Interval-Based Techniques for Variable-Structure and Backstepping Control of Nonlinear Multi-Input Multi-Output Systems

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- Barrier Lyapunov function techniques: Consideration of state constraints
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  - Control of an inverted pendulum
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First-Order Sliding Mode Control for SISO Systems (1)

System in nonlinear controller canonical form

\[
\dot{x}(t) = \begin{bmatrix}
\dot{x}_1(t) \\
\vdots \\
\dot{x}_{n-1}(t) \\
\dot{x}_n(t)
\end{bmatrix}^T = 
\begin{bmatrix}
x_2(t) \\
\vdots \\
x_n(t) \\
a(x(t), p) + b(x(t), p) \cdot v(t)
\end{bmatrix}^T
\]

with the state vector \( x(t) \in \mathbb{R}^n \)

Requirement for controllability

\( b(x(t), p) \neq 0 \) for any possible operating point and system parameter

Feedback linearizing control law for the output \( y(t) = x_1(t) \)

\[
v(t) = \frac{-a(x(t), p) + u(t)}{b(x(t), p)} \in \mathbb{R}
\]
First-Order Sliding Mode Control for SISO Systems (1)

**System in nonlinear controller canonical form**

\[
\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) & \ldots & \dot{x}_{n-1}(t) & \dot{x}_n(t) \end{bmatrix}^T = \begin{bmatrix} x_2(t) & \ldots & x_n(t) & a(x(t), p) + b(x(t), p) \cdot v(t) \end{bmatrix}^T
\]

with the state vector \(x(t) \in \mathbb{R}^n\)

**Feedback linearizing control law for the output** \(y(t) = x_1(t)\)

\[
v(t) = \frac{-a(x(t), p) + u(t)}{b(x(t), p)} \in \mathbb{R}
\]

System becomes a pure integrator chain of length \(n\) for perfect system and state information (trivially differentially flat system)
First-Order Sliding Mode Control for SISO Systems (2)

$n$-th order integrator chain model with the output $y(t) = x_1(t)$

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\vdots \\
\dot{x}_{n-1}(t) \\
\dot{x}_n(t)
\end{bmatrix}
= 
\begin{bmatrix}
x_2(t) \\
\vdots \\
x_n(t) \\
u(t)
\end{bmatrix}
\]

Definition of the tracking error and its $r$-th time derivative

\[
\tilde{\xi}_1^{(r)}(t) = x_1^{(r)}(t) - x_{1,d}^{(r)}(t)
\quad \text{with} \quad r \in \{0, 1, \ldots, n\}
\]

First-order sliding mode (Hurwitz polynomial of order $n - 1$)

\[
s := s(t) = \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r)}(t), \quad \alpha_{n-1} = 1 \quad \implies \quad s \to 0
\]
Derivation of the Control Law (1)

**Lyapunov function candidate (SISO case)**

\[ V = \frac{1}{2} s^2 > 0 \quad \text{for} \quad s \neq 0 \]

**Stability requirement**

\[ \dot{V} = s \cdot \dot{s} = \left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r)}(t) \right) \cdot \left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r+1)}(t) \right) < 0 \quad \text{for} \quad s \neq 0 \]
Derivation of the Control Law (1)

Lyapunov function candidate (SISO case)

\[ V = \frac{1}{2} s^2 > 0 \quad \text{for} \quad s \neq 0 \]

Stability requirement

\[ \dot{V} = s \cdot \dot{s} = \left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r)}(t) \right) \cdot \left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r+1)}(t) \right) < -\eta \cdot |s| \]
Derivation of the Control Law (1)

Lyapunov function candidate (SISO case)

\[ V = \frac{1}{2} s^2 > 0 \quad \text{for} \quad s \neq 0 \]

Stability requirement

\[
\left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r)}(t) \right) \cdot \left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r+1)}(t) \right) < -\eta \cdot \left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r)}(t) \right) \cdot \text{sign}(s)
\]
Derivation of the Control Law (2)

Control law (first-order sliding mode for SISO systems)

\[ u(t) = x_{1,d}^{(n)}(t) - \sum_{r=0}^{n-2} \alpha_r \tilde{\xi}_{1(r+1)}(t) - \tilde{\eta} \cdot \text{sign}(s) \]

Questions

- What are necessary extensions for the interval case?
- What are the implementation requirements for an interval-valued control signal?
Interval-Based Sliding Mode Control (1)

Definition of tracking error signals and sliding surface

- Specification of a sufficiently smooth desired output trajectory
  \[ y_d = x_{1,d} \]

- Interval definition of the tracking error and its derivatives
  \[ \tilde{\xi}_1^{(r)} \in \left[ \tilde{\xi}_1^{(r)} \right] = \left[ x_1^{(r)} \right] - x_{1,d}^{(r)}, \quad r \in \{0, 1, \ldots, n\} \]

- As before: Desired operating points are located on the sliding surface
  \[ s := \tilde{\xi}_1^{(n-1)}(t) + \sum_{r=0}^{n-2} \alpha_r \cdot \tilde{\xi}_1^{(r)}(t) = 0 \]

- \( \alpha_0, \ldots, \alpha_{n-2} \) are coefficients of a Hurwitz polynomial of order \( n - 1 \)
Interval-Based Sliding Mode Control (1)

Definition of tracking error signals and sliding surface

- Specification of a sufficiently smooth desired output trajectory
  \[ y_d = x_{1,d} \]

- Interval definition of the tracking error and its derivatives
  \[ \tilde{\xi}_1(r) \in \left[ \xi_1(r) \right] = \left[ x_1(r) \right] - x_{1,d}, \quad r \in \{0, 1, \ldots, n\} \]

- As before: Desired operating points are located on the sliding surface
  \[ s := \tilde{\xi}_1^{(n-1)}(t) + \sum_{r=0}^{n-2} \alpha_r \cdot \tilde{\xi}_1^{(r)}(t) = 0 \]

Guaranteed stabilizing control: Lyapunov function candidate

\[ V = \frac{1}{2} s^2 > 0 \quad \text{with} \quad \dot{V} = s \cdot \dot{s} < 0 \quad \text{for} \quad s \neq 0 \]
Interval-Based Sliding Mode Control (2)

Guaranteed stabilization despite uncertainty: Interval formulation of a variable-structure control law

\[
[v] := \frac{-a([x],[p]) + x_{1,d}^{(n)} - \sum_{r=0}^{n-2} \alpha_r \cdot [\tilde{\xi}_1^{(r+1)}] - \tilde{\eta} \cdot \text{sign}([s])}{b([x],[p])}
\]

with a suitably chosen parameter \(\tilde{\eta} > 0\) and \(0 \not\in b([x],[p])\)

Guaranteed stabilizing control: Extraction of suitable point values

\[
V := \{v - \epsilon, v + \epsilon, \bar{v} - \epsilon, \bar{v} + \epsilon\}
\]

with \(\underline{v} := \inf\{[v]\}, \bar{v} := \sup\{[v]\}\) and some small \(\epsilon > 0 \implies \dot{V} < 0\) needs to be satisfied with certainty
Interval-Based Sliding Mode Control (2)

Guaranteed stabilization despite uncertainty: Interval formulation of a variable-structure control law

\[
[v] := \frac{-a ([x], [p]) + x_{1,d}^{(n)} - \sum_{r=0}^{n-2} \alpha_r \cdot [\tilde{\xi}_1^{(r+1)}] - \tilde{\eta} \cdot \text{sign} ([s])}{b ([x], [p])}
\]

with a suitably chosen parameter \( \tilde{\eta} > 0 \) and \( 0 \notin b ([x], [p]) \)

Guaranteed stabilizing control: Extraction of suitable point values

- Guaranteed stabilization of system dynamics
- Extension: Guaranteed state constraints in terms of strict one- and two-sided barrier functions
- Inclusion of bounds on input variation rates (reduction of the effect of chattering)
Second-Order Sliding Mode Control

Second-order sliding mode (including integral tracking error feedback)


Generalization to mixed interval/stochastic uncertainty: Use of the Itô differential operator
Sliding Mode Control with One-Sided State Constraints

Specification of an upper state constraint

\[ x_1 < \bar{x}_{1,\text{max}} := x_{1,d} + \Delta x_{1,\text{max}} \quad \text{with} \quad \Delta x_{1,\text{max}} > 0 \]

Extension of the Lyapunov function candidate by a one-sided barrier function (repelling potential)

Extended ansatz for a Lyapunov function candidate

\[ \tilde{V}^{(A)} = V + V^{(A)} > 0 \quad \text{for} \quad s \neq 0 \quad \text{with} \]

\[ V^{(A)} = \rho_V \cdot \ln \left( \frac{\sigma V \cdot \bar{x}_{1,\text{max}}}{\bar{x}_{1,\text{max}} - x_1} \right) \quad \text{and} \quad x_1 < \bar{x}_{1,\text{max}} \]
Sliding Mode Control with One-Sided State Constraints

Specification of an upper state constraint

\[ x_1 < \bar{x}_{1,\text{max}} := x_{1,d} + \Delta x_{1,\text{max}} \quad \text{with} \quad \Delta x_{1,\text{max}} > 0 \]

Computation of the corresponding time derivative

Extended ansatz for a Lyapunov function candidate

\[ \dot{\tilde{V}}^{\langle A \rangle} = \dot{V} + \dot{\tilde{V}}^{\langle A \rangle} < 0 \quad \text{with} \]

\[ \dot{V}^{\langle A \rangle} = \frac{\rho V}{\bar{x}_{1,\text{max}}} \cdot \left( \frac{-x_1 \cdot \dot{x}_{1,\text{max}} + \dot{x}_1 \cdot \bar{x}_{1,\text{max}}}{\bar{x}_{1,\text{max}} - x_1} \right) , \quad \rho V > 0 , \quad \sigma V > 0 \]

Note

\( \dot{V} \) must have dominating influence in the neighborhood of \( s = 0 \)
Sliding Mode Control with One-Sided State Constraints

Specification of an upper state constraint

\[ x_1 < \bar{x}_{1, \text{max}} := x_{1,d} + \Delta x_{1, \text{max}} \quad \text{with} \quad \Delta x_{1, \text{max}} > 0 \]

Modified stability requirement

\[
s \cdot \left( \sum_{r=0}^{n-2} \alpha_r \tilde{\xi}_1^{(r+1)} + u - x_{1,d}^{(n)} + \eta \cdot \text{sign}(s) + \frac{1}{s} \cdot \dot{V}(A) \right) + \beta \cdot \text{sign}(s) < 0
\]
Sliding Mode Control with Two-Sided State Constraints

Specification of worst-case state deviations

\[ |x_1 - x_{1,d}| < \bar{\chi} > 0 \text{ , } \bar{\chi} = \text{const} \]

Extension of the Lyapunov function candidate by a two-sided barrier function

Extended ansatz for a Lyapunov function candidate

\[
\tilde{V}^{\langle B \rangle} = V + V^{\langle B \rangle} > 0 \quad \text{for } s \neq 0 \quad \text{with}

V^{\langle B \rangle} = \rho_V \cdot \ln \left( \frac{\bar{\chi}^{2l}}{\bar{\chi}^{2l} - (x_1 - x_{1,d})^{2l}} \right) \quad \text{and } l \in \mathbb{N} \]
Sliding Mode Control with Two-Sided State Constraints

Specification of worst-case state deviations

\[ |x_1 - x_{1,d}| < \bar{\chi} > 0 , \quad \bar{\chi} = \text{const} \]

Computation of the corresponding time derivative

Extended ansatz for a Lyapunov function candidate

\[ \dot{V}^{(B)} = \dot{V} + \dot{V}^{(B)} < 0 \quad \text{with} \]

\[ \dot{V}^{(B)} = \rho_V \cdot \frac{2l \cdot (x_1 - x_{1,d})^{2l-1} \cdot (\dot{x}_1 - \dot{x}_{1,d})}{\bar{\chi}^{2l} - (x_1 - x_{1,d})^{2l}} , \quad \rho_V > 0 \]

Note

\( \dot{V} \) must have dominating influence in the neighborhood of \( s = 0 \)
Extended First-Order Sliding Mode Control with State Constraints

New control signals for the first-order sliding mode control

\[
\begin{align*}
    u^{(A)} &= u - \frac{s}{s^2 + \tilde{\epsilon}} \cdot \dot{V}^{(A)} \\
    u^{(B)} &= u - \frac{s}{s^2 + \tilde{\epsilon}} \cdot \dot{V}^{(B)}
\end{align*}
\]

Remarks

- The approximation \( \frac{1}{s} \approx \frac{s}{s^2 + \tilde{\epsilon}} \) is only necessary for \(|s| \gg 0\).
- The variable-structure part is deactivated in a close vicinity of \(|s| = 0\) in the interval case.
- Note: For \(0 \in [s]\), the sign of \(s\) cannot be determined unambiguously.
Interval-Based Sliding Mode Control (continued)

Guaranteed stabilization despite uncertainty: Interval formulation of a variable-structure control law

\[
[v] := \frac{-a([x],[p]) + x_1^{(n)} - \sum_{r=0}^{n-2} \alpha_r \cdot \tilde{\xi}_1^{(r+1)} - \tilde{\eta} \cdot \text{sign}([s])}{b([x],[p])}
\]

Extension in the case of one-sided state constraints

\[
[v^{(A)}] = [v] - \frac{1}{b([x],[p])} \cdot \frac{[s]}{[s]^2 + \tilde{\epsilon}} \cdot [\dot{V}^{(A)}]
\]
Interval-Based Sliding Mode Control (continued)

Guaranteed stabilization despite uncertainty: Interval formulation of a variable-structure control law

\[
[v] := \frac{-a([x], [p]) + x_{1,d}^{(n)} - \sum_{r=0}^{n-2} \alpha_r \cdot [\tilde{\xi}_1^{(r+1)}] - \tilde{\eta} \cdot \text{sign}([s])}{b([x], [p])}
\]

Extension in the case of one-sided state constraints

\[
[v^{(A)}] = [v] - \frac{1}{b([x], [p])} \cdot \frac{[s]}{[s]^2 + \tilde{\epsilon}} \cdot [\dot{V}^{(A)}]
\]

Extension in the case of two-sided state constraints

\[
[v^{(B)}] = [v] - \frac{1}{b([x], [p])} \cdot \frac{[s]}{[s]^2 + \tilde{\epsilon}} \cdot [\dot{V}^{(B)}]
\]
Position Control of a Point Mass

System model

- Position: $x_1$
- Velocity: $x_2$
- Input force: $x_3$ (mass normalized to 1)

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
x_3 \\
p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 v
\end{bmatrix}
$$

with the uncertain parameters $p_i \in [-0.1; 0.1]$, $i \in \{1, 2, 3\}$, and $p_4 = 1$, containing both asymptotically stable and unstable realizations.
## Control Parameterizations

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
<th>Case 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V^{(I)}$, $\alpha_0 = 1$, $\alpha_1 = 0.9$</td>
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<td>$V^{(I)}$, $\alpha_0 = 15$, $\alpha_1 = 0.9$</td>
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</tr>
<tr>
<td>—</td>
<td>$\rho_V = 0.5$, $\sigma_V = 1$, $\Delta x_{1,\text{max}} = 0.01$</td>
<td>$\rho_V = 0.75$, $\sigma_V = 1$, $\Delta x_{1,\text{max}} = 0.01$</td>
<td>—</td>
<td>$\rho_V = 5$, $l = 1$, $\bar{\chi} = 0.165$</td>
</tr>
<tr>
<td>$p_i = 0$, $i = {1, 2, 3}$, $p_4 = 1$</td>
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<td>$p_i \in [-0.1; 0.1]$, $i = {1, 2, 3}$, $p_4 = 1$</td>
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<td>$0.0025 \cdot [-1; 1]$</td>
</tr>
</tbody>
</table>

**Lyapunov function $V$**

- $V^{(I)}$
- $V^{(I,A)}$
- $V^{(I)}$
- $V^{(I,B)}$

**Barrier**

- —
- $\rho_V$
- $\sigma_V$
- $\Delta x_{1,\text{max}}$
- —

**System parameters**

- $p_i = 0$
- $i = \{1, 2, 3\}$
- $p_4 = 1$
- $p_i \in [-0.1; 0.1]$
- $i = \{1, 2, 3\}$
- $p_4 = 1$
- $p_i \in [-0.1; 0.1]$
- $i = \{1, 2, 3\}$
- $p_4 = 1$
- $p_i \in [-0.1; 0.1]$
- $i = \{1, 2, 3\}$
- $p_4 = 1$

**Measurement tolerance for $x_1$**

- —
- $0.0025 \cdot [-1; 1]$
- $0.0025 \cdot [-1; 1]$
- $0.0025 \cdot [-1; 1]$

**Variable-structure gain**

- $\tilde{\eta} = 20$
- $\tilde{\eta} = 20$
- $\tilde{\eta} = 20$
- $\tilde{\eta} = 400$
- $\tilde{\eta} = 400$
Simulation Results

System output (Case 1)

![Graph of System output (Case 1)]

Tracking error (Case 1)

![Graph of Tracking error (Case 1)]

Violation of one-sided state constraint $\Rightarrow$ Barrier function is deactivated, parameters are assumed to be exactly known.
Simulation Results

No violation of one-sided state constraint $\implies$ Barrier function is activated, parameters are assumed to be exactly known.
Simulation Results

System output (Case 3)

Tracking error (Case 3)

No violation of one-sided state constraint $\Rightarrow$ Barrier function is activated, parameters and measured states are uncertain
Simulation Results

System output (Case 4)

Tracking error (Case 4)

Violation of two-sided state constraint $\implies$ Barrier function is deactivated, parameters and measured states are uncertain
Simulation Results

No violation of two-sided state constraint $\implies$ Barrier function is activated, parameters and measured states are uncertain
Backstepping Control for SISO Systems (1)

Interval-based backstepping control

- SISO system model in strict feedback form

\[
\begin{align*}
\dot{x}_1 &= a_1 (x_1, p) + b_1 (x_1, p) \cdot x_2 \\
\dot{x}_2 &= a_2 (x_1, x_2, p) + b_2 (x_1, x_2, p) \cdot x_3 \\
& \quad \vdots \\
\dot{x}_n &= a_n (x_1, \ldots, x_n, p) + b_n (x_1, \ldots, x_n, p) \cdot v
\end{align*}
\]

- Obvious prerequisite for controllability \( b_1 (x_1, p) \neq 0, \ldots, b_n (x_1, \ldots, x_n, p) \neq 0 \)
- Uncertain parameters \( p \in [p] \)
Interval-based backstepping control

- Successive stabilization of the dynamics for $x_i$, $i \in \{1, \ldots, n\}$
- Treatment of the states $x_{i+1}$ as *virtual control signals* $x^*_i$ in the equations for $\dot{x}_i$

Stabilizing control synthesis

- Definition of the Lyapunov function candidate

$$\tilde{V}_i = \sum_{j=1}^{i} V_j > 0 , \text{ e.g. } V_j = \frac{1}{2} (x_j - x_{j,d})^2$$

with the positive-definite summands $V_j > 0 (x \neq x_d)$

- Differentiability of the virtual control signals is required up to $x_n$
Backstepping Control for SISO Systems (3)

**Stability requirements**

- Computation of $x_{i+1}^*$ ($i \in \{1, \ldots, n-1\}$) and the actual control $v$ ($i = n$) by satisfying the stability condition

$$
\dot{\tilde{V}}_i = \sum_{j=1}^{i} \dot{V}_j < 0 \ , \ x \neq x_d \implies \dot{\tilde{V}}_i = \dot{\tilde{V}}_{i-1} + \dot{V}_i < 0
$$

- The inequality $\dot{\tilde{V}}_{i-1} < 0$ is ensured due to the previous design stage
- Straightforward extension to the Barrier Lyapunov function approach with

$$
\tilde{V}_i^{\langle \zeta \rangle} = V_{i}^{\langle \zeta \rangle} + \sum_{j=1}^{i} V_j > 0 \text{ with } \dot{\tilde{V}}_i^{\langle \zeta \rangle} = \dot{V}_i^{\langle \zeta \rangle} + \sum_{j=1}^{i} \dot{V}_j < 0 \ , \ \zeta \in \{A, B\}
$$
Backstepping Control for SISO Systems (4)

Stabilizing (virtual) control signals

- Control strategy inspired by the previous variable-structure techniques to ensure fast convergence to the desired trajectories
- Stabilization and perfect tracking according to

$$\left( \begin{array}{c} -a_i(x_1, \ldots, x_i, p) + \dot{x}_i,d - k_i \cdot (x_i - x_i,d) \\ b_i(x_1, \ldots, x_i, p) \\ -\tilde{\eta}_i \cdot \text{sat}(x_i - x_i,d) - \chi_i \\ b_i(x_1, \ldots, x_i, p) \end{array} \right) = \begin{cases} x_{i+1}^* & \text{for } i \leq n - 1 \\ v & \text{for } i = n \end{cases}$$

with $k_i > 0$, $\tilde{\eta}_i > 0$, and

$$\chi_i = \frac{(x_i - x_i,d) \cdot \dot{V}_i}{(x_i - x_i,d)^2 + \tilde{\epsilon}}$$
Backstepping Control for SISO Systems (4)

Stabilizing (virtual) control signals

- Control strategy inspired by the previous variable-structure techniques to ensure fast convergence to the desired trajectories
- Stabilization and perfect tracking according to

\[
\begin{align*}
\left( -a_i (x_1, \ldots, x_i, p) + \dot{x}_i,d - k_i \cdot (x_i - x_i,d) \right) \\
\frac{1}{b_i (x_1, \ldots, x_i, p)} \\
- \tilde{\eta}_i \cdot \text{sat} \left( x_i - x_i,d \right) - \chi_i \\
\right) = \begin{cases} 
  x_{i+1}^* & \text{for } i \leq n - 1 \\
  v & \text{for } i = n 
\end{cases}
\]

with \( k_i > 0, \tilde{\eta}_i > 0, \nu > 0 \) (sufficiently small) and

\[
\text{sat} \left( x_i - x_i,d \right) = \begin{cases} 
  \tanh \left( \frac{1}{\nu} (x_i - x_i,d) \right) & \text{for } i \leq n - 1 \\
  \text{sign} (x_i - x_i,d) & \text{for } i = n
\end{cases}
\]
Interval-Based Backstepping Control (1)

Stabilizing (virtual) control signals

- Interval-based generalization of the control law

\[
\begin{bmatrix}
-a_i ([x_1] , \ldots , [x_i] , [p]) + [\dot{x}_{i,d}] - k_i \cdot ([x_i] - [x_{i,d}]) \\
\frac{b_i ([x_1] , \ldots , [x_i] , [p])}{b_i ([x_1] , \ldots , [x_i] , [p])}
\end{bmatrix}
\]

\[-\tilde{\eta}_i \cdot \text{sat} ([x_i] - [x_{i,d}]) - [\chi_i] \]

\[
\begin{cases}
[x_{i+1}^*] & \text{for } i \leq n - 1 \\
 v & \text{for } i = n
\end{cases}
\]

with

\[
[\chi_i] = \frac{([x_i] - [x_{i,d}]) \cdot [\dot{V}_i^\langle \zeta \rangle]}{([x_i] - [x_{i,d}])^2 + \bar{\epsilon}}
\]
Interval-Based Backstepping Control (1)

Stabilizing (virtual) control signals

- Interval-based generalization of the control law

\[
\begin{aligned}
&( -a_i ([x_1], \ldots, [x_i], [p]) + [\dot{x}_{i,d}] - k_i \cdot ( [x_i] - [x_{i,d}] ) \bigg) \\
&\bigg( \\
&b_i ([x_1], \ldots, [x_i], [p]) \\
\end{aligned}
\]

\[
\begin{aligned}
&-\tilde{\eta}_i \cdot \text{sat} \left( [x_i] - [x_{i,d}] \right) - [\chi_i] \bigg) \\
&b_i ([x_1], \ldots, [x_i], [p]) \\
\end{aligned}
\]

\[
\begin{bmatrix}
[x_{i+1}] \\
[v]
\end{bmatrix} \quad \text{for } i \leq n - 1
\]

\[
\begin{bmatrix}
[x_{i+1}] \\
[v]
\end{bmatrix} \quad \text{for } i = n
\]

with

\[
\text{sat} \left( [x_i] - [x_{i,d}] \right) = \begin{cases}
\tanh \left( \frac{1}{\nu} ( [x_i] - [x_{i,d}] ) \right) & \text{for } i \leq n - 1 \\
\text{sign} \left( [x_i] - [x_{i,d}] \right) & \text{for } i = n
\end{cases}
\]
Interval-Based Backstepping Control (1)

Stabilizing (virtual) control signals

- Interval-based generalization of the control law

\[
\left( -a_i ([x_1], \ldots, [x_i], [p]) + [\dot{x}_i,d] - k_i \cdot ([x_i] - [x_i,d]) \right) / b_i ([x_1], \ldots, [x_i], [p])
- \tilde{\eta}_i \cdot \text{sat} ([x_i] - [x_i,d]) - [\chi_i] / b_i ([x_1], \ldots, [x_i], [p])
\]

\[
= \begin{cases} 
[x_{i+1}^*] & \text{for } i \leq n - 1 \\
[v] & \text{for } i = n 
\end{cases}
\]

with

\[
\text{sign} ([x_i] - [x_i,d]) = \begin{cases} 
-1 & \text{for } \sup ([x_i] - [x_i,d]) < 0 \\
+1 & \text{for } \inf ([x_i] - [x_i,d]) > 0 \\
0 & \text{else}
\end{cases}
\]
Interval-Based Backstepping Control (2)

Note

The third case in the interval $\text{sign}$ function

$$
\text{sign} ([x_i] - [x_{i,d}]) = \begin{cases} 
-1 & \text{for } \sup ([x_i] - [x_{i,d}]) < 0 \\
+1 & \text{for } \inf ([x_i] - [x_{i,d}]) > 0 \\
0 & \text{else}
\end{cases}
$$

reduces chattering, when the sign of the tracking error signal $[x_i] - [x_{i,d}]$ ($i = n$) cannot be determined unambiguously.
Control of an Inverted Pendulum

System model: Inverted pendulum on a carriage

- State vector \( \mathbf{z} = [\alpha \ z \ \dot{\alpha} \ \dot{z}]^T \); Control variable: Force \( F_C \)
- Derivation of a set of first-order state equations with

\[
\ddot{\alpha} = \frac{2g \sin(\alpha) (M + m) - ma\dot{\alpha}^2 \sin(2\alpha) + 2 \cos(\alpha) F_C}{a (2M + m (1 - \cos(2\alpha)))}
\]
\[
\dot{z} = \frac{mg \sin(2\alpha) - 2ma \sin(\alpha)\dot{\alpha}^2 + 2F_C}{2M + m (1 - \cos(2\alpha))}
\]

point mass: mass \( m \)
rod: massless, length \( a \)
carriage: mass \( M \)
Control of an Inverted Pendulum

System model: Inverted pendulum on a carriage

- State vector $\mathbf{z} = [\alpha \ z \ \dot{\alpha} \ \dot{z}]^T$; $y$: horizontal pendulum tip position
- Derivation of a set of first-order state equations with

\[
\ddot{\alpha} = \frac{2g \sin(\alpha) (M + m) - ma \dot{\alpha}^2 \sin(2\alpha) + 2 \cos(\alpha) F_C}{a (2M + m (1 - \cos(2\alpha)))}
\]

and

\[
\ddot{z} = \frac{mg \sin(2\alpha) - 2ma \sin(\alpha) \dot{\alpha}^2 + 2F_C}{2M + m (1 - \cos(2\alpha))}
\]
Approximate Transformation into Nonlinear Controller
Canonical Form (1)

Successive computation of the output’s Lie derivatives

\[
y = -a \sin(\alpha) + z, \quad \dot{y} = -a \cos(\alpha) \dot{\alpha} + \dot{z}
\]

\[
\ddot{y} = a \sin(\alpha) \dot{\alpha}^2 + mg \sin(2\alpha) - 2ma \sin(\alpha) \dot{\alpha}^2 + 2FC
\]

\[
\frac{2M + m (1 - \cos(2\alpha))}{2M + m (1 - \cos(2\alpha))}
\]

\[
\approx -p_1 g \alpha + p_2
\]

\[
\dddot{y} = p_1 g \ddot{\alpha} + p_3
\]

\[
y^{(4)} = p_1 g \dddot{\alpha} + p_4 = p_1 g \frac{2g \sin(\alpha) (M + m) - ma \dot{\alpha}^2 \sin(2\alpha) + 2 \cos(\alpha) FC}{a (2M + m (1 - \cos(2\alpha)))} + p_4
\]

\[
+ \frac{2p_1 g \cos(\alpha)}{a (2M + m (1 - \cos(2\alpha)))} FC =: a (x, p) + b (x, p) \cdot v
\]
Approximate Transformation into Nonlinear Controller
Canonical Form (2)

State-space representation for variable-structure control design

- State vector after transformation of coordinates $\mathbf{x} = [y \quad \dot{y} \quad \ddot{y} \quad \dddot{y}]^T$
- Control input $v := F_C$
- Interval parameters $[p_i]$, $i \in \{1, \ldots, 4\}$ (representation of approximation and modeling errors)
- Error for angle measurement $[-0.01 ; 0.01] \text{ rad}$ and carriage position measurement $[-0.01 ; 0.01] \text{ m}$

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= a(\mathbf{x}, \mathbf{p}) + b(\mathbf{x}, \mathbf{p}) \cdot v
\end{align*}
\]
Approximate Transformation into Nonlinear Controller
Canonical Form (3)

State-space representation for backstepping control design:

- State vector after transformation of coordinates \( x = [y \ y' \ y'' \ y''']^T \)
- Control input \( v := F_C \)
- Interval parameters \([p_i], i \in \{1, \ldots, 4\}\) (representation of approximation and modeling errors)
- Error for angle measurement \([-0.01 ; 0.01]\) rad and carriage position measurement \([-0.01 ; 0.01]\) m

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 + p_2 \\
\dot{x}_3 &= x_4 + p_3 \\
\dot{x}_4 &= a_4(x, p_1, p_4) + b_4(x, p_1, p_4) \cdot v
\end{align*}
\]
Simulation Results: Variable-Structure Control (VS)

Violation of two-sided state constraint $\Rightarrow$ Barrier function is deactivated, parameters and measured states are uncertain
Simulation Results: Variable-Structure Control (VS)

No violation of two-sided state constraint $\implies$ Barrier function is activated, parameters and measured states are uncertain.
Simulation Results: Backstepping Control (BS)

Position $y$ (Case 3)

Error $y_d - y$ (Case 3)

No violation of two-sided state constraint $\implies$ Barrier function is activated, parameters and measured states are uncertain
Simulation Results: Combined BS and VS Control

Position \( y \) (Case 4)

![Position graph](image)

Error \( y_d - y \) (Case 4)

![Error graph](image)

No violation of two-sided state constraint \( \implies \) Barrier function is activated, parameters and measured states are uncertain

Switching to VS as soon as \( 0 \in [s] \) has been reached for the first time
Sliding Mode Control for MIMO Systems (1)

State-space representation and fundamental MIMO extension

Consider the input affine nonlinear system

\[ \dot{x}(t) = a(x(t)) + B(x(t)) \cdot u(t) \]

with \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( a(x(t)) \in \mathbb{R}^n \), and \( B(x(t)) \in \mathbb{R}^{n \times m} \)

Specify \( m \) switching functions \( s(\tilde{x}) = [s_1(\tilde{x}) \ldots s_m(\tilde{x})]^T \)

General formulation of the variable structure controller

\[ u_{s,i}(\tilde{x}) = \begin{cases} 
    u_{s,i}^+(\tilde{x}) & \text{for } s_i(\tilde{x}) > 0 \\
    u_{s,i}^-(\tilde{x}) & \text{for } s_i(\tilde{x}) < 0 , \quad i \in \{1, \ldots, m\} 
\end{cases} \]

Goal: Reach the sliding surface \( s(\tilde{x}) \) in finite time

As in the SISO case, the dynamics on the sliding surface \( s(\tilde{x}) \) has an order smaller than \( n \).
Sliding Mode Control Control Example 1
Backstepping Control Example 2
Extension to MIMO Systems
Conclusions

Sliding Mode Control for MIMO Systems (2)

Equivalent control approach

- Derivation of the equivalent control (Utkin, 1971 and 1972) from the sliding condition ($\tilde{x}$ represents state/output tracking errors)

\[
\dot{s}(\tilde{x}) = \frac{\partial s(\tilde{x})}{\partial \tilde{x}} \cdot \frac{d\tilde{x}}{dt} = \frac{\partial s(\tilde{x})}{\partial \tilde{x}} \cdot (\dot{x} - \dot{x}_d)
\]

\[
= \frac{\partial s(\tilde{x})}{\partial \tilde{x}} \cdot (a(x) - \dot{x}_d) + \frac{\partial s(\tilde{x})}{\partial \tilde{x}} \cdot B(x) \cdot u_{eq}(\tilde{x}) = 0
\]

gives (under regularity assumptions for $\frac{\partial s(\tilde{x})}{\partial \tilde{x}} \cdot B(x)$)

\[
u = u_{eq}(\tilde{x}) = -\left(\frac{\partial s(\tilde{x})}{\partial \tilde{x}} \cdot B(x)\right)^{-1} \cdot \frac{\partial s(\tilde{x})}{\partial \tilde{x}} \cdot (a(x) - \dot{x}_d)
\]
Combination of equivalent and variable structure control components

- Stabilization of the desired trajectory using a (quadratic) Lyapunov function

\[
V(s(\tilde{x})) = \frac{1}{2} \sum_{i=1}^{m} s_i^2(\tilde{x})
\]

with

\[
\dot{V}(s(\tilde{x})) = \sum_{i=1}^{m} s_i(\tilde{x}) \dot{s}_i(\tilde{x}) \leq - \sum_{i=1}^{m} \eta_i |s_i(\tilde{x})| < 0 , \quad \eta_i > 0 \text{ for } s_i(\tilde{x}) \neq 0
\]

- Directly applicable if system model is already given in MIMO nonlinear controller canonical form \( \Longrightarrow \) Decoupling of the system dynamics

- All \( s_i \) are then parameterized as Hurwitz polynomials according to the respective relative degrees of each system output \( y_i \ (i \in \{1, \ldots, m\}) \)
Generalization to Systems Not Given in Controller
Canonical Form (1)

Approach 1: Cascaded control procedure

- (Approximate) State-Space transformation into integrator chains for the system outputs
- Necessary approximations result in interval variables which represent worst-case bounds for the approximation errors
- Successive stabilization of each chain up to the last but one element by the regularized control approach shown in the backstepping procedure
- The last stage corresponds to an interval variable-structure control with an (approximate) decoupling of the system dynamics
- Prerequisite: Interval equivalent of \[ \frac{\partial s(\tilde{x})}{\partial \tilde{x}} \cdot B(x) \] must be regular for all possible states and parameters
Generalization to Systems Not Given in Controller
Canonical Form (2)

Approach 2: Centralized control procedure

- Avoid state-space transformation by defining a vector-valued tracking error function for each of the original system states.
- Define a Lyapunov function candidate with the goal to obtain vanishing tracking error vectors.
- Instead of the original equivalent control, an underlying (linear) feedback control procedure with stationary or dynamic feedforward signals may be included which stabilizes the system in the close vicinity to the desired states.
- The complete control signal is then given as the underlying controller and a variable-structure component.
Generalization to Systems Not Given in Controller
Canonical Form (2)

Approach 2: Centralized control procedure

- Definition of the variable structure control in terms of
  \[ u_{i,s} = \eta_i^T \cdot \text{sign} (\tilde{x}) , \quad i \in \{1, \ldots, m\} \]
  for each of the inputs

- Compute the vectors \( \eta_i \) online so that stability of the error dynamics can be guaranteed despite interval uncertainty:

Conclusions and Outlook on Future Work

- Control-oriented modeling of dynamic systems
- Verified parameter identification as the basis for control design
- Stabilization of the error dynamics using interval techniques
- Handling of input and state constraints (guaranteed overshoot prevention, two-sided worst-case bounds for the system output)
- Use of interval analysis in real time
- Extension by backstepping-like control procedures
Conclusions and Outlook on Future Work

- Control-oriented modeling of dynamic systems
- Verified parameter identification as the basis for control design
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- Use of interval analysis in real time
- Extension by backstepping-like control procedures

- Extension by a sensitivity-based predictive controller
- Extension by a (sensitivity-based) state and disturbance observer
Thank you for your attention!
Merci beaucoup pour votre attention!
Спасибо за Ваше внимание!
Dziękuję bardzo za uwagę!
¡Muchas gracias por su atención!
Grazie mille per la vostra attenzione!
Vielen Dank für Ihre Aufmerksamkeit!