Set-membership estimation for discrete-time systems: the two-step method

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4 Conclusion and outlook
Objective:
To construct a set enclosing all admissible state values that are consistent with the system model and measurement data;

Assumption:
Model uncertainties are unknown but bounded;

Applications:
Fault diagnosis, constrained MPC (Model Predictive Control), bioprocess monitoring, etc.

Existing methods:
Geometrical methods: Ellipsoids, polytopes, zonotopes, etc.;
Interval observer: two sub-observers designed based on the monotone system theory.
Set-membership estimation

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Motivation

Existing problems

- Geometrical methods suffer from high computational complexity due to dealing with Minkowski sum and set intersection.
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- Design a set-membership estimation method for discrete-time systems with less computational complexity;
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- Geometrical methods suffer from high computational complexity due to dealing with Minkowski sum and set intersection.
- The over-approximation error in set operations will cause conservative estimation results.
- Interval observers suffer from the cooperative constraints, which may cause large conservatism.

Objectives:

- Design a set-membership estimation method for discrete-time systems with less computational complexity;
- Reduce the approximation error to increase the estimation accuracy.
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2. Set-membership estimation for regular discrete-time systems

3. Set-membership estimation for descriptor systems

4. Conclusion and outlook
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2 Set-membership estimation for regular discrete-time systems
   • The two-step set-membership estimation method
   • Interval estimation based on reachability analysis

3 Set-membership estimation for descriptor systems

4 Conclusion and outlook
Preliminaries

Definitions

Definition 1. Given two sets $S_1 \subset \mathbb{R}^n$ and $S_2 \subset \mathbb{R}^n$, their Minkowski sum is defined as

$S_1 \oplus S_2 = \{ s : s = s_1 + s_2 ; s_1 \in S_1 ; s_2 \in S_2 \}$

Definition 2. For a set $S \subset \mathbb{R}^n$, its interval hull is defined as the smallest interval vector containing it, which is denoted as

$S \subseteq \text{Box}(S) = [a; b]^T$ where $a = [a_1; \ldots; a_n]^T$ and $b = [b_1; \ldots; b_n]^T$. 
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where $a = [a_1, \ldots, a_n]^T$ and $b = [b_1, \ldots, b_n]^T$. 
Definitions

**Definition 3.** An $m$-order zonotope $\mathcal{Z} \subset \mathbb{R}^n$ is an affine transformation of a hypercube $\mathcal{B}^m$, which is defined as

$$\mathcal{Z} = \langle p, H \rangle = \{ p + Hz : z \in \mathcal{B}^m \},$$

where $p \in \mathbb{R}^n$ is the center of $\mathcal{Z}$ and $H \in \mathbb{R}^{n \times m}$ is called its generator matrix.
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**Figure:** A 4-order zonotope.
Properties

**Property 1. (Minkowski Sum)** Given two zonotopes \( \langle p_1, H_1 \rangle \subset \mathbb{R}^n \) and \( \langle p_2, H_2 \rangle \subset \mathbb{R}^n \), their Minkowski sum satisfies

\[
\langle p_1, H_1 \rangle \oplus \langle p_2, H_2 \rangle = \langle p_1 + p_2, [H_1, H_2] \rangle.
\]
Properties

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Properties

Property 2. (Linear Transformation) Given a zonotope $\mathcal{Z} = \langle p, H \rangle \subset \mathbb{R}^n$, its linear transformation associated with the matrix $L \in \mathbb{R}^{m \times n}$ satisfies

$$L\mathcal{Z} = \langle Lp, LH \rangle.$$
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Properties

**Property 3.** (Interval Hull) For an \( m \)-order zonotope \( \mathcal{Z} = \langle p, H \rangle \subset \mathbb{R}^n \), the components of \( \text{Box}(\mathcal{Z}) = [a, b] \) can be obtained from

\[
\begin{align*}
a_i &= p_i - \sum_{j=1}^{m} |H_{i,j}|, \quad i = 1, \ldots, n \\
b_i &= p_i + \sum_{j=1}^{m} |H_{i,j}|, \quad i = 1, \ldots, n
\end{align*}
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$$

\[x_1\] \[x_2\]

Figure: A zonotope.
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$$

**Figure:** The interval hull of the zonotope.
Consider the following discrete-time system:

\[
\begin{aligned}
    x_{k+1} &= Ax_k + Bu_k + Ew_k \\
    y_k &=Cx_k + Fv_k
\end{aligned}
\]  

(1)

where \( x_k \in \mathbb{R}^{n_x}, u_k \in \mathbb{R}^{n_u}, y_k \in \mathbb{R}^{n_y}, w_k \in \mathbb{R}^{n_w} \) and \( v_k \in \mathbb{R}^{n_v} \).
Problem formulation

System description

Consider the following discrete-time system:

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Assumption

\( x_0, w_k \) and \( v_k \) are unknown but bounded as follows

\[
    x_0 \in \langle p_0, H_0 \rangle, \quad w_k \in \mathcal{W} = \langle 0, H_w \rangle, \quad v_k \in \mathcal{V} = \langle 0, H_v \rangle,
\]

where \( p_0 \in \mathbb{R}^{n_x} \), \( H_0 \in \mathbb{R}^{n_x \times n_x} \), \( H_w \in \mathbb{R}^{n_w \times n_w} \) and \( H_v \in \mathbb{R}^{n_v \times n_v} \).
Set-membership estimation

Objective

- We aim to estimate a convex $X_k$ such that $x_k \in X_k$ for all $k \geq 0$. 
Set-membership estimation

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Robust observer design

Figure: The schematic of the two-step method.
Set-membership estimation

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Figure: The schematic of the two-step method.
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Figure: The schematic of the two-step method.
Two-step set-membership estimation

Figure: The demonstration of the two-step method.
Two-step set-membership estimation

Observer

\[
\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k) \quad \rightarrow \quad \text{state point estimation}
\]
Two-step set-membership estimation

Observer

\[ \hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k) \quad \rightarrow \quad \text{state point estimation} \]

Error system

\[ e_{k+1} = (A - LC)e_k + Ew_k - LFv_k \quad \rightarrow \quad e_k \in E_k \]

reachability analysis
Two-step set-membership estimation

**Observer**

\[
\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k) \quad \rightarrow \quad \text{state point estimation}
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**Error system**

\[
e_{k+1} = (A - LC)e_k + Ew_k - LFv_k \quad \rightarrow \quad e_k \in E_k
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**Set-membership estimation**

\[
x_k \in X_k = \hat{x}_k \oplus E_k
\]
Robust observer design

Size of $E_k$  $\rightarrow$ estimation accuracy
Robust observer design

Size of $E_k \rightarrow$ estimation accuracy

Robust observer design based on $H_\infty$ technique

Given a scalar $\gamma > 0$, if there exist a positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$ and a matrix $Y \in \mathbb{R}^{n_x \times n_y}$ such that

$$
\begin{bmatrix}
I_{n_x} - P & * & * & * \\
0 & -\gamma^2 I_{n_w} & * & * \\
0 & 0 & -\gamma^2 I_{n_v} & * \\
PA - YC & PE & -YF & -P
\end{bmatrix} < 0
$$

and $L = P^{-1}Y$, then the transfer function $G_{ed}(z) = (zl_{n_x} - A_e)^{-1}B_e$ satisfies $\|G_{ed}(z)\|_\infty < \gamma$. Moreover, $e_k$ is bounded.
Robust observer design

Size of $E_k \rightarrow$ estimation accuracy

Robust observer design based on $H_\infty$ technique

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PA - YC & PE & -YF & -P
\end{bmatrix} \preceq 0
\]

and $L = P^{-1}Y$, then the transfer function $G_{ed}(z) = (zl_{n_x} - A_e)^{-1}B_e$ satisfies $\|G_{ed}(z)\|_\infty < \gamma$. Moreover, $e_k$ is bounded.

Minimize $\gamma$ to increase estimation accuracy
Set-membership estimation

Reachability analysis

\[ e_{k+1} \in (A - LC)\langle 0, \tilde{H}_k \rangle \oplus EW \oplus (-LF)V \quad \rightarrow \quad e_k \in \langle 0, H_k \rangle \]
Set-membership estimation

Reachability analysis

\[ e_{k+1} \in (A - LC)\langle 0, \tilde{H}_k \rangle \oplus EW \oplus (-LF)V \implies e_k \in \langle 0, H_k \rangle \]

\[ H_{k+1} = [(A - LC)\tilde{H}_k \quad EH_w \quad -LFH_v] \]
Set-membership estimation

Reachability analysis

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Set-membership estimation

\[ x_k \in X_k = \langle \hat{x}_k, H_k \rangle, \quad \forall k \geq 0 \]
Set-membership estimation

Reachability analysis

\[ e_{k+1} \in (A - LC)\langle 0, \tilde{H}_k \rangle \oplus EW \oplus (-LF)V \quad \longrightarrow \quad e_k \in \langle 0, H_k \rangle \]

\[ H_{k+1} = [(A - LC)\tilde{H}_k \quad EH_w \quad -LFH_v] \]

Set-membership estimation

\[ x_k \in X_k = \langle \hat{x}_k, H_k \rangle, \quad \forall k \geq 0 \]

Order reduction

Limit the dimension of \( \langle 0, H_k \rangle \):

\[ \tilde{H}_k = \begin{cases} H_k, & m \leq s; \\ \mathcal{R}_s(H_k), & m > s. \end{cases} \]
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Interval estimation

\[ e_k \leq e_k \leq \bar{e}_k \quad \rightarrow \quad e_k + \hat{x}_k \leq x_k \leq \hat{x}_k + \bar{e}_k \]
Interval estimation

\[ e_k \leq e_k \leq \bar{e}_k \quad \quad \Rightarrow \quad \quad e_k + \hat{x}_k \leq x_k \leq \hat{x}_k + \bar{e}_k \]

Estimation error

\[ e_k = (A - LC)^k e_0 + \sum_{i=0}^{k-1} (A - LC)^i Ew_{k-1-i} + \sum_{i=0}^{k-1} (A - LC)^i (-LFv_{k-1-i}) \]
Interval estimation

\[ e_k \leq e_k \leq \bar{e}_k \quad \rightarrow \quad e_k + \hat{x}_k \leq x_k \leq \hat{x}_k + \bar{e}_k \]

Estimation error

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Reachable set

\[ E_k = (A - LC)^k E_0 \bigoplus \bigoplus_{i=0}^{k-1} (A - LC)^i E W \bigoplus \bigoplus_{i=0}^{k-1} (A - LC)^i (-LFV) \]
Interval hull

Properties

- **Property 1.** Give sets $S_i \subset \mathbb{R}^n$ ($i = 1, \ldots, m$), the interval hull of their Minkowski sum satisfies

$$\text{Box}(\bigoplus_{i=1}^{m} S_i) = \bigoplus_{i=1}^{m} \text{Box}(S_i)$$
Interval hull

Properties

- **Property 1.** Give sets $S_i \subset \mathbb{R}^n$ ($i = 1, \ldots, m$), the interval hull of their Minkowski sum satisfies

$$\text{Box}(\bigoplus_{i=1}^{m} S_i) = \bigoplus_{i=1}^{m} \text{Box}(S_i)$$

- **Property 2.** Given two interval vectors $[a, b] \subset \mathbb{R}^n$ and $[c, d] \subset \mathbb{R}^n$, their Minkowski sum satisfies

$$[a, b] \oplus [c, d] = [a + c, b + d]$$
Interval estimation

Interval hull of error set

\[
\text{Box}(E_k) = \bigoplus_{i=0}^{k-1} \text{Box}((A-LC)^i EW) \bigoplus_{i=0}^{k-1} \text{Box}((A-LC)^i EW) \bigoplus_{i=0}^{k-1} ((A-LC)^i (-LFV))
\]
Interval estimation

Interval hull of error set

\[
\text{Box}(\mathbf{E}_k) = \bigoplus_{i=0}^{k-1} \text{Box}\left( (A-LC)^i E W \right) \oplus \bigoplus_{i=0}^{k-1} \text{Box}\left( (A-LC)^i E W \right) \oplus \bigoplus_{i=0}^{k-1} ((A-LC)^i (-LFV))
\]

Interval estimation

- Interval estimation of \( e_k \): \([e_k, \overline{e}_k] = \text{Box}(\mathbf{E}_k)\)
Interval estimation

Interval hull of error set

\[
\text{Box}(E_k) = \text{Box}((A-LC)^i EW) \bigoplus_{i=0}^{k-1} \text{Box}((A-LC)^i EW) \bigoplus_{i=0}^{k-1} ((A-LC)^i (-LFV))
\]

Interval estimation

- Interval estimation of \( e_k \): \([e_k, \bar{e}_k] = \text{Box}(E_k)\)
- Interval estimation of \( x_k \): \[
\begin{align*}
\underline{x}_k &= \hat{x}_k + e_k, \\
\overline{x}_k &= \hat{x}_k + \bar{e}_k
\end{align*}
\]
Algorithm 1 Interval estimation based on reachability analysis

Input: $u_k$, $y_k$
Output: $\underline{x}_k$, $\overline{x}_k$

1: **Initialization**:
2: $\hat{x}_0 = p_0$, $D_{w_0} = EW$, $D_{v_0} = -LFV$
3: $S_{x_0} = \langle 0, H_0 \rangle$, $S_{w_0} = \emptyset$, $S_{v_0} = \emptyset$
4: **for** $k \geq 0$ **do**
5: $[e_k, \overline{e}_k] = \text{Box}(S_{x_k}) \oplus S_{w_k} \oplus S_{v_k}$
6: $\overline{x}_k = \hat{x}_k + \overline{e}_k$
7: $\underline{x}_k = \hat{x}_k + e_k$
8: $\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - C\hat{x}_k)$
9: $S_{x_{k+1}} = (A - LC)S_{x_k}$
10: $S_{w_{k+1}} = S_{w_k} \oplus \text{Box}(D_{w_k})$
11: $S_{v_{k+1}} = S_{v_k} \oplus \text{Box}(D_{v_k})$
12: $D_{w_{k+1}} = (A - LC)D_{w_k}$
13: $D_{v_{k+1}} = (A - LC)D_{v_k}$
Comparison with other methods

Comparison (theoretically provable)

Under the same conditions:

1. The interval estimation obtained by Algorithm 1 is more accurate than that by the zonotope-based method [1];
2. The interval estimation obtained by Algorithm 1 is more accurate than that by the regular interval observer [2];
3. The interval estimation obtained by Algorithm 1 is more accurate than that by the interval observer based on coordinate transformation [3].

Comparison with other methods

Comparison (theoretically provable)

Under the same conditions:

- The interval estimation obtained by Algorithm 1 is more accurate than that by the zonotope-based method\([1]\);

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Comparison with other methods

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Under the same conditions:

- The interval estimation obtained by Algorithm 1 is more accurate than that by the zonotope-based method\(^1\);
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Comparison with other methods

Comparison (theoretically provable)

Under the same conditions:

- The interval estimation obtained by Algorithm 1 is more accurate than that by the zonotope-based method\(^1\);
- The interval estimation obtained by Algorithm 1 is more accurate than that by the regular interval observer\(^2\);
- The interval estimation obtained by Algorithm 1 is more accurate than that by the interval observer based on coordinate transformation\(^3\);

Simulation results

Consider a DC motor\(^1\):

\[
\begin{bmatrix}
\dot{\theta} \\
\dot{n} \\
\dot{i}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & -\frac{\mu}{J} & \frac{K_t}{J} \\
0 & -\frac{K_e}{L} & -\frac{R}{L}
\end{bmatrix}
\begin{bmatrix}
\theta \\
n \\
i
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
\frac{1}{L}
\end{bmatrix} u.
\]

---

Simulation results

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\theta \\
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i
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
\frac{1}{L}
\end{bmatrix} u.
\]

The discrete-time model parameters:

\[
A = \begin{bmatrix} 1 & 0.1 & 0 \\ 0 & 0.8495 & 0.4977 \\ 0 & -0.0357 & 0.9995 \end{bmatrix},
B = \begin{bmatrix} 0 \\ 0 \\ 0.0729 \end{bmatrix},
C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},
E = l_3, F = l_2.
\]

Simulation results

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\]

The discrete-time model parameters:

\[
A = \begin{bmatrix}
1 & 0.1 & 0 \\
0 & 0.8495 & 0.4977 \\
0 & -0.0357 & 0.9995
\end{bmatrix},
B = \begin{bmatrix}
0 \\
0 \\
0.0729
\end{bmatrix},
C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix},
E = I_3, F = I_2.
\]

Disturbance and measurement noise:

\[
|w_k| \leq \begin{bmatrix}
0.0225 \\
0.0225 \\
0.0404
\end{bmatrix},
|v_k| \leq \begin{bmatrix}
0.0564 \\
0.0564
\end{bmatrix}.
\]

Simulation results

$[\bar{x}_k, \underline{x}_k]$—Algorithm 1;
$[\bar{x}^z_k, \underline{x}^z_k]$—zonotope-based method$^1$;
$[\bar{x}^i_k, \underline{x}^i_k]$—regular interval observer$^2$;
$[\bar{x}^t_k, \underline{x}^t_k]$—interval observer based on coordinate transformation$^3$

$L_1$—the observer gain designed by the proposed method;
$L_2$—the observer gain designed under interval observer constraint
Simulation results

\[ [x_k, \bar{x}_k] \text{— Algorithm 1;} \]
\[ [x^z_k, \bar{x}^z_k] \text{— zonotope-based method}^{[1]}; \]
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\[ [\underline{x}_k, \overline{x}_k] \text{– Algorithm 1; } \]

\[ [\underline{x}^z_k, \overline{x}^z_k] \text{– zonotope-based method} \cite{1}; \]

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1 Context

2 Set-membership estimation for regular discrete-time systems

3 Set-membership estimation for descriptor systems

4 Conclusion and outlook
Problem formulation

System description

Consider the following descriptor system:

\[
\begin{aligned}
Ex_{k+1} &= Ax_k + Bu_k + D_w w_k \\
y_k &= Cx_k + D_v v_k
\end{aligned}
\]  

(2)
Problem formulation

System description

Consider the following descriptor system:

\[
\begin{align*}
E x_{k+1} &= A x_k + B u_k + D_w w_k \\
y_k &= C x_k + D_v v_k
\end{align*}
\]  \hspace{1cm} (2)

Assumptions

- **Assumption 1.** (UBB, Unknown But Bounded):

\[
x_0 \in \langle \hat{x}_0, H_0 \rangle, \quad w_k \in W = \langle 0, H_w \rangle, \quad v_k \in V = \langle 0, H_v \rangle.
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Problem formulation

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Consider the following descriptor system:

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  \[x_0 \in \langle \hat{x}_0, H_0 \rangle, \quad w_k \in W = \langle 0, H_w \rangle, \quad v_k \in V = \langle 0, H_v \rangle.\]

- **Assumption 2.** (Observable):
  
  \[\text{rank } \begin{bmatrix} E \\ C \end{bmatrix} = n_x, \quad \text{rank } \begin{bmatrix} zE - A \\ C \end{bmatrix} = n_x, \quad z \in \mathbb{C}.\]
Two-step set-membership estimation

Observer

\[
\hat{x}_k = T\hat{x}_{k-1} + TBu_{k-1} + L(y_{k-1} - \hat{x}_{k-1}) + Ny_k \quad \rightarrow \quad \text{state point estimation}
\]
Two-step set-membership estimation

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\[ TE + NC = I_{nx} \]

Error system

\[ e_k = (TA - LC)e_{k-1} + TD_w w_{k-1} - LD_v v_{k-1} - ND_v v_k \quad \rightarrow \quad e_k \in E_k = \langle 0, H_k \rangle \]
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Set-membership estimation

\[ x_k \in X_k = \hat{x}_k \oplus E_k \]
Robust observer design based on $H_\infty$ technique

Given a scalar $\gamma > 0$, if there exist a positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$ and matrices $W \in \mathbb{R}^{n_x \times n_y}$, $Y \in \mathbb{R}^{n_x \times (n_x+n_y)}$ such that

$$
\begin{bmatrix}
I_{n_x} - P & * & * & * & * \\
0 & -\gamma^2 I_{n_w} & * & * & * \\
0 & 0 & -\gamma^2 I_{n_v} & * & * \\
0 & 0 & 0 & -\gamma^2 I_{n_v} & * \\
\Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & -P
\end{bmatrix} \prec 0,
$$

$$
\Omega_1 = P\Theta^\dagger \alpha_1 A + Y\psi \alpha_1 A - WC,
\Omega_2 = P\Theta^\dagger D_w + Y\psi \alpha_1 D_w,
\Omega_3 = -WD_v,
\Omega_4 = -P\Theta^\dagger \alpha_2 D_v - Y\psi \alpha_2 D_v,
\Theta = \begin{bmatrix} E \\ C \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} I_{n_x} \\ 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix},
\psi = I_{n_x+n_y} - \Theta \Theta^\dagger
$$
Robust observer design based on $H_\infty$ technique

and $T$, $N$, $L$ satisfy

\[
T = \Theta^\dagger \alpha_1 + P^{-1} Y \psi \alpha_1, \\
N = \Theta^\dagger \alpha_2 + P^{-1} Y \psi \alpha_2, \\
L = P^{-1} W
\]

then, estimation error is robust against disturbance and noise, and satisfies $\| G_{ed}(z) \|_\infty < \gamma$. 

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then, estimation error is robust against disturbance and noise, and satisfies $\|G_{ed}(z)\|_\infty < \gamma$.

Optimization

Minimize $\gamma$ to increase estimation accuracy
Set-membership estimation

Reachability analysis

\[ e_{k+1} \in (TA - LC)\langle 0, \tilde{H}_k \rangle \oplus TD_w\langle 0, H_w \rangle \oplus (-LD_v)\langle 0, H_v \rangle \oplus (-ND_v)\langle 0, H_v \rangle \]
Set-membership estimation

Reachability analysis

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\[ e_k \in E_k = \langle 0, H_k \rangle \rightarrow H_{k+1} = \begin{bmatrix} (TA - LC)\tilde{H}_k & TD_wH_w & -LD_vH_v & -ND_vH_v \end{bmatrix} \]
Set-membership estimation for descriptor systems

Reachability analysis

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Set-membership estimation

\[ x_k \in X_k = \langle \hat{x}_k, H_k \rangle, \quad \forall k \geq 0 \]
Interval estimation

\[ e_k \leq e_k \leq \bar{e}_k \quad \rightarrow \quad \hat{x}_k + e_k \leq x_k \leq \hat{x}_k + \bar{e}_k \]
Interval estimation

\[ e_k \leq e_k \leq \bar{e}_k \quad \rightarrow \quad \hat{x}_k + e_k \leq x_k \leq \hat{x}_k + \bar{e}_k \]

Estimation error

\[ e_k = (TA - LC)^k e_0 + \sum_{i=0}^{k-1} (TA - LC)^i TD_w w_{k-1-i} + \sum_{i=0}^{k-1} (TA - LC)^i (-LD_v v_{k-1-i}) \]

\[ + \sum_{j=0}^{k-1} (TA - LC)^j (-ND_v v_{k-j}) \]
Set-membership estimation for descriptor systems

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\[ e_k \leq e_k \leq \bar{e}_k \quad \rightarrow \quad \hat{x}_k + e_k \leq x_k \leq \hat{x}_k + \bar{e}_k \]

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\[
+ \sum_{j=0}^{k-1} (TA - LC)^j (-ND_v v_{k-j})
\]
\[ e_k = (TA - LC)^k e_0 + \sum_{i=0}^{k-1} (TA - LC)^i TD_w w_{k-i} - (TA - LC)^{k-1} LD_v v_0 \]
\[ - ND_v v_k + \sum_{i=0}^{k-2} (TA - LC)^i [(LC - TA) ND_v - LD_v] v_{k-1-i}, \quad k \geq 2 \]
\[ e_1 = (TA - LC)e_0 + TD_w w_0 - LD_v v_0 - ND_v v_1 \]
\[e_k = (TA - LC)^k e_0 + \sum_{i=0}^{k-1} (TA - LC)^i TD_w w_{k-i} - (TA - LC)^{k-1} LD_v v_0 \]

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\[ - ND_v v_k + \sum_{i=0}^{k-2} (TA - LC)^i [(LC - TA) ND_v - LD_v] v_{k-1-i}, \quad k \geq 2 \]

\[ e_1 = (TA - LC) e_0 + TD_w w_0 - LD_v v_0 - ND_v v_1 \]

Reachable sets

\[ E_k = (TA - LC)^k E_0 \oplus \bigoplus_{i=0}^{k-1} (TA - LC)^i TD_w W \oplus \bigoplus_{i=0}^{k-2} (TA - LC)^i [(LC - TA) ND_v - LD_v] V, \quad k \geq 2 \]

\[ E_1 = (TA - LC) E_0 \oplus TD_w W \oplus (-LD_v) V \oplus (-ND_v) V \]
Interval estimation of $e_k$

$$[e_k, \bar{e}_k] = \text{Box}(E_k), \quad k \geq 0$$
Interval estimation

Interval estimation of $e_k$

\[ [e_k, \bar{e}_k] = \text{Box}(E_k), \quad k \geq 0 \]

Interval estimation of $x_k$

\[
\begin{cases}
\underline{x}_k = \hat{x}_k + \underline{e}_k \\
\overline{x}_k = \hat{x}_k + \bar{e}_k
\end{cases}
\]
Algorithm 1 Interval estimation based on reachability analysis

**Input:** $u_k, y_k$

**Output:** $\underline{x}_k, \overline{x}_k$

1: **Initialization:**

   2: $X_0 = \langle 0, H_0 \rangle$, $\hat{x}_0 = p_0$, $M_0 = D_w W$, $D_0 = \emptyset$, $N_0 = \emptyset$, $S_{a0} = -LD_v V$, $S_{b0} = -ND_v V$, $S_{c0} = \emptyset$, $[\overline{c}_0, \underline{c}_0] = Box(X_0)$, $\overline{x}_0 = \hat{x}_0 + \overline{c}_0$, $\underline{x}_0 = \hat{x}_0 + \underline{c}_0$.

3: **for** $k \geq 0$ **do**

   4: $X_k = (TA - LC)X_{k-1}$

   5: $D_k = D_{k-1} \oplus Box(M_{k-1})$

   6: $N_k = N_{k-1} \oplus Box(S_{ak-1}) \oplus Box(S_{bk-1}) \oplus Box(S_{ck-1})$

   7: $M_k = (TA - LC)M_{k-1}$

   8: $S_{ak} = (TA - LC)S_{ak-1}$

   9: $S_{bk} = (TA - LC)S_{bk-1}$

10: **if** $k = 1$ **then**

11: $S_{ck} = ((LC - TA)ND_v - LD_v)V$

12: $S_{ck} = (TA - LC)S_{ck-1}$

13: $[\overline{c}_k, \underline{c}_k] = Box(X_k) \oplus D_k \oplus N_k$

14: $\hat{x}_k = TA\hat{x}_{k-1} + TBu_{k-1} + L(y_{k-1} - C\hat{x}_{k-1}) + Ny_k$

15: $\overline{x}_k = \hat{x}_k + \overline{c}_k$

16: $\underline{x}_k = \hat{x}_k + \underline{c}_k$
Simulation results

Consider a numerical system with parameters as follows:

\[
E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.8 & 0.95 & 0 \\ -1 & 0.5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad D_w = I_3, \quad D_v = I_2.
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Disturbance and measurement noise: \( w_k \in \langle 0, 0.1I_3 \rangle, \: v_k \in \langle 0, 0.1I_2 \rangle. \)
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\]

Disturbance and measurement noise: \( w_k \in \langle 0, 0.1l_3 \rangle, \ v_k \in \langle 0, 0.1l_2 \rangle. \)

\[
T = \begin{bmatrix}
0.8941 & 0.1059 & 0.3783 \\
0.6901 & 0.3099 & 0.5364 \\
-0.8941 & -0.1059 & -0.3783
\end{bmatrix}, \quad N = \begin{bmatrix}
0 & 0.1059 \\
0 & -0.6901 \\
1 & -0.1059
\end{bmatrix},
\]

\[
L = \begin{bmatrix}
0.1546 & -0.1744 \\
0.2156 & -0.3246 \\
-0.1545 & 0.1743
\end{bmatrix}.
\]
Simulation results

![Graph showing state and proposed methods]

Simulation results

![Simulation results graph](image)

Simulation results

![Graph showing simulation results](image)

Context

Set-membership estimation for regular discrete-time systems

Set-membership estimation for descriptor systems

Conclusion and outlook
We propose a two-step set-membership estimation method by combining robust observer design with reachability analysis technique; under the same condition, the proposed method can obtain more accurate estimation than the exiting methods; the proposed two-step method is extended to descriptor systems.

Outlook:
- Extend the proposed two-step method to nonlinear systems;
- Apply the two-step method to robust control for instance the constrained MPC.
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Thanks for your attention!