Vector set inversion interval filtering based fault observer design

International Online Seminar on Interval Methods in Control Engineering

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February 26, 2021
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SET-MEMBERSHIP FILTERING

Feasible set
state, measurement output, disturbance, noise, ...

Geometry
interval, ellipsoid, zonotope, paralotope, ...

Introduction
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SET-MEMBERSHIP FILTERING

- Assume that the disturbances and noises of system are unknown but bounded
- Express feasible set with simple geometry
- Eliminate errors caused by inaccurate system models
- The computational complexity increasing with the increase of the system dimensions and high conservative
- It is widely used in signal processing, fault diagnosis, robot and other fields
**Design criterion**

The error system between the estimated state value and the true value of system is stable.

Common design methods:
- Coordinate transformation
- Pole placement
- LMI
WRAPPING EFFECT

- The axis-aligned box used in interval recursive calculation process brings errors in feasible solution set.
- As the amount of calculation increases, the accumulation of errors will cause too large enclosures.
Given a linear time-invariant system described by:

$$\begin{cases}
x_{k+1} = Ax_k + Bu_k + Ew_k + Ff_k \\
y_k = Cx_k + Dv_k
\end{cases}$$

- $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^p$, $y_k \in \mathbb{R}^q$ are the state, input, and output vectors, respectively
- $f_k \in \mathbb{R}^m$ is actuator fault vector
- $w_k \in \mathbb{R}^s$ and $v_k \in \mathbb{R}^l$ denote system disturbances and the measurement noises
- $A$, $B$, $C$, $D$, $E$ and $F$ are constant matrices with appropriate dimensions
Augmented state vector

By combining the actuator fault $f_k$ and the state vector $x_k$:

$$\bar{x}_k = \begin{bmatrix} x_k \\ f_k \end{bmatrix}.$$ 

Augmented system

Construct the augmented system:

$$\begin{cases} 
\bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{B}u_k + \bar{E}w_k + \bar{G}\Delta f_k, \\
y_k = \bar{C}\bar{x}_k + \bar{D}v_k, 
\end{cases}$$

where $\bar{A} = \begin{bmatrix} A & F \\ 0 & I_m \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $\bar{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}$, $\bar{G} = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$, $\bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}$, $\bar{D} = D$, $\Delta f_k = f_{k+1} - f_k$. 
Interval observer

Construct state interval observer of augmented system*:

\[ \hat{x}_k = T \bar{A} \hat{x}_{k-1} + T \bar{B} u_{k-1} + L (y_{k-1} - \bar{C} \hat{x}_{k-1}) + N y_k \]

- \( \hat{x}_k \in \mathbb{R}^{n+m} \) is estimated vector of augmented state vector \( \bar{x}_k \)
- \( L \in \mathbb{R}^{(n+m) \times q} \) is observer gain
- \( T \in \mathbb{R}^{n+m} \) and \( N \in \mathbb{R}^{(n+m) \times q} \) are constant matrices

*see in Wang Z H, et al., Systems & Control Letters 2018

Error system

Define error system:

\[ e_k = \bar{x}_k - \hat{x}_k \]

\[ = (T \bar{A} - L \bar{C}) e_{k-1} + T \bar{E} w_{k-1} + T \bar{G} \Delta f_{k-1} - L \bar{D} v_{k-1} - N \bar{D} v_k. \]
Given a scalar $\gamma > 0$, if there are positive definite matrices $P \in \mathbb{R}^{n+m}$, $Y \in \mathbb{R}^{(n+m) \times n+m+q}$, $Z \in \mathbb{R}^{(n+m) \times q}$ satisfy

$$
\begin{bmatrix}
I_{n+m} - P & * & * & * & * & * & * \\
0 & -\gamma^2 I_s & * & * & * & * & * \\
0 & 0 & -\gamma^2 I_m & * & * & * & * \\
0 & 0 & 0 & -\gamma^2 I_l & * & * & * \\
0 & 0 & 0 & 0 & -\gamma^2 I_l & * & * \\
P(T\bar{A} - L\bar{C}) & PTE & PT\bar{G} & -PL\bar{D} & -P\bar{N}\bar{D} & -P
\end{bmatrix} < 0,
$$

the error system is stable.
Using the proposed theorem:

- $e_k$ is robust to disturbance and noise
- Design interval observer by $H_{\infty}$ technique
- Calculate error interval $[e_k]$ and observer state estimation interval $[\bar{x}_k]^o = \hat{x}_k + [e_k]$ by interval operation

**Question:** How to reduce the impact of the wrapping effect?
Vector set inversion problem

\[ X = \{ [\bar{x}_k]^v \in \mathbb{IR}^n | O_{(k:k+s)}[\bar{x}_k]^v \subset [Y_k] \} = O_{(k:k+s)}^{-1}[Y_k] \]

where, \( X \) is solution set, \([\bar{x}_k]^v \) is vector set inversion interval

\[ [Y_k] = y_{(k:k+s)} - O_{u(k:k+s)} u_{(k:k+s)} - O_{f(k:k+s)}[\Delta f_{(k:k+s)}] - O_{w(k:k+s)}[w_{(k:k+s)}] - O_{v(k:k+s)}[v_{(k:k+s)}], \]

\( y_{(k:k+s)} \), \( u_{(k:k+s)} \), \( [\Delta f_{(k:k+s)}] \), \( [w_{(k:k+s)}] \) and \( [v_{(k:k+s)}] \) are the output, input, fault difference interval, disturbance interval and noise interval of the system from time instant \( k \) to \( k + s \), respectively
Vector set inversion problem

\[
X = O^{-1}_{(k:k+s)}(y(k:k+s) - O_{u(k:k+s)} u(k:k+s) - O_f(k:k+s) \Delta f(k:k+s) \\
- O_w(k:k+s) \bar{w}(k:k+s) - O_v(k:k+s) \bar{v}(k:k+s))
\]

where,

\[
O_{(k:k+s)} = \begin{bmatrix}
\bar{C} \\
\bar{C} \bar{A} \\
\vdots \\
\bar{C} \bar{A}^s
\end{bmatrix},
O_{v(k:k+s)} = \begin{bmatrix}
\bar{D} & 0 & \ldots & 0 \\
0 & \bar{D} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \bar{D}
\end{bmatrix}
\]

\[
O_{u(k:k+s)} = \begin{bmatrix}
0 & 0 & \ldots & 0 \\
\bar{C} \bar{B} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\bar{C} \bar{A}^{s-1} \bar{B} & \bar{C} \bar{A}^{s-2} \bar{B} & \ldots & \bar{C} \bar{B}
\end{bmatrix}
\]
Vector set inversion problem

\[ X = O^{-1}_{(k:k+s)} \left( y(k:k+s) - O_{u(k:k+s)} u(k:k+s) - O_{f(k:k+s)} [\Delta f(k:k+s)] ight) 
- O_{w(k:k+s)} [w(k:k+s)] - O_{v(k:k+s)} [v(k:k+s)] \]

where,

\[ O_{f(k:k+s)} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \overline{C} \overline{G} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{C} \overline{A}^{s-1} \overline{G} & \overline{C} \overline{A}^{s-2} \overline{G} & \ldots & \overline{C} \overline{G} \end{bmatrix} \]

\[ O_{w(k:k+s)} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \overline{C} \overline{E} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{C} \overline{A}^{s-1} \overline{E} & \overline{C} \overline{A}^{s-2} \overline{E} & \ldots & \overline{C} \overline{E} \end{bmatrix} \]
Use SIVIA algorithm to solve the above problem. The larger the time length $s$, the higher the accuracy and the greater the amount of calculation.

**Solution:** Convert the interval boxes to the form of row vectors
\( \mathcal{L} \) is a vector group representing all interval boxes in the solution process. There are four different situations:

1. \( O_{(k:k+s)} \mathcal{L}_i \) and \( [Y_k] \) intersect, but not completely belong to \( [Y_k] \)
2. \( O_{(k:k+s)} \mathcal{L}_i \) and \( [Y_k] \) have no intersection
3. \( O_{(k:k+s)} \mathcal{L}_i \) belongs entirely to \( [Y_k] \)
4. \( O_{(k:k+s)} \mathcal{L}_i \) have partial intersection with \( [Y_k] \) and the width of the corresponding interval box is less than the precision parameter
Test function

\[ [t](\cdot) = \begin{cases} 
  \text{in}, & O_{(k:k+s)} \mathcal{L}_i \subset [Y_k] \\
  \text{out}, & O_{(k:k+s)} \mathcal{L}_i \cap [Y_k] = \emptyset \\
  \text{eps}, & W(\mathcal{L}_i) < \varepsilon 
\end{cases} \]

- \text{in}, \text{out} and \text{eps} are all column vectors of Boolean variables equal to the dimension of \( \mathcal{L} \)
- \( W(\mathcal{L}_i) \) is the column vector composed of the width of each interval box in \( \mathcal{L}_i \)
- \( \varepsilon \) is precision parameter
Test function

\[ [t](\cdot) = \begin{cases} 
  \text{in}, & O_{(k:k+s)}L_i \subset [Y_k] \\
  \text{out}, & O_{(k:k+s)}L_i \cap [Y_k] = \emptyset \\
  \text{eps}, & W(L_i) < \varepsilon 
\end{cases} \]

1. If \( O_{(k:k+s)}L_i \subset [Y_k] \), \( \text{in}(i) = 1 \). Otherwise, \( \text{in}(i) = 0 \). Push the vector group \( L(\text{in}) \) into feasible set \( \mathcal{N} \)
2. If \( O_{(k:k+s)}L_i \cap [Y_k] = \emptyset \), \( \text{out}(i) = 1 \), \( L(\neg \text{in} \land \neg \text{out}) \) belongs to the uncertain vector group \( \mathcal{U} \)
3. If \( W(L_i) < \varepsilon \), \( \text{eps}(i) = 1 \), push \( \mathcal{U}(\text{eps}) \) into uncertain set \( \mathcal{E} \)
4. Bisect the remaining interval boxes in \( \mathcal{U} \)
5. Loop until \( \mathcal{L} \) is empty
The solution set $[\overline{x}_k]^v$ obtained by the vector set inversion interval filtering algorithm satisfies:

$$[\overline{x}_k]^v \subset X \subset [\overline{x}_k]^v \cup \mathcal{E}.$$  

**Proof.** In the process of solving, $O_{(k:k+s)}[\overline{x}_k]^v_i$ completely belongs to $[Y_k]$, $[\overline{x}_k]^v_i$ is a feasible subset satisfying

$$[\overline{x}_k]^v_i \subset \mathcal{N}$$

The union of all feasible subsets in $\mathcal{N}$ is $[\overline{x}_k]^v$, namely

$$\bigcup_{i=1,2,...} [\overline{x}_k]^v_i = [\overline{x}_k]^v \subset O^{-1}_{(k:k+s)}[Y_k] = X$$
Theorem

The solution set $[\bar{x}_k]^{\mathcal{V}}$ obtained by the vector set inversion interval filtering algorithm satisfies:

$$[\bar{x}_k]^{\mathcal{V}} \subset X \subset [\bar{x}_k]^{\mathcal{V}} \cup \mathcal{E}. \quad \Box$$

Similarly, when $O_{(k:k+s)}[\bar{x}_k]^{\mathcal{V}}_i$ and $[\mathcal{Y}_k]$ have a partial intersection and the width of $[\bar{x}_k]^{\mathcal{V}}_i$ is less than the precision parameter $\varepsilon$, $[\bar{x}_k]^{\mathcal{V}}_i$ is an uncertain subset satisfying

$$[\bar{x}_k]^{\mathcal{V}}_i \subset \mathcal{E}$$

All uncertain subsets form an uncertain layer $\mathcal{E}$ satisfying

$$X \setminus [\bar{x}_k]^{\mathcal{V}} \subset \mathcal{E}$$

Therefore, $[\bar{x}_k]^{\mathcal{V}} \subset X \subset [\bar{x}_k]^{\mathcal{V}} \cup \mathcal{E}. \quad \Box$
Consider the linear system model with:

- state matrix \( A = \begin{bmatrix} 0.9842 & 0.0407 \\ 0 & 0.9590 \end{bmatrix} \)
- input matrix \( B = \begin{bmatrix} 0.0831 & 0.0007 \\ 0 & 0.0352 \end{bmatrix} \)
- output matrix \( C = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix} \)
- disturbance matrix \( E = \begin{bmatrix} 0.9842 & 0.0407 \\ 0 & 0.9590 \end{bmatrix} \)
- noise matrix \( D = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.05 \end{bmatrix} \)
- actuator fault matrix \( F = [0.8 \quad 0]^T \)
Obtain the interval observer matrix parameters:

\[
T = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0.6550 & 0 \\
-1.2496 & 0.0086 & 1
\end{bmatrix},
\]
\[
L = \begin{bmatrix}
0 & 0 \\
-0.0393 & 0.3780 \\
-1.2241 & -0.0225
\end{bmatrix},
\]
\[
N = \begin{bmatrix}
2 & 0 \\
0 & 0.6900 \\
2.4992 & -0.0173
\end{bmatrix}
\]

In simulation,

- initial observation state \( \hat{x}_0 = [0 \ 0]^T \)
- initial error interval \([e_0] = \begin{bmatrix} [e_{0,1}] \\ [e_{0,2}] \end{bmatrix} = \begin{bmatrix} [-0.1 \ 0.1] \\ [-0.1 \ 0.1] \end{bmatrix} \)
- unknown disturbance \(|w_k| \leq [0.2 \ 0.2]^T \)
- unknown noise \(|v_k| \leq [0.2 \ 0.2]^T \)
- input \( u = [3 \ 3]^T \)
Actuator fault:

\[ f_k = \begin{cases} 
0, & k < 50, k \geq 100 \\
10, & 50 \leq k < 100 
\end{cases} \]

Fig: Fault estimation results

Fig: Comparison of state estimation results at 
\( k=38 \sim 45 \) time instant
Conclusions

- The actuator fault observation method of linear system with unknown but bounded disturbance and noise is studied.
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- Use multi-time output data to reduce the wrapping effect of interval calculation.
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• Use multi-time output data to reduce the wrapping effect of interval calculation.
• Solve the problem that the calculation time of the traditional interval filtering algorithm increases exponentially as the interval dimension increases.
Conclusions

- The actuator fault observation method of linear system with unknown but bounded disturbance and noise is studied.
- Use multi-time output data to reduce the wrapping effect of interval calculation.
- Solve the problem that the calculation time of the traditional interval filtering algorithm increases exponentially as the interval dimension increases.
- Vector set inversion interval filtering based fault observer can be extended to deal with fault diagnosis problems in aircraft systems, multi-machine node systems, servo motor systems, diode circuits and other engineering fields.
Thank you for your kind attention!