Observability of Nonlinear Systems and Injectivity

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Introduction

Systems

\[ \dot{x}(t) = f(x(t)), \quad x(0) = x^0 \]
\[ y(t) = h(x(t)) \]

Definitions

\[ y^1 \text{ corresponds to } x^0 = z^1 \]
\[ y^2 \text{ corresponds to } x^0 = z^2 \]

Observability

For all \( z^1 \neq z^2 \) we have \( y^1 \neq y^2 \).
How to check observability?

Find computational condition!

Simpler problem: Check
For all $z^1 \neq z^2 \in x^I$ we have $y^1 \neq y^2$.

$x^I$ interval vector
Lie Series

Taylor series expansion of $y$

$$y(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left( L^k_f h \right)(x^0)$$

Lie derivative

$$\begin{align*}
(L_f h)(x) &= \left( \frac{\partial h}{\partial x} \right) f(x) \\
(L^0_f h)(x) &= h(x) \\
(L^{l+1}_f h)(x) &= \left( L_f \left( L^l_f h \right) \right), l \geq 1
\end{align*}$$

$f, h$ real analytic $\Rightarrow y(t)$ real analytic.
Reformulation

\[ y^i(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left( L^k f h \right) (z^i) \]

When is \( y^1(t) = y^2(t) \)?

\[ \sum_{k=0}^{\infty} \frac{t^k}{k!} \left( L^k f h \right) (z^1) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left( L^k f h \right) (z^2) \]

\[ \Downarrow \]

\[ \left( L^k f h \right) (z^1) = \left( L^k f h \right) (z^2) , k = 0, 1, 2, ... \]

Observability mapping

\[ F(x) := \left( \begin{array}{c} h(x) \\ (L_f h)(x) \\ (L^2_f h)(x) \\ \vdots \end{array} \right) \]
Injectivity Condition

\[ F(z^1) = F(z^2) \Rightarrow z^1 = z^2 \text{ uniquely} \]

\[ \Rightarrow \text{System is observable.} \]

- This is injectivity of \( F \)!

- First step: finite number of columns
Injectivity Condition

- $F(z^1) = F(z^2)$ for $z^1, z^2 \in x^I \Rightarrow z^1 = z^2$ uniquely
- $F(z^1) - F(z^2) = 0$
- Idea: Apply mean value theorem

$$g(z^1) - g(z^2) = \frac{\partial g}{\partial x}(\xi)(z^1 - z^2),$$

g scalar, $\xi$ between $z^1$ and $z^2$

$$g = \left( \begin{array}{c} g_1 \\ \vdots \\ g_p \end{array} \right), \
g_i(z^1) - g_i(z^2) = \frac{\partial g_i}{\partial x}(z^1 - z^2)$$

$$g(z^1) - g(z^2) = \left( \begin{array}{c} \frac{\partial g_1}{\partial x}(\xi_1) \\ \vdots \\ \frac{\partial g_p}{\partial x}(\xi_p) \end{array} \right)(z^1 - z^2)$$

$$M = \left( \begin{array}{c} \frac{\partial g_1}{\partial x}(\xi_1) \\ \vdots \\ \frac{\partial g_p}{\partial x}(\xi_p) \end{array} \right), \
M \neq \frac{\partial g}{\partial x}(\xi)$$

$M \in \frac{\partial g}{\partial x}(x^I) = M^I$ Interval matrix

$M$ full rank $\Rightarrow z^1 - z^2 = 0 \Rightarrow g$ injective
Injectivity Condition

- Interval matrix $M^I$ full rank $\Rightarrow M$ full rank $\Rightarrow F$ injective $\Rightarrow$ system observable on $x^I$

- $M^I$ full rank?

- Apply result of Jiri Rohn (sufficient condition)

- In general NP hard

- Application in the talk by Thomas Paradowski
Thank you for your attention