple roots are $\alpha_i = \lambda_i - \lambda_{i+1}$ for $i=1,...,n-1$; $\alpha_n = \lambda_n$
- Cn =sp(2n) The simple roots are $\alpha_i = \lambda_i - \lambda_{i+1}$ for $i=1,...,n-1$; $\alpha_n = 2\lambda_n$
- Dn =so(2n) The simple roots are $\alpha_i = \lambda_i - \lambda_{i+1}$ for $i=1,...,n-1$; $\alpha_n = \lambda_{n-1} + \lambda_n$

6 Classical and exceptional Lie algebras

Four classical Lie algebras

The classical Lie algebras are finite-dimensional Lie algebras over a field which can be classified into four types An , Bn , Cn and Dn , where gl(n) is the general linear Lie algebra and I_n = n×n identity matrix

An =su(n+1)=sl(n+1,C) $=\{x\in gl(n+1,C),\overline{x}^T=-x,tr(x)=0\}$ the special linear complex Lie algebra

Bn:=s*o(2n+1)*= $\{x\in gl(2n+1),x^T=-x\}$, the odd orthogonal Lie algebra;

Cn:=*sp(2n)*= $\left\{x\in gl(2n),x^T J_n=-J_n x,J_n=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}\right\}$, the symplectic Lie algebra;

Dn:=s*o(2n)*= $\{x\in gl(2n),x^T=-x\}$ the even orthogonal Lie algebra.

Except for the low-dimensional cases D_1=so(2) and D_2=so(4) , the classical Lie algebras are simple .

Five Exceptional Lie Algebras

- g_2 (Dimension 14): The smallest of the exceptional Lie algebras, it is the automorphism group of the octonions and the stabilizer of a generic 3-form in 7 dimensions.
- f_4 (Dimension 52): The Lie algebra of the exceptional group F_4, it is related to the symmetries of the Albert algebra (a 27-dimensional exceptional Jordan algebra).
- e_6 (Dimension 78): Can be understood through the stabilizer of a certain cubic form and appears in the classification of certain projective planes over division algebras.
- e_7 (Dimension 133): Leaves invariant a symmetric rank-4 tensor in its fundamental 56-dimensional representation.
- e_8 (Dimension 248): The largest of the exceptional Lie algebras. It is heavily associated with the E_8 lattice, which models the densest possible sphere packing in 8-dimensional space.

6.1 General examples

- **Four-dim nilpotent Lie-algebra**

Every 4-dimensional nilpotent Lie algebra over a field K of characteristic zero is either abelian or isomorphic to the direct sum $n_3(K)\oplus K$, where $n_3(K)$ denotes the 3-dimensional Heisenberg Lie algebra, or to the filiform nilpotent Lie algebra $f_4(K)$ with adapted basis (e_1,…e_4) and brackets determined by $[e_1,e_i]=e_{i+1}$ for i=2,3. This is well known and not difficult to show. Obviously, both algebras contain an abelian ideal of dimension 3. For example, $\langle e_2,e_3,e_4\rangle$ is an abelian ideal in $f_4(K)$.

- **Lie4(ε) algebra**

The 4 generators T_i are $(T_j)_{ik}=\varepsilon_{ijk}$.

Commutator relations are

$[T_k,T_m]=\varepsilon_{kmi}T_i$

where ε_{kmi} is the 3-dimensional Levi-Civitta (antisymmetric) 4x4x4 tensor with indices k,m,i=1,...,4 .

The structure constants $c^i_{km}=\varepsilon_{kmi}$ satisfy

antisymmetry $c^i_{km}=-c^i_{mk}$

and Jacobi-identity $c^m_{ij}c^k_{mk}+c^m_{jk}c^i_{mi}+c^m_{ki}c^j_{mj}=0$

so they define a Lie algebra [19].

However, the four generators T_i are linearly dependent, so the Lie algebra is 3-dimensional and has to be reformulated.

The 4x4 Killing matrix is $\begin{pmatrix}-6 & -2 & 2 & -2\\ -2 & -6 & -2 & 2\\ 2 & -2 & -6 & -2\\ -2 & 2 & -2 & -6\end{pmatrix}$ [22]

has eigenvalues (-8,-8,-8,0),

so the reduced 3x3 Killing matrix has eigenvalues (-8,-8,-8)

- **3-dimensional Lie3(ε) algebra**

Lie3(ε) is a reformulation of the Lie4(ε) algebra, which results from the replacement of the generator T_4 by a linear combination of (T_1,T_2,T_3) .
Commutator relations are
$[T_1,T_2]=T_1-T_2+2T_3$, $[T_1,T_3]=T_1-2T_2+T_3$, $[T_2,T_3]=2T_1-T_2+T_3$
with generic formula $[T_a,T_b]=\sum_i (1+\varepsilon_{abi})T_i$
The 3-dimensional Lie algebra defined by these commutator relations is isomorphic to so(3) over reals R or sl(2,C) over complex nmbers C.

The Killing matrix is $\begin{pmatrix} -6 & -2 & 2 \\ -2 & -6 & -2 \\ 2 & -2 & -6 \end{pmatrix}$ and has eigenvalues (-8,-8,-2).

Characterization of Lie3(ε)
The derived algebra [g,g] in orthonormal basis e_i is spanned by the vectors
$e_1-e_2+2e_3$, $e_1-2e_2+e_3$, $2e_1-e_2+e_3$

The corresponding coefficient matrix is $M=\begin{pmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$, det(M)=4, so dim([g,g])=3 .
Killing form is negative-definite, and by Cartan's criterion, a semisimple real Lie algebra is compact if and only if its Killing form is negative-definite.
Cartan subalgebra has dim=1, the simple root is λ_1, the 2 roots are $\pm\lambda_1$
The unique 3-dimensional compact simple real Lie algebra is so(3), isomorphic to su(2).

- **Heisenberg algebra**

The Lie algebra $\mathfrak{h}$ of the Heisenberg group H (over the real numbers) is known as the Heisenberg algebra.
It may be represented using the space of 3×3 matrices of the form
$$\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$
with a,b,c∈R.
The following three generators form a basis for $\mathfrak{h}$:
$$x=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ p=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ z=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
they are all nilpotent $p^2=y^2=z^2=0$
These basis elements satisfy the commutation relations
$[x,p]=z$, $[x,z]=0$, $[p,z]=0$
Cartan subalgebra is generated by Z, so rank(H)=1.
The Heisenberg algebra H is the Lie algebra of the conjugated variables in quantum mechanics, i.e. it has the Heisenberg commutation relations
$[x,p]=i\hbar I$, $[x,i\hbar I]=0$, $[p,i\hbar I]=0$
where x,p are the position resp. the momentum operator, and $\hbar$ is the Planck constant, $z=i\hbar I$ is the Cartan generator (i.e. a multiple of the identity operator).
The Heisenberg group generated by the 3 generators has the form
$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$
with a,b,c∈R.

The Heisenberg group H has the special property that the exponential map is a bijective (one-to-one) map from the Lie algebra $\mathfrak{h}$ to the Heisenberg group H

$$\exp\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a & c+ab/2 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

• Heisenberg algebra in $\mathbf{R^{n+2}}$

Heisenberg algebra with dimension $2n+1$ has the form of (n+2)x(n+2) matrices

$$\begin{pmatrix} 0 & a & c \\ 0_n & 0_{nxn} & b \\ 0 & 0_n & 0 \end{pmatrix}$$ with n-vectors a, b, and scalar c $\in$R .

With the canonical basis $(e_1, e_2, ..., e_n)$ in R^n , we obtain for the generators

$$x_i = \begin{pmatrix} 0 & e_i & 0 \\ 0_n & 0_{nxn} & 0_n \\ 0 & 0_n & 0 \end{pmatrix}, \; p_i = \begin{pmatrix} 0 & 0_n & 0 \\ 0_n & 0_{nxn} & e_i \\ 0 & 0_n & 0 \end{pmatrix}, \; z = \begin{pmatrix} 0 & 0_n & 1 \\ 0_n & 0_{nxn} & 0_n \\ 0 & 0_n & 0 \end{pmatrix}$$

with the commutators

$$\left[x_i, p_j\right] = \delta_{ij} z \; , \; \left[p_j, z\right] = 0 \; , \; \left[x_i, z\right] = 0$$

Here, p_i are the generalized momentums, and x_i are the generalized positions, z is the Cartan operator corresponding to $i\hbar I$ in quantum mechanics.

• Heisenberg algebra in quantum mechanics

The generalized momentums p_i are derived from the Lagrangian L and have the properties

$$p_i = \frac{\partial L}{\partial(\partial x_i / \partial t)}$$, Euler-Lagrange equation: $$\frac{\partial}{\partial t} p_i = \frac{\partial L}{\partial x_i}$$

The Heisenberg algebra yields for them the commutator relations

$$\left[x_i, p_j\right] = i\hbar\delta_{ij}$$

and the uncertainty relations $\sigma(x_i)\sigma(p_j) \geq \frac{\hbar}{2}\delta_{ij}$ where $\sigma(x) = \sqrt{\left\langle (x-\bar{x})^2 \right\rangle}$ is the standard deviation.

6.2 Special linear su-Lie-algebras su(n+1), Cartan type An

[21]

The special unitary Lie algebra su(n+1) consists of (n+1)x(n+1) skew-Hermitian matrices with a trace of zero. It is the compact real form of the complex special linear Lie algebra sl(n+1,C). Its Cartan structure corresponds to the classical An.

su(n+1) is compact and semi-simple.

su(n+1) has the properties:

-dim=$(n+1)^2$-1

-rank=n , dimension of the Cartan subalgebra

-the n(n+1) roots are all permutations of (1, -1, 0, ..., 0)

The simple roots are $\alpha_i = e_i - e_{i+1}$ for i=1,...,n+1

The generators are traceless skew-symmetric matrices Ti , i=1...$(n+1)^2$-1

with commutator relations $[T_a, T_b] = c_{ab}^k T_k$

In the $((n+1)^2 - 1)$-dimensional adjoint representation, the generators are represented by $((n+1)^2 - 1) \times ((n+1)^2 - 1)$ matrices, whose elements are defined by the structure constants themselves.

$(T_a)_{jk} = c_{jk}^a$

The Killing form is $\kappa(x, y) = 2(n+1)\, tr(x\, y)$

Roots in ei-basis

We use as basis the matrices $E_{ij} = (\delta_{ik}\delta_{jl})_{kl}$.

We make the ansatz for the Cartan subalgebra $H = \sum_{i=1}^{n+1} \lambda_i E_{ii}$, $\sum_{i=1}^{n+1} \lambda_i = 0$,

and for the root vectors $e_i = E_{i,i+1}$, *i=1...n*

We obtain the commutator relations

$[H, e_i] = (\lambda_i - \lambda_{i+1}) e_i$

and the simple roots $\alpha_i = (\lambda_i - \lambda_{i+1})$, *i=1...n*

- **su(2) algebra**

A Lie algebra with structure constants proportional to the totally antisymmetric tensor ε_{abc} is the su(2) algebra. Its generators satisfy $[T_a, T_b] = 2i\varepsilon_{abc} T_c$, where ε_{abc} is the Levi-Civita symbol. This algebra describes the rotational symmetry of 3D space and the spin of quantum particles.

The generators T_a are the Pauli matrices σ_i

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The rank is 1, meaning the maximal abelian subalgebra (Cartan subalgebra h) is 1-dimensional.

Cartan generator: It is typically chosen as the diagonal Pauli matrix:

$H = \sigma_z$

The remaining dimensions of the 3-dimensional algebra form the root spaces. They are spanned by the ladder (raising and lowering) operators

raising operator $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E = \sigma_x + i\sigma_y$

lowering operator $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $F = \sigma_x - i\sigma_y$

The action of H on E, F is

$[H, E] = 2E$, $[H, F] = -2F$

The roots in Weyl-Cartan representation are $\alpha_i = \{+2, -2\}$, normalized simple root $\alpha = 1$ (one-dimensional)

In orthonormal basis we have : the simple root is $\alpha_1 = \lambda_1$, the 2 roots are $\pm\lambda_1$

- **su(3) algebra**

[21]
The su(3) Lie algebra is the real vector space of 3 x 3 skew-Hermitian, traceless matrices. It is an 8-dimensional Lie algebra of rank 2. Its generators are $T_a = i\,\lambda_a/2$, a=1...8, where λ_a are the eight traceless and Hermitian Gell-Mann matrices. The fundamental representations are used heavily in particle physics, such as the Eightfold Way and Quantum Chromodynamics.
The commutator relations are $[T_a, T_b] = c_{ab}^k T_k$

The 8 Gell-Mann matrices λ_a are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The structure constants are [22]

$$\left(\begin{array}{cccccccc}
\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\-1\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\-\frac{1}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\\frac{1}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\-\frac{1}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\\frac{1}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}-1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\-\frac{1}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\-\frac{1}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\\frac{1}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\\frac{1}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\-1\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\-\frac{1}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\\frac{1}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\\frac{1}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\-\frac{1}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\0\\0\\0\\0\\\frac{1}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\\frac{1}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\\frac{1}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\-\frac{1}{2}\\0\\0\\0\\0\\-\frac{\sqrt{3}}{2}\end{pmatrix} & \begin{pmatrix}0\\-\frac{1}{2}\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}-\frac{1}{2}\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\\frac{\sqrt{3}}{2}\\0\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\0\\0\\0\\-\frac{1}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\\frac{1}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\-\frac{1}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\\frac{1}{2}\\0\\0\\0\\0\\\frac{\sqrt{3}}{2}\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}\frac{1}{2}\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\-\frac{1}{2}\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\-\frac{\sqrt{3}}{2}\\0\\0\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\0\\0\\\frac{1}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\-\frac{1}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\-\frac{1}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\\frac{1}{2}\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}-\frac{1}{2}\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\\frac{1}{2}\\0\\0\\0\\0\\-\frac{\sqrt{3}}{2}\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\\frac{\sqrt{3}}{2}\\0\end{pmatrix}
\end{array}\right)$$

$$\left(\begin{array}{cccccccc}
\begin{pmatrix}0\\0\\0\\-\frac{1}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\-\frac{1}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\\frac{1}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}\frac{1}{2}\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\\frac{1}{2}\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\-\frac{1}{2}\\0\\0\\0\\0\\\frac{\sqrt{3}}{2}\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\-\frac{\sqrt{3}}{2}\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\-\frac{\sqrt{3}}{2}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\\frac{\sqrt{3}}{2}\\0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\-\frac{\sqrt{3}}{2}\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\\frac{\sqrt{3}}{2}\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}
\end{array}\right)$$

The root system for the su(3) Lie algebra is of type A2.
It consists of 6 non-zero roots spanning a two-dimensional Cartan subalgebra.

Geometrically, these roots point to the vertices of a regular hexagon, with two simple roots α1 and α2 subtending an angle of $\pi/3$.

- Rank: n-1 = 2
- Dimension: n^2-1 = 8 generators (2 Cartan diagonal generators, 6 non-diagonal root generators)

Cartan matrix $A_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ is $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Roots in Cartan-Weyl basis

Cartan-Weyl basis with integer coefficients is (2 Cartan operators $\hat{I}_3, \hat{Y}$, and 2x3 ladder operators $\hat{I}_\pm$, $\hat{U}_\pm$, $\hat{V}_\pm$)

$\hat{F}_i = \frac{1}{2}\lambda_i$, $\hat{V}_\pm = \hat{F}_4 \pm i\hat{F}_5$ $\hat{U}_\pm = \hat{F}_6 \pm i\hat{F}_7$, $\hat{I}_\pm = \hat{F}_1 \pm i\hat{F}_2$

$\hat{Y} = 2\hat{F}_8$, $\hat{I}_3 = 2\hat{F}_3$

has the commutator action of the two Cartan operators $\hat{Y}$, $\hat{I}_3$

$\left[\hat{Y},\hat{I}_\pm\right] = 0$, $\left[\hat{Y},\hat{U}_\pm\right] = \pm\sqrt{3}\hat{U}_\pm$, $\left[\hat{Y},\hat{V}_\pm\right] = \pm\sqrt{3}\hat{V}_\pm$

$\left[\hat{I}_3,\hat{I}_\pm\right] = \pm 2\hat{I}_\pm$, $\left[\hat{I}_3,\hat{U}_\pm\right] = \mp\hat{U}_\pm$, $\left[\hat{I}_3,\hat{V}_\pm\right] = \pm\hat{V}_\pm$

raw simple roots (3-vectors) are , $\beta_1 = (2,-1,1)$ $\beta_2 = \sqrt{3}(0,1,1)$ [22]

from this we obtain 2 simple roots

$\alpha_1 = (2,0)$, $\alpha_2 = \left(-1,\sqrt{3}\right)$

Positive Roots: α1, α2, (α1+ α2)

Negative Roots: -α1, -α2, -(α1+ α2)

Roots representation in e_i-basis

the 2 simple roots are $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_2 - \lambda_3$

the 6 roots are $\pm(\lambda_1 - \lambda_2)$, $\pm(\lambda_1 - \lambda_3)$, $\pm(\lambda_2 - \lambda_3)$

The Killing matrix [22] $\kappa_{ij} = c^a_{ib} c^b_{ja}$ of su(3) is $\kappa_{ij} = -diag_8\left(\{3,...,3\}\right)$

- **su(4) algebra**

[21]

The su(4) Lie algebra is the mathematical framework for the special unitary group SU(4).

It is a 15-dimensional real Lie algebra consisting of 4 × 4 traceless, skew-Hermitian matrices.

In physics, it is most commonly represented using traceless Hermitian matrices. It is isomorphic to the orthogonal algebra so(6) .

The dimension is n=15 .

The generators are $T_a = i\,\lambda_a/2$, a=1...15, where λ_a a=1...15 are extended 4 x 4 Gell-Mann matrices .

Exponentiating these generators yields the continuous group transformations of SU(4) .

The commutator relations are $\left[T_a, T_b\right] = c^k_{ab} T_k$.

su(4) is a compact, semi-simple Lie algebra with properties:

- Rank 3: There are 3 simultaneously diagonalizable matrices in the Cartan Subalgebra , denoted H_1, H_2, H_3.
- 12 Roots: The remaining 12 generators act as raising and lowering operators.
- Dynkin Diagram: Its roots map geometrically into R^3, structurally identical to the A_3 Dynkin diagram, which consists of three interconnected nodes.

The λ_a a=1...15 are extended 4 x 4 Gell-Mann matrices

for su(2) subgroup a=1...3

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

remaining Gell-Mann matrices a=4...8

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

remaining index-4 matrices a=9...15

$$\lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\lambda_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

The first 3 structure constant matrices are [22]

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\begin{pmatrix}
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0\\
0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0\\
0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
\begin{pmatrix}
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

Cartan matrix $A_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ is $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Root system Weyl-Cartan representation

We obtain from the 3x2x6 commutator relations with 2x6 ladder operators [22]:

3 action vectors of Cartan operators on ladder operators (6-vectors)

$\beta_1 = (2,1,-1,1,-1,0)$,

$\beta_2 = \sqrt{3}(0,1,1,1/3,1/3,-2/3)$,

$\beta_3 = \left(\sqrt{1/6}+\sqrt{3/2}\right)(0,0,0,1,1,1)$

we obtain 3 simple root vectors α1 α2 α3 in Weyl-Cartan representation

$\alpha_i = \left((2,0,0),\left(1,1/\sqrt{3},0\right),\left(1,1/\sqrt{3},\gamma\right)\right)$, $\gamma = \left(\sqrt{1/6}+\sqrt{3/2}\right)$

All 12 roots are $\pm\{\alpha_1,\alpha_2,\alpha_3,\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_1+\alpha_2+\alpha_3\}$

Representation in ei-basis

In orthonormal basis $\{e_1,e_2,e_3,e_4\}$ we obtain:

the 3 simple roots are $\alpha_1 = \lambda_1-\lambda_2$, $\alpha_2 = \lambda_2-\lambda_3$, $\alpha_3 = \lambda_3-\lambda_4$,

the 12 roots are $\pm(\lambda_1-\lambda_2)$, $\pm(\lambda_1-\lambda_3)$, $\pm(\lambda_1-\lambda_4)$, $\pm(\lambda_2-\lambda_3)$, $\pm(\lambda_2-\lambda_4)$, $\pm(\lambda_3-\lambda_4)$

The Killing matrix [22] $\kappa_{ij} = c^a_{ib}c^b_{ja}$ of su(4) is $\kappa_{ij} = -diag_{15}(\{4,\ldots,4\})$

- **so(1,3) Lie algebra**

The Lie algebra of the Lorentz group O(1,3) is a 6-dimensional, non-compact, and semi-simple real Lie algebra denoted as so(1,3). It consists of infinitesimal Lorentz transformations, which are generated by three spatial rotations Ji and three velocity boosts Ki .

It is semi-simple and non-compact.

Complex so(1,3)$_C$ is isomorphic to direct sum of sl(2,C) , $so(1,3)_C \cong sl(2,C)\oplus sl(2,C)$

The Killing form is indefinite rather than negative-definite, it defines a hyperbolic space, which is not bounded, therefore so(1,3) is non-compact.

The matrices X of so(1,3) satisfy the condition $X^T\eta+\eta X=0$

With the matrices M_{kl} , $(M_{kl})_{mn} = \delta_{km}\delta_{ln}-\delta_{kn}\delta_{lm}$, k,l=1,...4, k<l ,

and the matrices $\bar{M}_{kl}$, $(\bar{M}_{kl})_{mn} = \delta_{km}\delta_{ln}+\delta_{kn}\delta_{lm}$, k,l=1,...4, k<l ,

the generators are [24]

$J_i = -(M_{34},M_{42},M_{23})$, $K_i = (\bar{M}_{12},\bar{M}_{13},\bar{M}_{14})$

$$J_1 = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\quad J_2 = \begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}\quad J_3 = \begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$$

$$K_1 = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad K_2 = \begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}\quad K_3 = \begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}$$

with the commutators $[J_i,J_j] = \varepsilon_{ijk}J_k$, $[K_i,K_j] = -\varepsilon_{ijk}J_k$, $[J_i,K_j] = \varepsilon_{ijk}K_k$

Ki-commutator differs in sign from the corresponding relation in the so(4) algebra.

Cartan subalgebra $h = \langle J_1,K_1\rangle$ is 2-dimensional (so(1,3) has rank 2), type A1

In Weyl-Cartan representation, the simple roots are $\alpha_1 = (i,1)$, $\alpha_2 = (1,i)$ [22].

In the ei-basis the simple roots are $\alpha_1 = \lambda_1-\lambda_2$, $\alpha_2 = \lambda_1+\lambda_2$

The 4 roots are $\pm\lambda_1\pm\lambda_2$,

Killing matrix is $\kappa_{ij} = diag_6(\{-4,-4,-4,4,4,4\})$

6.3 Odd orthogonal so-Lie-algebras so(2n+1) , Cartan type Bn
[21]
The odd-dimensional orthogonal Lie algebras so(2n+1) compact simple Lie algebras of Cartan type Bn. They correspond to the Lie algebras of the Special Orthogonal groups SO(2n+1) and consist of skew-symmetric (2n+1) × (2n+1) matrices, the dimension of so(2n+1) is n(2n+1) .
The rank of the algebra is n, which the dimension of the Cartan subalgebra.
Generators are M_{kl} , $(M_{kl})_{mn} = \delta_{km}\delta_{\ln} - \delta_{kn}\delta_{1m}$, k,l=1,...2n+1, k<l
with commutators $[M_{mn}, M_{rs}] = \delta_{nr}M_{ms} - \delta_{mr}M_{ns} + \delta_{ms}M_{nr} - \delta_{ns}M_{mr}$
Cartan subalgebra is spanned by $H_i = M_{2i-1,1}$ i=1,...,n
Roots in ei-basis
with index 0=2n+1;
We use the Kronecker matrices $E_{ij} = \left(\delta_{ik}\delta_{jl}\right)_{kl}$, and we use as basis the extended Kronecker matrices
$E^1_{jk} = E_{jk} - E_{k+n,j+n}$, $E^2_{jk} = E_{j,k+n} - E_{k,j+n}$, $E^3_{jk} = E_{j+n,k} - E_{k+n,j}$, $E^3_{jk} = E_{j+n,k} - E_{k+n,j}$,
$E^4_j = E_{0j} - E_{j+n,0}$, $E^5_j = E_{j0} - E_{0,j+n}$,
We make the ansatz for the Cartan subalgebra $H = \sum_{i=1}^{n} \lambda_i E_{ii}$,
we obtain the commutator relations
$\left[H, E^1_{jk}\right] = \left(\lambda_j - \lambda_k\right) E^1_{jk} \quad i \neq j$
$\left[H, E^2_{jk}\right] = \left(\lambda_j + \lambda_k\right) E^2_{jk} \quad i \leq j$
$\left[H, E^3_{jk}\right] = -\left(\lambda_j + \lambda_k\right) E^3_{jk} \quad i \leq j$
$\left[H, E^4_j\right] = -\lambda_j E^4_j$
$\left[H, E^5_j\right] = \lambda_j E^5_j$
and for the root vectors $e_i = E^1_{i,i+1}$, *i=1...n-1* , $e_n = E^5_n$
The simple root vectors are then
$\left(\lambda_1 - \lambda_2\right) E^1_{12}$
$\left(\lambda_2 - \lambda_3\right) E^1_{23}$
...
$\lambda_n E^5_n$
with n simple roots $\alpha_i = \left(\lambda_i - \lambda_{i+1}\right)$, *i=1...n-1* , $\alpha_n = \lambda_n$
$2n^2$ roots: $\pm\lambda_i \pm \lambda_j$, *1<=i<j<=n* , $\pm\lambda_i$ *1<=i<=n*

- **so(5) Lie algebra**

The so(5) Lie algebra is the real, 10-dimensional Lie algebra associated with the 5-dimensional special orthogonal group SO(5), which describes rotations in 5-dimensional Euclidean space. It is a classical simple Lie algebra of rank 2 and is mathematically isomorphic to the symplectic Lie algebra sp(4).
The standard representation is by 5x5 real, skew-symmetric matrices.
It is of Cartan type B2, its Cartan subalgebra has dimension 2, i.e. it has rank=2.
The generator basis for so(5) consists of the 10 matrices S_{kl} k,l=1,...5, k<l
$(M_{kl})_{mn} = \delta_{km}\delta_{\ln} - \delta_{kn}\delta_{1m}$
$M_{12}, M_{13}, M_{14}, M_{15}, M_{23}, M_{24}, M_{25}, M_{34}, M_{35}, M_{45}$
The commutator relations are $[M_{mn}, M_{rs}] = \delta_{nr}M_{ms} - \delta_{mr}M_{ns} + \delta_{ms}M_{nr} - \delta_{ns}M_{mr}$
In the adjoint representation of a Lie algebra, ad(Sij) is a linear operator $\mathrm{ad}(S_i)(X) = [S_i, X]$. .
Roots in Weyl-Cartan representation

Cartan subalgebra is $h = \langle M_{12}, M_{34} \rangle$

the remaining space is spanned by the 8 ladder operators $h_\perp = \langle M_{13} \pm M_{23}, M_{14} \pm M_{24}, M_{15} \pm M_{25}, M_{35} \pm M_{45} \rangle$

In the basis $h_\perp$, we have the eigenvalues and eigenvectors [22]:

for H_1

```
{i, i, i, -i, -i, -i, 0, 0}
```

```
{{0, 0, 0, 0, -i, 1, 0, 0}, {0, 0, -i, 1, 0, 0, 0, 0}, {-i, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, i, 1, 0, 0},
 {0, 0, i, 1, 0, 0, 0, 0}, {i, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 0, 0, 0, 1, 0}}
```

for H_2

```
{i, i, i, -i, -i, -i, 0, 0}
```

```
{{0, 0, 0, 0, 0, 0, -i, 1}, {0, i, 0, 1, 0, 0, 0, 0}, {i, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, i, 1},
 {0, -i, 0, 1, 0, 0, 0, 0}, {-i, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 1, 0, 0, 0}}
```

We obtain common eigenvectors for H_1 and H_2 for the eigenvalue doublets (= simple roots) $(\lambda(H_1), \lambda(H_2))$ = $(0,i)$, $(-i,0)$, so the simple roots are $\alpha_1 = (0,i)$, $\alpha_2 = (-i,0)$

and all 8 roots are $\pm\alpha_1$, $\pm\alpha_2$, $\pm\alpha_1 \pm \alpha_2$

Roots in ei basis

In orthonormal basis e_i, the 2 simple roots are $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_2$

The root system consists of 8 roots $\pm\lambda_1$, $\pm\lambda_2$, $\pm\lambda_1 \pm \lambda_2$

- **so(3) Lie algebra , isomorphic to su(2)**

The so(3) Lie algebra is the tangent space at the identity of the SO(3) rotation group. It consists of all 3x3 real, skew-symmetric matrices.

The algebra is spanned by three standard basis matrices, which correspond to infinitesimal rotations (J_1, J_2, J_3) about the x,y,z axes

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with commutator relations of angular momentum $[J_i, J_j] = \varepsilon_{ijk} J_k$

Cartan subalgebra is generated by J3, it has dim=1,

the simple root is λ_1, the 2 roots are $\pm\lambda_1$

6.4 Symplectic Lie algebras sp(2n) , Cartan type Cn

[21]

The symplectic Lie algebra sp(2n) is a simple non-compact Lie algebra corresponding to the Cartan classification type Cn. It consists of the infinitesimal symmetries of a non-degenerate, skew-symmetric bilinear form (the symplectic form), with dim=n(2n+1).

sp(2n) is realized as the set of 2n x 2n block matrices X that satisfy the condition

$$X^T J_n = -J_n X$$

where J is the canonical symplectic matrix:

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Matrices are of the form

$X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$, with symmetric nxn matrices B, C .

Generators are of three types [23]

where the basis elements are:

n² generators A_{ij}

$A_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}$ i,j=1...n

n(n+1)/2 generators for symmetric B_{ij}

$B_{ij} = \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix}$, i<=j

n(n+1)/2 generators for symmetric C_{ij}

$C_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{pmatrix}$, i<=j

Here, the E_{ij} are nxn ij-Kronecker matrices $\left(E_{ij}\right)_{kl} = \delta_{ik}\delta_{jl}$

The generators are $A_{ij} = E^1_{ij} \quad i, j = 1...n$, $B_{(ij)} = E^2_{ij} \quad 1 \le i \le j \le n$, $C_{(ij)} = E^3_{ij} \quad 1 \le i \le j \le n$

with extended Kronecker matrices

$E^1_{ij} = E_{jk} - E_{k+n,j+n}$, $E^2_{ij} = E_{j,k+n} - E_{k,j+n}$, $E^3_{ij} = E_{j+n,k} - E_{k+n,j}$

The commutator relations are:

$\left[A_{ij}, A_{kl}\right] = \delta_{jk} A_{il} - \delta_{li} A_{kj}$,

$\left[A_{ij}, B_{kl}\right] = \delta_{jk} B_{il} + \delta_{lj} B_{ik}$, $\left[A_{ij}, C_{kl}\right] = -\delta_{il} C_{jk} - \delta_{ik} C_{jl}$, $\left[B_{ij}, C_{kl}\right] = \delta_{jk} A_{il} + \delta_{il} A_{jk} + \delta_{lj} A_{ik} + \delta_{ik} A_{jl}$

$\left[B_{ij}, A_{kl}\right] = -\left(\delta_{li} B_{kj} + \delta_{lj} B_{ki}\right)$, $\left[C_{ij}, A_{kl}\right] = \delta_{kj} C_{li} + \delta_{ki} C_{lj}$, $\left[C_{ij}, B_{kl}\right] = -\left(\delta_{li} A_{kj} + \delta_{kj} A_{li} + \delta_{jl} A_{ki} + \delta_{ki} A_{lj}\right)$

Cartan subalgebra is spanned by $h = \langle A_{11}, ..., A_{kk} \rangle$

Roots in ei-basis

With the ansatz for Cartan elements $H = \sum_{j=1}^{k} \lambda_j A_{jj}$

we obtain the commutator relations

$\left[H, A_{jk}\right] = \left(\lambda_j - \lambda_k\right) A_{jk} \quad i \ne j$

$\left[H, B_{jk}\right] = \left(\lambda_j + \lambda_k\right) B_{jk} \quad i \le j$

$\left[H, C_{jk}\right] = -\left(\lambda_j + \lambda_k\right) C_{jk} \quad i \le j$

The simple root vectors are then

$\left(\lambda_1 - \lambda_2\right) A_{12}$

$\left(\lambda_2 - \lambda_3\right) A_{23}$

...

$2\lambda_n B_{nn}$

The root system of type Cn has $2n^2$ roots,,
in ei basis they are:

n simple roots are $\alpha_i = (\lambda_i - \lambda_{i+1})$ for i=1,...,n-1 ; $\alpha_n = 2\lambda_n$

$2n^2$ roots are : $\pm(\lambda_i \pm \lambda_j)$, $\pm\lambda_j$ $1 \le i < j \le n$

- **sp(4) symplectic Lie algebra**

The sp(4) Lie algebra is the 10-dimensional, rank-2 simple Lie algebra associated with the 4 × 4 symplectic group.

sp(2n) is realized as the set of 4x4 block matrices X that satisfy the condition

$X^T J_2 = -J_2 X$

where J_2 is the canonical symplectic matrix:

$$J_2 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

Matrices are of the form

$X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$, with symmetric nxn matrices B, C .

4 generators A_{ij} , 3 generators B_{ij} , 3 generators C_{ij}

$A_{ij} = \begin{pmatrix} A_{2,ij} & 0 \\ 0 & -A_{2,ij}{}^T \end{pmatrix}$, $B_{ij} = \begin{pmatrix} 0 & B_{2,ij} \\ 0 & 0 \end{pmatrix}$, $C_{ij} = \begin{pmatrix} 0 & 0 \\ C_{2,ij} & 0 \end{pmatrix}$

generators 4 $A_{ij} = E^1_{ij}$, 3 $B_{(ij)} = E^2_{ij}$, 3 $C_{(ij)} = E^3_{ij}$

with extended Kronecker matrices

$E^1_{jk} = E_{jk} - E_{k+n,j+n}$, $E^2_{jk} = E_{j,k+n} - E_{k,j+n}$, $E^3_{jk} = E_{j+n,k} - E_{k+n,j}$, $E^3_{jk} = E_{j+n,k} - E_{k+n,j}$

The commutator relations are:

$\left[A_{ij}, A_{kl}\right] = \delta_{jk} A_{il} - \delta_{li} A_{kj}$, $\left[A_{ij}, B_{kl}\right] = \delta_{jk} B_{il} + \delta_{jl} B_{ik}$, $\left[A_{ij}, C_{kl}\right] = -\delta_{ik} C_{lj} - \delta_{ik} C_{jl}$,

$\left[B_{ij}, C_{kl}\right] = \delta_{ik} A_{lj} + \delta_{jk} A_{li} + \delta_{il} A_{kj} + \delta_{jl} A_{ki}$

The 10 generators are

$A_{11}, A_{22}, A_{12}, A_{21}, B_{11}, B_{12}, B_{22}, C_{11}, C_{12}, C_{22}$

Killing matrix is $\begin{pmatrix} 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 24 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 24 & 0 & 0 & 0 \end{pmatrix}$ [22]

with eigenvalues $\{-24,-24,24,24,-12,-12,12,12,12,12\}$

Cartan algebra $h = \langle A_{11}, A_{22} \rangle$ has dim=2, so sp(4) has rank=2.

Roots in the Weyl-Cartan representation

Cartan subalgebra $h = \langle H_1, H_2 \rangle$ is spanned by $H_1 = A_{11}$, $H_2 = A_{22}$

the remaining space is spanned by ladder operators

$h_\perp = \langle A_{12} + A_{21}, A_{12} - A_{21}, B_{11} + B_{12}, B_{11} - B_{12}, B_{22} + C_{11}, B_{22} - C_{11}, C_{12} + C_{22}, C_{12} - C_{22} \rangle$

In the basis $h_\perp$, we have the eigenvalues and eigenvectors of ad[H_i] [22]:

for H_1

$\{-\sqrt{14}, \sqrt{14}, 2, 2, 0, 0, 0, 0\}$

$\left\{\left\{0, 0, 0, 0, 0, 0, -\sqrt{\frac{2}{7}}, 1\right\}, \left\{0, 0, 0, 0, 0, 0, \sqrt{\frac{2}{7}}, 1\right\}, \{0, 0, 0, 0, 0, 1, 0, 0\}, \{0, 0, 0, 0, 1, 0, 0, 0\}, \{0, 0, 0, 1, 0, 0, 0, 0\}, \{0, 0, 1, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0, 0\}, \{1, 0, 0, 0, 0, 0, 0, 0\}\right\}$

for H_2

$\{-2, -2, -2, 2, 1, 0, 0, 0\}$

$\{\{0, 0, 0, 0, 0, 0, 0, 1\}, \{0, 0, 0, 0, -1, -1, 3, 0\}, \{0, 0, -1, 1, 0, 0, 0, 0\}, \{0, 0, 1, 1, 0, 0, 0, 0\}, \{0, 0, 0, 0, 1, 1, 0, 0\}, \{0, 0, 0, 0, 1, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0, 0\}, \{1, 0, 0, 0, 0, 0, 0, 0\}\}$

We obtain common eigenvectors for H_1 and H_2 for the simple eigenvalue doublets $\left(\lambda(H_1), \lambda(H_2)\right) = (0,2)$, $(2,0)$

Roots in ei basis

In orthonormal basis the simple roots are

$\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = 2\lambda_2$,

8 roots are $\pm\lambda_1 \pm \lambda_2$, $\pm 2\lambda_1$, $\pm 2\lambda_2$

6.5 Even orthogonal so-Lie-algebras so(2n) , Cartan type Dn
[21]
The even-dimensional orthogonal Lie algebra so(2n) corresponds to the Cartan type Dn. It consists of all 2n × 2n skew-symmetric matrices. These algebras are simple for n ≥ 4 and represent infinitesimal rotations and reflections that preserve an even-dimensional non-degenerate symmetric bilinear form.
so(2n) has dim=n(2n-1), it is a simple compact (n>=2) Lie algebra.
Cartan subalgebra h , dimension *n*, consists of block-diagonal matrices with 2 × 2 skew-symmetric blocks along the diagonal.
Generators are M_{kl} , $(M_{kl})_{mn} = \delta_{km}\delta_{\ln} - \delta_{kn}\delta_{1m}$, k,l=1,...2n, k<l
with commutators $[M_{mn}, M_{rs}] = \delta_{nr}M_{ms} - \delta_{mr}M_{ns} + \delta_{ms}M_{nr} - \delta_{ns}M_{mr}$
Cartan subalgebra is spanned by $H_i = M_{2i-1,2i}$ i=1,...,n
Cartan type Dn has 2n(n-1) roots.
Roots in ei-basis
We use as basis the matrices $E_{ij} = (\delta_{ik}\delta_{jl})_{kl}$, extended to
$E^1_{jk} = E_{jk} - E_{k+n,j+n}$, $E^2_{jk} = E_{j,k+n} - E_{k,j+n}$, $E^3_{jk} = E_{j+n,k} - E_{k+n,j}$, $E^3_{jk} = E_{j+n,k} - E_{k+n,j}$,
We make the ansatz for the Cartan subalgebra $H = \sum_{i=1}^{n} \lambda_i E_{ii}$,
we obtain the commutator relations
$[H, E^1_{jk}] = (\lambda_j - \lambda_k) E^1_{jk} \quad i \neq j$
$[H, E^2_{jk}] = (\lambda_j + \lambda_k) E^2_{jk} \quad i \leq j$
$[H, E^3_{jk}] = -(\lambda_j + \lambda_k) E^3_{jk} \quad i \leq j$
The simple root vectors are then
$(\lambda_1 - \lambda_2) E^1_{12}$
$(\lambda_2 - \lambda_3) E^1_{23}$
...
$(\lambda_{n-1} + \lambda_n) E^2_{n-1,n}$
with n simple roots $\alpha_i = (\lambda_i - \lambda_{i+1})$, *i=1...n-1* , $\alpha_n = (\lambda_{n-1} + \lambda_n)$
the 2n(n-1) roots are $\pm(\lambda_i \pm \lambda_j)$ 1<=i<j<=n

- **so(4) Lie algebra**

The so(4) Lie algebra is the tangent space of the SO(4) group (4D rotations) and consists of 4 × 4 real, skew-symmetric matrices. It has a dimension of 6 and is uniquely characterized by its isomorphism to the direct sum of two su(2) (or equivalently so(3)) Lie algebras:
$so(4) = so(3) \oplus so(3)$
so(4) is a semi-simple and compact Lie algebra.
Elements of so(4) are skew-symmetric and have the generic form
$$X = \begin{pmatrix} 0 & -b_3 & b_2 & -a_1 \\ b_3 & 0 & -b_1 & -a_2 \\ -b_2 & b_1 & 0 & -a_3 \\ a_1 & a_2 & a_3 & 0 \end{pmatrix}$$
The six generators are the 3-dimensional rotations of so(3) A_1, A_2, A_3

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the 3 boost rotations B_1, B_2, B_3 (in physics: velocity transformations, including the last (time) coordinate)

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

with the commutator relations

$$\left[A_i, A_j\right] = \varepsilon_{ijk} A_k, \quad \left[B_i, B_j\right] = \varepsilon_{ijk} A_k, \quad \left[A_i, B_j\right] = \varepsilon_{ijk} B_k$$

Roots in the Weyl-Cartan representation

Cartan subalgebra $h = \langle H_1, H_2 \rangle$ is spanned by $H_1 = A_3 + B_3$, $H_2 = A_3 - B_3$

the remaining space is $h_\perp = \langle A_1, A_2, B_1, B_2 \rangle$,

In the basis $h_\perp$, we have the eigenvalues and eigenvectors [22]:

for H_1

```
{2 i, -2 i, 0, 0}
{{-i, 1, -i, 1}, {i, 1, i, 1}, {0, -1, 0, 1}, {-1, 0, 1, 0}}
```

for H_2

```
{2 i, -2 i, 0, 0}
{{i, -1, -i, 1}, {-i, -1, i, 1}, {0, 1, 0, 1}, {1, 0, 1, 0}}
```

We obtain common eigenvectors for H_1 and H_2 for the simple eigenvalue doublets

$$\left(\lambda(H_1), \lambda(H_2)\right) = (0, 2i), \ (2i, 0)$$

Equivalent ladder generators and commutators

An equivalent ladder generator set is

$$X_i = \left(A_i + B_i\right)/2, \quad Y_i = \left(A_i - B_i\right)/2$$

with the commutator relations

$$\left[X_i, X_j\right] = \varepsilon_{ijk} X_k, \quad \left[Y_i, Y_j\right] = \varepsilon_{ijk} Y_k, \quad \left[X_i, Y_j\right] = 0$$

which are the basis for the direct sum $so(4) = so(3) \oplus so(3)$

with the structure constants $c_{ab}^{d} = \begin{cases} \varepsilon_{ijk,} & a = i, b = j, d = k \\ \varepsilon_{ijk,} & a = i+3, b = j+3, d = k+3 \\ 0, & else \end{cases}$

The Killing form is $\kappa_{ij} = c_{ib}^{a} c_{ja}^{b}$ and becomes here $\kappa_{ij} = -2\delta_{ij}$

Cartan subalgebra is 2-dimensional (so(4) has rank 2) and is spanned by

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Roots representation in e_i-basis

In orthonormal basis $\{e_1, e_2\}$

the simple roots are $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = \lambda_1 + \lambda_2$

all 4 roots are $\pm\lambda_1 \pm \lambda_2$.

- **so(2) Lie algebra**

The **so(2)** Lie algebra is the tangent space at the identity for the Lie group SO(2), which represents 2D rotations. It is a 1-dimensional, abelian (commutative) Lie algebra. The elements of **so(2)** are 2×2 real, skew-symmetric matrices.

The generator is $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

6.6 sl(n,C) algebras

sl(n,C) is Lie algebra is the set of all complex n × n matrices with trace of zero. It is a simple Lie algebra of dimension $n^2 - 1$.

Cartan algebra (dimension n-1) consists of traceless diagonal matrices.

The root system is A_{n-1}, with n(n-1) roots.

- sl(2,C) Lie algebra

The Lie algebra sl(2,C)is the set of all 2x2 matrices with complex entries and a trace of zero.

generators $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

commutator relations $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$

Cartan subalgebra is generated by *H*, has dim=1,

in Weyl-Cartan representation the roots are {2,-2},

in orthonormal basis the roots are $\pm\lambda_1$.

6.7 Exceptional Lie algebras

• g_2 (Dimension 14): is the automorphism group of the octonions and the stabilizer of a generic 3-form in 7 dimensions, rank=2

Cartan matrix $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$, rank=2

2 simple roots $e_1-e_2 = (1,-1,0)$, $-e_1+2e_2-e_3 = (-1,2,-1)$

12 roots

(1,−1,0), (−1,1,0) (2,−1,−1), (−2,1,1)
(1,0,−1), (−1,0,1) (1,−2,1), (−1,2,−1)
(0,1,−1), (0,−1,1) (1,1,−2), (−1,−1,2)

• f_4 (Dimension 52): The Lie algebra of the exceptional group F_4, is the automorphism group of the Albert algebra (a 27-dimensional exceptional Jordan algebra), rank=4

Albert algebra is the space of 3×3 self-adjoint 3x3 matrices over the octonions .

f4 is the automorphism group of Albert algebra.

f4 is the group of automorphisms of the following set of 3 polynomials in 27 variables

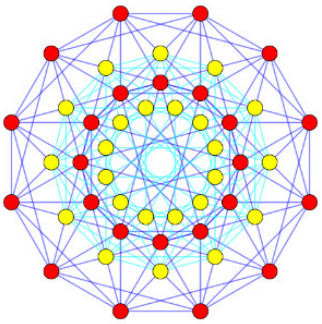
$$C_1 = x + y + z$$
$$C_2 = x^2 + y^2 + z^2 + 2X\overline{X} + 2Y\overline{Y} + 2Z\overline{Z}$$
$$C_3 = xyz - xX\overline{X} - yY\overline{Y} - zZ\overline{Z} + XYZ + \overline{XYZ}$$

where x,y,z are reals and X,Y,Z are octonions.

Cartan matrix $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, rank=4

4 simple roots $\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

8 roots , red and yelllow vertices in the diagram



• e_6 (Dimension 78): is the stabilizer of a certain cubic form and appears in the classification of certain projective planes over division algebras, rank=6

The minimal representation of e6 is the Albert algebra.

The fundamental group of the adjoint form of e6 ad(e6) is the cyclic group Z3 .

Cartan matrix is $\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$

The rank=6, simple roots are

(0,0,0;0,0,0;0,1,−1)

(0,0,0;0,0,0;1,−1,0)

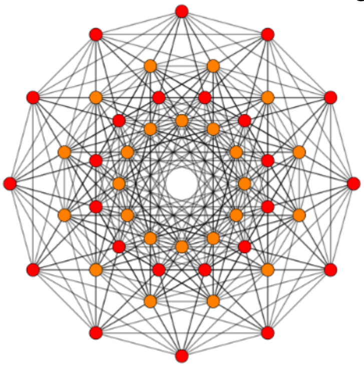
(0,0,0;0,1,−1;0,0,0)

(0,0,0;1,−1,0;0,0,0)

(0,1,−1;0,0,0;0,0,0)

$\left(\frac{1}{3},-\frac{2}{3},\frac{1}{3};-\frac{2}{3},\frac{1}{3},\frac{1}{3};-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)$

72 roots , vertices in the diagram



- e_7 (Dimension 133): Leaves invariant a symmetric rank-4 tensor in its fundamental 56-dimensional representation, rank=7

The rank=7, simple roots are

(0,−1,1,0,0,0,0,0)

(0,0,−1,1,0,0,0,0)

(0,0,0,−1,1,0,0,0)

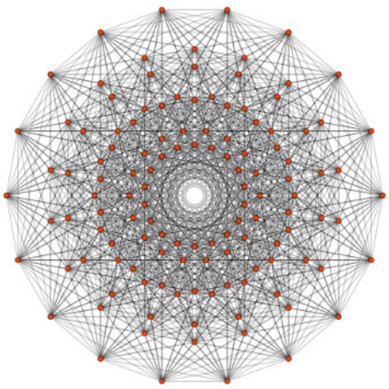
(0,0,0,0,−1,1,0,0)

(0,0,0,0,0,−1,1,0)

(0,0,0,0,0,0,−1,1)

$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$

126 roots , vertices in the diagram



- e_8 (Dimension 248): Its root diagram is the E_8 lattice, which models the densest possible sphere packing in 8-dimensional space, rank=8

e8 has rank=8

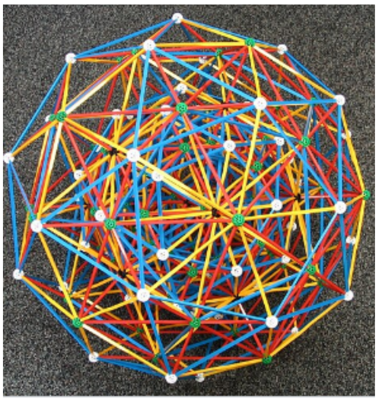
Cartan matrix is

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

simple roots are

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

240 roots are the vertices of the 3-dim diagram



7 Clifford algebras

A Clifford algebra [25] is an associative algebra generated by a vector space with a quadratic form, generalizing complex numbers and quaternions, and widely used in geometry and physics.

Definition and Construction

A **Clifford algebra** is constructed from a vector space V over a field K equipped with a quadratic form Q. Formally, it is defined as the quotient of the **tensor algebra** T(V) by the ideal generated by elements of the form $v \otimes v - Q(v)1$ for all $v \in V$.

This ensures that the square of each vector in the algebra corresponds to its quadratic form value. The algebra is **associative but generally non-commutative**, and its dimension is $2^{\dim V}$.

Generators and Relations

The algebra is generated by the elements of V, called **generators**, with the defining relation:

$v^2 = Q(v)$ for all $v \in V$

or equivalently, using the symmetric bilinear form B associated with Q:

$vw + wv = 2B(v,w)1$ for all $v,w \in V$. When the quadratic form is zero, the Clifford algebra reduces to the ***exterior algebra***.

Applications

Clifford algebras have broad applications in both mathematics and physics:

Geometry: They provide a framework for **geometric algebra**, allowing the representation of rotations, reflections, and orthogonal transformations ..

Spinors and Quantum Physics:

Clifford algebras define **spinor spaces** and the **Spin group**, which are essential in describing fermions like electrons and in constructing spinor bundles on manifolds.

Representation Theory: They are used to study **Lie algebras**, root systems, and weights, providing a natural setting for constructing spinor modules.

Combinatorics and Algebra:

Clifford algebras generalize complex numbers, quaternions, and octonions, and are useful in defining hypercomplex numbers and various algebraic operations .

Structure and Properties

Grading: Clifford algebras are often **$\mathbb{Z}$2-graded**, separating even and odd elements.

Tensor Products: The Clifford algebra of a direct sum of vector spaces can be expressed as a tensor product of the Clifford algebras of the summands .

Normed Division Algebras: Real Clifford algebras are closely related to normed division algebras, connecting them to quaternions and octonions .

Clifford algebras thus provide a unifying algebraic framework that extends classical number systems and underpins many modern mathematical and physical theories.

Abstract definition

A Clifford algebra is a unital (with 1) associative algebra that contains and is generated by a vector space V over a field K, where V is equipped with a bilinear form $\langle u,v \rangle$: $V \times V \to K$. , which becomes a quadratic form $Q(v)$ $V \to K$ for $v \in V$

The Clifford algebra Cl(V, Q) is the unital associative algebra generated by V , defined by multiplication $u \cdot v$ with the anticommutator condition (Clifford product)

$\{u,v\} = u \cdot v + v \cdot u = 2\langle u,v \rangle 1$ for $u,v \in V$

here, $Q(v) = \langle v,v \rangle$ and $\langle u,v \rangle = \frac{1}{2}\left(Q(u+v) - Q(u) - Q(v)\right)$

If Q(v)=0, Cl(V, Q) becomes the external algebra $\wedge V$.

Usually, the bilinear form is the inner product $\langle u,v \rangle = u_1 v_1 + \ldots + u_n v_n$

A graded Clifford algebra $Cl_{p,q}(V, Q)$ has the quadratic form $\langle v,v \rangle = v_1^2 + \ldots + v_p^2 - v_{p+1}^2 - \ldots - v_{p+q}^2$ with p positive and q negative terms.

The canonical Clifford construction

The Clifford algebra Cl(V, Q) can be formulated in a standard orthogonal basis $(e_1, e_2, \ldots., e_n)$ with the relations $e_i \cdot e_j = -e_j \cdot e_i$, $e_i \cdot e_i = 1$, and the quadratic form $\langle v,v \rangle = v_1^2 + \ldots + v_p^2 - v_{p+1}^2 - \ldots - v_{p+q}^2$.

The general element of the Clifford algebra is given by

$$q = q_0 + \sum_{i_1<i_2} q_{i_1 i_2} e_{i_1} e_{i_2} + \sum_{i_1<i_2<i_3} q_{i_1 i_2 i_3} e_{i_1} e_{i_2} e_{i_3} + \ldots + q_{i_1 \ldots i_n} e_{i_1} \ldots e_{i_n}$$

The basis $(e_1, e_2, \ldots, e_n)$ are the n generators of the Clifford algebra, with dimension dim(Cl(V, Q))=n .

The linear combination of the even degree elements of Cl(V, Q) defines the even subalgebra $Cl^{[0]}$(V, Q) with the general element

$$q = q_0 + \sum_{i_1<i_2} q_{i_1 i_2} e_{i_1} e_{i_2} + \sum_{i_1<i_2<i_3<i_4} q_{i_1 i_2 i_3 i_4} e_{i_1} e_{i_2} e_{i_3} e_{i_4} + \ldots$$

The quaternion algebra

The even subalgebra $Cl_{3,0}$[0](R, $\langle v, v \rangle$) with the generic form

$$q = q_0 + q_{12} e_1 e_2 + q_{23} e_2 e_3 + q_{31} e_3 e_1$$

is Hamilton's real quaternion algebra H with quaternions (i, j, k) and the relations

$i^2 = \alpha$, $j^2 = \beta$, $i\,j = k$, $j\,i = -k$, $k^2 = i\,j\,i\,j = -ii\,j\,j = -\alpha\beta$

where ($\alpha = \beta = -1$) for Hamilton's quaternions H

and ($\alpha = -1$, $\beta = +1$) for split-quaternions, isomorphic to the ring of the 2×2 real matrices .

Hamilton's quaternions have the properties

$i^2 = j^2 = k^2 = -1$, $i\,j = k = -j\,i$, $j\,k = i = -k\,j$, $k\,i = j = -ik$

H has the generic form $q = a + bi + c\,j + d\,k$

H is actually a field, with the inverse $q^{-1} = (a + bi + c\,j + d\,k)^{-1} = \dfrac{a - bi - c\,j - d\,k}{a^2 + b^2 + c^2 + d^2}$

The conjugated element is $q^* = -\frac{1}{2}(q + iqi + jq\,j + kqk)$

the norm is $\|q\| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}$

According to the Frobenius theorem, there are only two division-free algebras over the field of real numbers R : complex numbers C and Hamilton's quaternions H .

A 2x2 matrix representation of H is by Pauli matrices {1, i, j, k} $\rightarrow \{I, i\sigma_3, i\sigma_2, i\sigma_1\}$.

The dual quaternion algebra

With $V=R^4$ and the degenerate bilinear form $d(v, w) = v_1 w_1 + v_2 w_2 + v_3 w_3$

and the Clifford product $vw + wv = -2d(v, w)$, we obtain the Clifford algebra Cl_4(**R**, d) with 16 components and the basis relations

$e_1{}^2 = e_2{}^2 = e_3{}^2 = -1$, $e_4{}^2 = 0$, $e_m e_n = -e_n e_m$ $m \neq n$

The linear combination of the even degree elements defines the even subalgebra $Cl^{[0]}$(R4, d) with the general element

$$h = h_0 + h_1 e_2 e_3 + h_2 e_3 e_1 + h_3 e_1 e_2 + h_4 e_4 e_1 + h_5 e_4 e_2 + h_6 e_4 e_3 + h_7 e_1 e_2 e_3 e_4$$

The basis elements can be identified with the quaternion basis elements i, j, k and the dual unit ε as

$i = e_2 e_3$, $j = e_3 e_1$, $k = e_1 e_2$, $\varepsilon = e_1 e_2 e_3 e_4$

This is an isomorphism with the dual quaternion algebra.

Dual quaternions H_d are an 8-dimensional real algebra isomorphic to the tensor product of the quaternions and the dual numbers with the generic form $A + \varepsilon B$, where A, B are quaternions, and ε is the dual unit, which commutes with all quaternions and is nilpotent: $\varepsilon^2=0$.

Super algebra

Clifford algebras are superalgebras, i.e. the reflection α: v ↦ −v preserves the quadratic form Q and an algebra automorphism.α:Cl(V,Q)→Cl(V,Q).

We can decompose Cl(V, Q) into direct sum of even and odd subspaces

$$Cl(V,Q) = Cl^{[0]}(V,Q) \oplus Cl^{[1]}(V,Q)$$

where $Cl^{[i]}(V,Q) = \left\{ x \in Cl(V,Q) \middle| \alpha(x) = (-1)^i x \right\}$

Transposition

A transpose of an element reverses the order in product terms

$$q = q_0 + \sum_{i_1<i_2} q_{i_1 i_2} e_{i_1} e_{i_2} + \sum_{i_1<i_2<i_3} q_{i_1 i_2 i_3} e_{i_1} e_{i_2} e_{i_3} + \dots$$

$$q^t = q_0 + \sum_{i_1<i_2} q_{i_1 i_2} e_{i_2} e_{i_1} + \sum_{i_1<i_2<i_3} q_{i_1 i_2 i_3} e_{i_3} e_{i_2} e_{i_1} + \dots$$

The Clifford scalar product is then

$\langle x, y\rangle = \langle x^{\mathrm{t}} y\rangle_0$, where $\langle x^{\mathrm{t}} y\rangle_0$ is the scalar part q_0 of the product $x^t y$

Pin and Spin-group

$Pin_{p,q} = \left\{ v_1 \dots v_{2r} \,\middle|\, \|v_i\| = \pm 1 \right\}$ consists of products of invertible unit vectors , where the spinor norm is $\|v\| = \left(v^t v\right)_0$,

$Pin_{p,q}$ is the full rotation group O(p,q) ,

$Spin_{p,q}$ is the proper (without reflections) rotation group SO(p,q) , it consists of even-number-products of unit vectors

$$Spin_{p,q} = \left\{ v_1 \dots v_{2r} \,\middle|\, \|v_i\| = \pm 1 \right\} = \{ q \in Cl(V) \,|\, q^t q = I, \|q\| = 1 \}$$

8 Dirac matrices, spinors and Lorentz Lie algebra

Dirac Clifford algebra

Dirac algebra is the algebra with the basis of Dirac matrices matrices γ_0, ..., γ_3 with the anticommutator $\{\gamma^\mu,\gamma^\mu\}=2\eta_{\mu\nu}I$ in (1,3) space, Minkowski signature (+,---)

This is the complexification $Cl_{1,3}(R)_C$ of the Clifford algebra $Cl_{1,3}(R)$, which is isomorphic to the 4x4 complex matrices $M_4(C)$.

The Dirac matrices (Dirac basis) are explicitly given by the Pauli matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

with the relations $[\sigma_j,\sigma_k]=2i\varepsilon_{jkl}\sigma_l$, $\{\sigma_j,\sigma_k\}=2\delta_{jk}I$

in the form

$$\gamma^0=\begin{pmatrix}I&0\\0&-I\end{pmatrix},\quad \gamma^j=\begin{pmatrix}0&\sigma_j\\-\sigma_j&0\end{pmatrix},\quad \gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}0&I\\I&0\end{pmatrix}$$

$$\gamma^0=\sigma_3\otimes I_2,\quad \gamma^j=i\sigma_2\otimes\sigma_j,\quad \gamma^5=\sigma_1\otimes I_2$$

$$\gamma^0=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix},\quad \gamma^2=i\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix},\quad \gamma^3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix},$$

$$\gamma^5=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$$

In chiral Weyl basis they are

$$\gamma^0=\begin{pmatrix}0&I\\I&0\end{pmatrix},\quad \gamma^j=\begin{pmatrix}0&\sigma^j\\-\sigma^j&0\end{pmatrix},\quad \gamma^5=\begin{pmatrix}-I&0\\0&I\end{pmatrix}$$

The corresponding spin Lie algebra so(1,3) is generated by the spin generators $\Sigma^{\mu\nu}=\frac{i}{4}[\gamma^\mu,\gamma^\nu]$, with the Lie bracket

$$[\Sigma^{\mu\nu},\Sigma^{\rho\tau}]=-i\left(\eta^{\tau\mu}\Sigma^{\rho\nu}+\eta^{\nu\tau}\Sigma^{\mu\rho}-\eta^{\rho\mu}\Sigma^{\tau\nu}-\eta^{\nu\rho}\Sigma^{\mu\tau}\right)$$

so(1,3) Lie algebra

The Lie algebra of the Lorentz group O(1,3) is a 6-dimensional, non-compact, and semi-simple real Lie algebra denoted as so(1,3). It consists of infinitesimal Lorentz transformations, which are generated by three spatial rotations Ji and three velocity boosts Ki .

It is semi-simple and non-compact.

Complex $so(1,3)_C$ is isomorphic to direct sum of sl(2,C) , $so(1,3)_C\cong sl(2,C)\oplus sl(2,C)$

The Killing form is indefinite rather than negative-definite, it defines a hyperbolic space, which is not bounded, therefore so(1,3) is non-compact.

With the matrices M_{kl} , $(M_{kl})_{mn}=\eta_{km}\delta_{\ln}-\eta_{kn}\delta_{1m}$, k,l=1,...4, k<l ,

and the matrices $\bar{M}_{kl}$, $(\bar{M}_{kl})_{mn}=\eta_{km}\delta_{\ln}+\eta_{kn}\delta_{1m}$, k,l=1,...4, k<l ,

the generators are [24] $J_k=\frac{1}{2}\varepsilon_{ijk}M_{ij}$, $K_j=M_{0j}$

$$J_i=-(M_{34},M_{42},M_{23}),\quad K_i=(\bar{M}_{12},\bar{M}_{13},\bar{M}_{14})$$

$$J_1=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\quad J_2=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}\quad J_3=\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$$

$$K_1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad K_2=\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}\quad K_3=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}$$

with the commutators , $\left[K_i,K_j\right]=-\varepsilon_{ijk}J_k$, $\left[J_i,K_j\right]=\varepsilon_{ijk}K_k$

Lorentz group SO(1,3)

The rotations Λ [24] of the Lorentz groups $x'=\Lambda x$ can be written in the Kronecker matrices with 6 elementary rotations $\omega_{\mu\nu}$ and Kronecker matrices $M_{\mu\nu}$

$$\Lambda=\exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)$$

or in the rotation and boost generators $\left(J_i\right),\left(K_i\right)$ with rotation vector $\vec{\varphi}=\left(\varphi_1,\varphi_2,\varphi_3\right)$ and boost vector $\vec{v}=\left(v_1,v_2,v_3\right)$ we obtain with

$\omega_{0i}=v_i=-\omega_{i0}$, $\omega_{ij}=\varepsilon_{ijk}\varphi_k$

in the form $\Lambda=\exp(-i(\vec{\varphi}\vec{J}+\vec{v}\vec{K}))$

rotation operator $\Phi=(\vec{\varphi}\vec{J}+\vec{v}\vec{K})$

Spinor group Spin(1,3)

The corresponding spin rotations of the Dirac spinor $\Psi=\begin{pmatrix}\Psi_L\\\Psi_R\end{pmatrix}$, $\Psi'=S\Psi$

can be formulated with the spin generators $\Sigma^{\mu\nu}$, $\Sigma^{\mu\nu}=\frac{i}{4}\left[\gamma^\mu,\gamma^\nu\right]=\begin{pmatrix}\sigma^{\mu\nu}&0\\0&\tilde{\sigma}^{\mu\nu}\end{pmatrix}$

in the form $S=exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)$

or in the rotation-boost generators

$$S=\begin{pmatrix}A_L&0\\0&A_R\end{pmatrix}=\begin{pmatrix}exp\left(-\frac{i}{2}(\vec{\varphi}-i\vec{v})\vec{\sigma}\right)&0\\0&exp\left(-\frac{i}{2}(\vec{\varphi}+i\vec{v})\vec{\sigma}\right)\end{pmatrix},$$

with Pauli spin-tensor $\sigma^{\mu\nu}=\frac{i}{4}[\sigma^\mu,\tilde{\sigma}^\nu]$, $\tilde{\sigma}^{\mu\nu}=\frac{i}{4}[\tilde{\sigma}^\mu,\sigma^\nu]$, $\sigma^\mu=\frac{1}{2}(1,\vec{\sigma})$, $\tilde{\sigma}^\mu=\frac{1}{2}(1,-\vec{\sigma})$

and Pauli 3-vector $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$

$S=\begin{pmatrix}A_L&0\\0&A_R\end{pmatrix}$ with the left- and right Weyl-spinor rotations

$A_L=exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)=exp\left(-\frac{i}{2}(\vec{\varphi}-i\vec{v})\vec{\sigma}\right)$, left-handed Weyl-spinor

$A_R=exp\left(-\frac{i}{2}\omega_{\mu\nu}\tilde{\sigma}^{\mu\nu}\right)=exp\left(-\frac{i}{2}(\vec{\varphi}+i\vec{v})\vec{\sigma}\right)$, right-handed Weyl-spinor

rotation operator $\Phi_s=\frac{1}{2}(\vec{\varphi}\vec{\sigma}\pm i\vec{v}\vec{\sigma})$

General Dirac matrices

General Dirac matrices Γ_a [26] generate the Dirac group G in dim=d . metrics (p,q) in the form

$x = i^n \Gamma_{a_1}\Gamma_{a_2}...\Gamma_{a_d}$, $x \in G$, in canonical order $a_1 < a_2 < ... < a_d$

with the anti-commutator $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} I$ for Cl(p,q)(C)

In SO(4) with the Euclidean metric $\delta_{\mu\nu}$, the Dirac matrices become

$$\gamma_E^0 = \gamma^0 \ , \ \gamma_E^j = i\gamma^j \ , \ \gamma_E^5 = -i\gamma^5$$

$$\gamma_E^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} , \ \gamma_E^j = i\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} , \ \gamma_E^5 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

compactly $\gamma_E^0 = \sigma_3 \otimes I_2$, $\gamma_E^j = -\sigma_2 \otimes \sigma_j$, $\gamma_E^5 = -i\sigma_1 \otimes I_2$

with the anticommutator $\{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta^{\mu\nu} I_4$

In chiral Weyl basis the Euclidean Dirac matrices are

$$\gamma_E^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} , \ \gamma_E^j = i\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} , \ \gamma_E^5 = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

compactly $\gamma_E^0 = \sigma_1 \otimes I_2$, $\gamma_E^j = -\sigma_2 \otimes \sigma_j$, $\gamma_E^5 = i\sigma_3 \otimes I_2$

In SO(2,2) (bi-Lorentzian) they are

$\gamma_{(2,2)}^0 = \gamma^0$, $\gamma_{(2,2)}^1 = i\gamma^1$ (time-like)

$\gamma_{(2,2)}^2 = \gamma^2$, $\gamma_{(2,2)}^3 = \gamma^3$ (space-like)

General Dirac matrices Γ_a are nxn matrices, where $n = \begin{cases} 2^{d/2} & , d = 2m \\ 2^{(d-1)/2} & , d = 2m+1 \end{cases}$ for even and odd dimension

In Euclidean space we have in Cl(d)(C) the Dirac generators

for d=2, $\Gamma_a = (i\sigma_1, i\sigma_2)$

for d=3, $\Gamma_a = (i\sigma_1, i\sigma_2, i\sigma_3)$

for d=4, $\Gamma_a^{(4)} = (\gamma_E^1, \gamma_E^2, \gamma_E^3, \gamma_E^0)$, $\gamma_E^0 = \sigma_1 \otimes I_2$, $\gamma_E^j = -\sigma_2 \otimes \sigma_j$, $\gamma_E^5 = i\sigma_3 \otimes I_2$,

for d=5, $\Gamma_a^{(5)} = (\gamma_E^1, \gamma_E^2, \gamma_E^3, \gamma_E^0)$, $\Gamma_5^{(5)} = \gamma_E^5$,

for d=6, $\Gamma_a^{(6)} = -\sigma_2 \otimes \Gamma_a^{(5)}$ $a = 1,2,3,4,5$; $\Gamma_6^{(6)} = \sigma_1 \otimes I$

for d=7, $\Gamma_a^{(7)} = -\sigma_2 \otimes \Gamma_a^{(5)}$ $a = 1,2,3,4,5$; $\Gamma_6^{(7)} = \sigma_1 \otimes I$, $\Gamma_7^{(7)} = i\sigma_3 \otimes I$

This scheme is continued accordingly for higher dimensions.

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