Robust output feedback MPC using interval observers

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Outline

1. Motivation
2. Problem statement
3. Design of interval observer and predictor
4. Interval MPC
5. Numerical example
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1 Motivation
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4 Interval MPC
5 Numerical example
Motivation

- Constrained systems are **recurrent**: physical limitations, performance and safety;

![Chemical reactor diagram](image1)

![Power electronics diagram](image2)

![Vehicle control diagram](image3)

- Usual feedback solutions based on Lyapunov methods often **fail** to ensure constraint satisfaction → **Model Predictive Control**

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Chemical reactors [Wikipedia]  
Power electronics [Elprocus]  
Vehicle control [MPC and VDL Labs]
Motivation

- What about robustness?
  - Model uncertainties and noises → discrepancies between prediction and real system;
  - Unavailable states → state estimation;
  - How to ensure constraint satisfaction and feasibility?

Classical solutions: Tubes (rigid, homothetic), error set-membership estimation, moving-horizon estimation (MHE), minmax optimization, multi-stage MPC, …
Motivation

• What about *robustness*?

Illustration of loss of feasibility due to uncertainty
Outline

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Consider the following discrete-time LPV system:

\begin{align*}
x_{k+1} &= A(\theta_k)x_k + B(\theta_k)u_k + w_k \\
y_k &= Cx_k + v_k
\end{align*}

(1)

where \( x_k \) is the state vector, \( u_k \) is the control input, \( y_k \) is the measurement vector, \( w_k \) and \( v_k \) are process and measurement noises, respectively.

**Assumption 1:** The additive perturbations \( w_k \in [\underline{w}_k, \overline{w}_k] \) and \( v_k \in [\underline{v}_k, \overline{v}_k] \) for all \( k \in \mathbb{Z}_+ \), where \( \underline{w}, \overline{w} \in \ell^\infty \) and \( \underline{v}, \overline{v} \in \ell^\infty \) are known signals. The scheduling parameter is unmeasured, but takes values in a known bounded set \( \Theta \).

**Assumption 2:** Initial conditions of (1) are bounded such as \( \underline{x}_0 \leq x_0 \leq \overline{x}_0 \), for some known \( \underline{x}_0, \overline{x}_0 \in \mathbb{R}^n \).
Assumption 3: There exist matrices $A_0 \in \mathbb{R}^{n \times n}, B_0 \in \mathbb{R}^{n \times m}$ and $\Delta A_i \in \mathbb{R}^{n \times n}, \Delta B_i \in \mathbb{R}^{n \times m}, i = 1, \ldots, \nu$ for some $\nu \in \mathbb{Z}_+$, such that the following relations are satisfied for all $\theta \in \Theta$:

$$A(\theta) = A_0 + \sum_{i=1}^{\nu} \lambda_i(\theta) \Delta A_i, \quad B(\theta) = B_0 + \sum_{i=1}^{\nu} \lambda_i(\theta) \Delta B_i,$$

$$\sum_{i=1}^{\nu} \lambda_i(\theta) = 1, \quad \lambda_i(\theta) \in [0, 1].$$

Assumption 4: Let $C \geq 0$. 
Problem 1 (*Robust constrained control*) Design an output feedback control that stabilizes (1) while respecting the following constraints

\[ x_k \in X, \quad u_k \in U, \quad \forall k \in \mathbb{Z}_+ \]

having \( X \) and \( U \) as known convex bounded sets, for any possible realization of disturbances \( w_k \) and \( v_k \), and of the scheduling parameter \( \theta_k \).
For our developments, we will need the following lemmas:

**Lemma 1:** [Efimov et al. 2013] Let $x \in \mathbb{R}^n$ be a vector variable, $x \preceq x \preceq \overline{x}$ for some $x, \overline{x} \in \mathbb{R}^n$. Then,

(1) if $A \in \mathbb{R}^{m \times n}$ is a constant matrix, then

$$A^+ x - A^- \overline{x} \leq Ax \leq A^+ \overline{x} - A^- x$$  \hspace{1cm} (2)

(2) if $A \in \mathbb{R}^{m \times n}$ is a matrix variable and $A \preceq \underline{A} \preceq A \preceq \overline{A}$ for some $A, \underline{A}, \overline{A} \in \mathbb{R}^{m \times n}$, then

$$A^+ x^+ - \overline{A}^+ x^- - \overline{A}^- \overline{x}^+ + \overline{A}^- \overline{x}^- \leq Ax \leq \overline{A}^+ \overline{x}^+ - A^+ \overline{x}^- - \overline{A}^- \overline{x}^+ + A^- x^-$$  \hspace{1cm} (3)
Preliminaries

Lemma 2: [Smith, 1995] For $A \in \mathbb{R}_{+}^{n \times n}$, the system

$$x_{k+1} = Ax_k + \omega_k, \quad \omega : \mathbb{Z}_+ \to \mathbb{R}_+^n, \quad \omega \in \mathcal{L}_\infty^n, \quad k \in \mathbb{Z}_+$$

has a non-negative solution $x_k \in \mathbb{R}_+^n$ for all $k \in \mathbb{Z}_+$ provided that $x_0 \geq 0$.

Lemma 3: [Farina and Rinaldi, 2000] A matrix $A \in \mathbb{R}_+^{n \times n}$ is Schur stable iff there exists a diagonal matrix $P \in \mathbb{R}^{n \times n}$, $P > 0$, such that $A^TPA - P < 0$. 
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Interval estimators

An *interval observer* is a two-point set-membership estimator, with stability guarantees. Under *cooperativity conditions*, they produce the following bounds:

\[ \underline{x}_k \leq x_k \leq \overline{x}_k \]

**Main idea:** use the relation above to check constraints, since

\[ [x_k, \overline{x}_k] \subset X \implies x_k \in X. \]

**Main features:** low computation complexity and ease of design (LMIs).
**Interval estimators**

Using the measurement $y_k$:

$$x_{k+1} = (A_0 - LC)x_k + \sum_{i=1}^{v} \lambda_i(\theta) \Delta A_i x_k + Ly_k + (B_0 + \sum_{i=1}^{v} \lambda_i(\theta) \Delta B_i) u_k - Lv_k + w_k$$

the following IO can be proposed:

$$\bar{x}_{k+1} = (A_0 - L_o C) \bar{x}_k + \Delta A_+ \bar{x}_k^+ + \Delta A_- \bar{x}_k^- + B_0 u_k + \Delta B u_k^+ + L_o y_k - L_o^+ v_k + L_o^- \bar{v}_k + \bar{w}_k$$

$$\underline{x}_{k+1} = (A_0 - L_o C) \underline{x}_k - \Delta A_+ \underline{x}_k^+ - \Delta A_- \underline{x}_k^- + B_0 u_k - \Delta B u_k^- + L_o y_k - L_o^+ \bar{v}_k + L_o^- v_k + \underline{w}_k$$

where $L_o$ is the observer gain to be designed. Define the observer estimation errors $e_k = x_k - \underline{x}_k$ and $\bar{e}_k = \bar{x}_k - x_k$.

**Lemma 4:** Let assumptions 1–3 be satisfied. Then, provided that $A_0 - L_o C$ is non-negative, the estimation errors are non-negative, i.e., $e_k, \bar{e}_k \geq 0$ for all $k > 0$. 

IO-MPC
Interval estimators

In order to derive stability conditions for IO (4), let us rewrite it as:

\[ \chi_{k+1} = (A_0 - \bar{L}_o C_1) \chi_k + A_+ \chi^+_k + A_- \chi^-_k + \delta_k \]

where \( A_0 = \text{diag}(A_0, A_0) \in \mathbb{R}^{2n \times 2n} \), \( \bar{L}_o = \text{diag}(L_o, L_o) \in \mathbb{R}^{2n \times 2p} \), \( C_1 = \text{diag}(C, C) \in \mathbb{R}^{2p \times 2n} \), \( \delta_k = \text{vec}(\bar{\delta}_k, \underline{\delta}_k) \), and

\[
A_+ = \begin{bmatrix}
\Delta A_+ & 0 \\
-\Delta A_- & 0
\end{bmatrix}, \quad A_- = \begin{bmatrix}
0 & \Delta A_- \\
0 & -\Delta A_+
\end{bmatrix},
\]

\[
\bar{\delta}_k = B_0 u_k + \Delta B u^+_k + L_0 y_k - L_0^+ v_k + L_0^- \bar{v}_k + \bar{w}_k,
\]

\[
\underline{\delta}_k = B_0 u_k - \Delta B u^-_k + L_0 y_k - L_0^+ \bar{v}_k + L_0^- v_k + \underline{w}_k.
\]
Interval estimators

The next result verifies stability:

Theorem 1: Let assumptions 1–3 be satisfied. If there exist diagonal matrices \( \bar{P}, Q_1, Q_2, Q_3, \Omega_+, \Omega_- \), matrices \( \Gamma \in \mathbb{R}^{2n \times 2n} \) and \( \bar{U} \in \mathbb{R}^{2n \times p} \), such that the following LMIs are verified:

\[
\begin{bmatrix}
\bar{P} - Q_1 & -\Omega_+ & -\Omega_- & 0 & A_0^\top \bar{P} - C_1^\top \bar{U}^\top \\
* & -Q_2 & -\Psi & 0 & A_1^\top \bar{P} \\
* & * & -Q_3 & 0 & A_2^\top \bar{P} \\
* & * & * & \Gamma & \bar{P} \\
* & * & * & * & \bar{P}
\end{bmatrix} \succeq 0
\]

\( \bar{P} > 0, \quad \Gamma \succ 0, \quad Q_1, Q_2, Q_3, \Omega_+, \Omega_- \geq 0, \)

\( Q_1 + \min\{Q_2, Q_3\} + 2 \min\{\Omega_+, \Omega_-\} > 0 \)

then system (4) with a gain \( L_o = P^{-1}U \) is an IO for system (1), i.e., relation \( x_k \leq x_k \leq \bar{x}_k \) is satisfied for all \( k \in \mathbb{Z}_+ \) and, in addition, \( \chi \in \ell_2^n \) provided that \( \delta \in \ell_2^n \).
Interval estimators

To better illustrate the developments of this section, consider the following prototype model:

\[
x_{k+1} = \begin{bmatrix} 0.5 & 0.6 + \theta_k \\ \theta_k & 0.3 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + w_k
\]

\[
y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k + v_k
\]

\[
\mathbb{W} = [-0.1, 0.1] \times [-0.1, 0.1], \quad \mathbb{V} = [-0.1, 0.1], \quad \text{and} \quad \Theta = [0, -0.3].
\]

Interpolating functions \( \lambda_1 = \frac{\theta_k - \theta_k}{\theta_k - \theta_k} \) and \( \lambda_2 = \frac{\theta_k - \theta_k}{\theta_k - \theta_k} \).
Interval estimators

Simulate the IO

\[ u_k = 1, \text{ for } k = [0, \ldots, 49] \]
\[ u_k = -1, \text{ for } k = [50, \ldots, 100] \]
\[ \theta_k = -|0.3 \sin(0.1k)| \]
\[ w_k = 0.1 \sin(k), \quad v_k = 0.1 \sin(k) \]
Interval estimators

As seen in (4), the IO requires the measurement $y_k \rightarrow \text{unsuitable}$ for prediction.

**Solution:** propose an *interval predictor* $\rightarrow$ an open-loop *framer*, i.e., independent of $y_k$.

Recalling that $y_k = Cx_k + v_k$, we can write the following relation under Lemma 1 and Assumption 4:

$$L_p^+ Cz_k - L_p^- C\bar{z}_k \leq L_p Cz_k \leq L_p^+ C\bar{z}_k - L_p^- Cz_k. \quad (5)$$

then the terms $L_p y_k - L_p v_k = L_p Cx_k$ can be replaced by the bounding relations above.
Interval estimators

The proposed IP:

\[
\begin{align*}
\bar{z}_{k+1} &= (A_0 - L_p C)\bar{z}_k + \Delta A_+\bar{z}_k^+ + \Delta A_-\bar{z}_k^- + L_p^+ C\bar{z}_k - L_p^- C\bar{z}_k + B_0 u_k + \Delta B u_k^+ + \bar{w}_k \\
\tilde{z}_{k+1} &= (A_0 - L_p C)\tilde{z}_k - \Delta A_+\tilde{z}_k^+ - \Delta A_-\tilde{z}_k^- + L_p^+ C\tilde{z}_k - L_p^- C\tilde{z}_k + B_0 u_k - \Delta B u_k^- + \tilde{w}_k
\end{align*}
\]

Define the prediction estimation errors \( \varepsilon_k = x_k - \bar{z}_k \) and \( \bar{\varepsilon}_k = \tilde{z}_k - x_k \).

**Lemma 5:** Let assumptions 1–4 be satisfied. Then, provided that \( A_0 - L_p C \) is non-negative, the prediction errors are non-negative, i.e., \( \varepsilon_k, \bar{\varepsilon}_k \geq 0 \) for all \( k \in \mathbb{Z}_+ \).
Interval estimators

In order to derive stability conditions for IP (6), let us rewrite it as:

$$ Z_{k+1} = (A_0 + \tilde{L}_p C_2) Z_k + A_+ Z_k^+ + A_- Z_k^- + \varrho_k, $$

where $A_0$, $A_+$ and $A_-$ are the same as for IO (4), $\tilde{L}_p = \text{diag}(L_p^-, L_p^-) \in \mathbb{R}^{2n \times 2p}$, $\varrho_k = \text{vec}(\rho_k, \rho_k^-)$ and

$$ C_2 = \begin{bmatrix} C & -C \\ -C & C \end{bmatrix}, $$

$$ \overline{\rho}_k = B_0 u_k + \Delta B u_k^+ + \overline{w}_k, \quad \underline{\rho}_k = B_0 u_k - \Delta B u_k^- + \underline{w}_k. $$
Interval estimators

Theorem 2: Let assumptions 1–4 be satisfied. If there exist diagonal matrices \( \tilde{P}_2, Q_1, Q_2, Q_3, \Omega_+, \Omega_-, \Psi, \Gamma \in \mathbb{R}^{2n \times 2n} \) and \( U^+, U^- \in \mathbb{R}^{n \times p} \), such that

\[
\tilde{P}_2 A_0 - \tilde{U}^+ C_1 + \tilde{U}^- C_1 \geq 0
\]

\[
\begin{bmatrix}
\tilde{P}_2 - Q_1 & -\Omega_+ & -\Omega_- & 0 & (\tilde{P}_2 A_0 + \tilde{U}^- C_2)^\top \\
* & -Q_2 & -\Psi & 0 & (\tilde{P}_2 A_+)^\top \\
* & * & -Q_3 & 0 & (\tilde{P}_2 A_-)^\top \\
* & * & * & \Gamma & \tilde{P}_2 \\
* & * & * & * & \tilde{P}_2 
\end{bmatrix} \succeq 0
\]

\( Q_1, Q_2, Q_3, \Omega_+, \Omega_-, U^+, U^- \geq 0, \quad \Gamma > 0, \quad P_2 > 0 \)

\( \tilde{P}_2 = \text{diag}(P_2, P_2), \quad \tilde{U}^+ = \text{diag}(U^+, U^+), \quad \tilde{U}^- = \text{diag}(U^-, U^-), \quad Q = Q_1 + \min\{Q_2, Q_3\} + 2 \min\{\Omega_+, \Omega_-\} > 0 \)

then (6) with gains \( L_p^- = P_2^{-1}U^- \) and \( L_p^+ = P_2^{-1}U^+ \) is an IP for system (1), i.e., \( z_k \leq x_k \leq \bar{z}_k \) holds for all \( k \in \mathbb{Z}_+ \), and (6) is ISS with respect to the input \( \varphi \in \ell_\infty^{2n} \).
Interval estimators

Simulate the IP

\[ u_k = 1, \text{ for } k = [0, \ldots 49] \]
\[ u_k = -1, \text{ for } k = [50, \ldots 100] \]
\[ \theta_k = -|0.3 \sin(0.1k)| \]
\[ w_k = 0.1 \sin(k), \quad v_k = 0.1 \sin(k) \]
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Recall on MPC

How to prove stability → stabilizing ingredients:

- **Terminal set** $\mathcal{X}_f$: the set that the endpoint of the prediction must reach;
- **Terminal gain** $\kappa_f$: there exists a stabilizing controller;
- **Terminal cost** $V_f$. 
Recall on MPC

How to prove stability → stabilizing ingredients. Recall the classic axioms of Mayne et al.:

Definition 1 The stabilizing ingredients are such that the following axioms are verified:

1. $X_f \subset X$, closed and $0 \in X_f$: the state constraint is satisfied in $X_f$;
2. $\kappa_f(x) \in U$, $\forall x \in X_f$: the control constraint is satisfied in $X_f$;
3. $f(x,\kappa_f(x)) \in X_f$, $\forall x \in X_f$: $X_f$ is positively invariant under $\kappa_f(x)$;
4. $[V_f + \ell](x,\kappa_f(x)) \leq 0$, $\forall x \in X_f$: $V_f$ is a local Lyapunov function.
IP: Control design

How to design a feedback controller for the IP? Let us consider:

\[ u_k = K \mathcal{Z}_k + K_+ \mathcal{Z}_k^+ + K_- \mathcal{Z}_k^- + R \mathcal{W} \]  \hspace{1cm} (7)

where \( \mathcal{W}_k = \text{vec}(\mathbf{w}_k, \bar{\mathbf{w}}_k) \). This control leads to the following closed-loop:

\[ \mathcal{Z}_{k+1} = \mathcal{K} \mathcal{Z}_k + \mathcal{K}_+ \mathcal{Z}_k^+ + \mathcal{K}_- \mathcal{Z}_k^- + \tilde{D} \mathcal{W} \]  \hspace{1cm} (8)

where \( \mathcal{K} = \mathcal{A}_0 + \tilde{L}_p \mathcal{C}_2 + \mathcal{B}_0 \mathcal{K}, \quad \mathcal{K}_* = \mathcal{A}_* + \mathcal{B}_0 \mathcal{K}_* \quad \tilde{D} = \mathbb{I}_{2n} + \mathcal{B}_0 \mathcal{R} \) and \( \mathcal{B}_0 = [\mathcal{B}_0^\top, \mathcal{B}_0^\top] \).
This brings us to the following result:

**Theorem 3:** Let assumptions 1–4 be satisfied. If there exist matrices $P, Q_1, Q_2, Q_3, \Gamma, \Omega_+, \Omega_- \in \mathbb{R}^{2n \times 2n}$ and $W_1, W_2, W_3, W_4 \in \mathbb{R}^{m \times 2n}$ such that

$$
\begin{bmatrix}
P - Q_1 & -\Omega_+ & -\Omega_- & 0 & W_1^\top B_0^\top & + PD_z^\top \\
* & -Q_2 & -\Psi & 0 & W_2^\top B_0^\top & + PA_+^\top \\
* & * & -Q_3 & 0 & W_3^\top B_0^\top & + PA_-^\top \\
* & * & * & \Gamma & W_4^\top B_0^\top & + P \\
* & * & * & * & P
\end{bmatrix} > 0
$$

$P > 0, \quad \Gamma > 0, \quad Q_1, Q_2, Q_3, \Omega_+, \Omega_- \geq 0,$

$Q = Q_1 + \min\{Q_2, Q_3\} + 2\min\{\Omega_+, \Omega_-\} > 0,$

then IP (6) under control (7) with gains $K = W_1P^{-1}, K_+ = W_2P^{-1}, K_- = W_3P^{-1}, R = W_4P^{-1}$ is ISS with respect to the inputs $\mathcal{W} \in \ell^{2n}_\infty.$
*IP: Control design*

How to ensure that $u_k \in \mathcal{U}$?

**Corollary 1:** Let there exist symmetric and positive definite matrices $S \in \mathbb{R}^{m \times m}$ and $Z \in \mathbb{R}^{2n \times 2n}$ such that $\mathcal{U} = \{u \in \mathbb{R}^m : u^\top Su \leq 1\}$ and $\mathcal{W}_k \in \{\mathcal{W} \in \mathbb{R}^{2n} : \mathcal{W}^\top Z \mathcal{W} \leq 1\}$, and the conditions of Theorem 4 be satisfied with additional inequalities:

$$\frac{\eta}{\alpha \kappa} \Gamma \leq \min\{\kappa^{-1}Z, P\}, \ P \geq \kappa Z^{-1},$$

$$\begin{bmatrix}
\frac{\eta}{3}P & 0 & 0 & W_1^\top + W_2^\top \\
0 & \frac{\eta}{3}P & 0 & W_3^\top - W_1^\top \\
0 & 0 & \frac{\kappa}{3}P & W_4^\top \\
W_1 + W_2 & W_3 - W_1 & W_4 & S^{-1}
\end{bmatrix} \succeq 0$$

for some constants $\eta > 0$ and $\kappa > 0$, then control (7) satisfies the constraint $u_k \in \mathcal{U}$ for all $\mathcal{Z}_k \in \mathcal{X}_f \times \mathcal{X}_f$. 
The predictive controller

Determine $S_n = \{s_0, \ldots, s_{N-1}\}$ solving the OCP

$$S_N^k := \arg\min_{S_N} V_N(\mathcal{Z}_{k,0}, \ldots, \mathcal{Z}_{k,N}, S_N)$$

with a cost function

$$V_N(\mathcal{Z}_{k,0}, \ldots, \mathcal{Z}_{k,N}, S_N) = V_f(\mathcal{Z}_{k,N}) + \sum_{i=0}^{N-1} \ell(\mathcal{Z}_{k,i}, s_i).$$

under the following constraints:

\begin{align*}
\mathcal{Z}_{k,0} &= \min\{\bar{x}_k, \bar{z}_{k-1,1}\}, & \bar{z}_{k,0} &= \max\{x_k, \bar{z}_{k-1,1}\} & (9a) & \rightarrow \text{initialization} \\
\mathcal{Z}_{k,i+1} &\text{ computed by } X & (9b) & \rightarrow \text{prediction using the IP} \\
\mathcal{Z}_{k,i+1} &\subset X \times X, & s_i &\subset U, & (9c) & \rightarrow \text{state and input constraint} \\
\mathcal{Z}_{k,N} &\in X_f \times X_f & (9d) & \rightarrow \text{terminal constraint}
\end{align*}
The predictive controller

Why initialize using information from both IO and IP? Let $V = [-0.5, 0.5]$.

Comparison IO/IP in case of big meas. noise

![Graph comparing IO/IP](image)
The predictive controller

**Algorithm 1: IO-MPC**

**Offline:** Solve LMIs, estimate $X_f$ and select $\Psi_1 = P^{-1}, \Psi_2 \leq \frac{\alpha}{2} P^{-1}$ and $\Psi_3 \leq \frac{\alpha}{8} P^{-1}$.

**Input:** Initial conditions $x_0, \bar{x}_0$ and prediction horizon $N$.

**Online:**

1. **for** each decision instant $k \in \mathbb{Z}_+$ **do**

2. Measure $y_k$ and update IO (4).

3. Initialize IP (6).

4. Solve OCP (17) under constraints (9a)-(9d).

5. Assign $u_k = s_0^k$ and apply to the system.

6. **end for**
The predictive controller

Theorem 4: Let \([x_0, \bar{x}_0] \subset X\) and assumptions 1–4 be satisfied with \([w_{k+1}, \bar{w}_{k+1}] \subseteq [w_k, \bar{w}_k]\) for all \(k \in \mathbb{Z}_+\). Then, following Algorithm 1, the closed-loop system composed by (1), (4) and (6) has the following features:

1. Recursive feasibility of reaching the terminal set in \(N\) steps;
2. ISS of dynamics (8) in \(X_f\) and practical ISS for (1);
3. Constraint satisfaction.
The LTI and the TD case

The same ideas were applied to linear time-invariant (LTI) and time-delayed systems (TD):

\[ x_{k+1} = A_0 x_k + A_1 x_{k-h} + Bu_k + w_k, \quad k \in \mathbb{Z}_+ \]
\[ x_k = \phi_k, \quad k \in [-h, \ldots, 0] \]
\[ y_k = C x_k + v_k \]

Main differences:

- Optimization of gains made through the interval width \( \delta x_k = \bar{x}_k - x_k \).
- Control design made regarding the interval center \( x_k^* = \frac{\bar{x}_k + x_k}{2} \).
- For the TD case, the Lyapunov-Krasovskii framework is required;
Complexity

One of the main advantages of using IO/IP is their fixed complexity.

Assume that the number of hyperplanes needed to define $X$, $U$ and $X_f$ depends linearly on $n$, and that $m = n$. Therefore, the worst-case number of variables for solving the constrained OCP is $10Nn$ ($8Nn$ for the linear cases).
Outline

1 Motivation
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5 Numerical example
Numerical example (LPV)

Recall the LPV prototype example:

\[
x_{k+1} = \begin{bmatrix} 0.5 & 0.6 + \theta_k \\ \theta_k & 0.3 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + \omega_k
\]

\[
y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k + v_k
\]

Constraints: \( \mathcal{X} = [-12, 3] \times [-12, 3], \quad \mathcal{U} = [-2, 2] \)

Disturbances: \( \mathcal{W} = [-0.1, 0.1]^2, \quad \mathcal{V} = [-0.1, 0.1] \)

Interpolating functions \( \lambda_1 = \frac{\theta_k - \bar{\theta}_k}{\theta_k - \theta_k} \) and \( \lambda_1 = \frac{\theta_k - \theta_k}{\bar{\theta}_k - \theta_k}, \quad \Theta = [-0.1, 0.1] \)

Select \( x_0 = \text{vec}(-7, -12) \) and \( \bar{x}_0 = \text{vec}(-6, -10) \). Prediction horizon \( N = 20 \), simulation time span \( T = 20 \) steps \( \times \) 100 runs.
Numerical example (LPV)

Evolution of the states

Control Input

Constraints
IP trajectory
Real trajectories

Time steps $[k]$

Time steps $[k]$
Numerical example (LPV)

Solver: *fmincon* (active set method)

For $N = 10$, computation time $0.22 \pm 0.0313$ second/step with a maximum of $0.7725$ second.
Numerical example (LTI)

Consider the (linearized) CSTR model, given by the following matrices:

\[
A = \begin{bmatrix}
0.745 & -0.002 \\
5.610 & 0.780 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
5.6 \times 10^{-6} \\
0.464 \\
\end{bmatrix}, \quad C = \begin{bmatrix}
0 \\
1 \\
\end{bmatrix}
\]

Constraints: \( X = [-2,2] \times [-10,5] \) and \( U = [-4.5,4.5] \)

Disturbances: \( W = [-0.02,0.02] \times [-0.2,0.2] \) and \( V = [-0.3,0.3] \)

For a later comparison, the Tube-MPC from [Mayne et al, 2009] will be implemented, taking an LQR controller for its design with matrices \( Q_{LQ} = 0.1 I_2 \) and \( R_{LQ} = 0.1 \).

Solver: \textit{quadprog}, computation time: \( 0.0032 \pm 0.0021 \) second/step, maximum of 0.1358.
Numerical example (LTI)

States evolution (IO-MPC)

Control input

Time steps [k]

Time steps [k]

IO-MPC
Numerical example (LTI)

Feasible Regions (OPC)

Constraint Set
- IO-MPC
- Tube-MPC
Numerical example (TD)

Consider the following TD system:

\[
x_{k+1} = \begin{bmatrix} 0.5 & -0.1 \\ 0.5 & 0.2 \end{bmatrix} x_k + \begin{bmatrix} 0.1 & -0.3 \\ 0 & -0.1 \end{bmatrix} x_{k-h} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + w_k
\]

\[
y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k + v_k
\]

Constraints: \( X = [-9,3] \times [-7,4] \) and \( U = [-1,1] \)

Disturbances: \( W = [-0.2,0.2]^2 \) and \( V = [-0.5,0.5] \)

and a known time-delay \( h = 10 \).

**Solver:** quadprog, computation time: \( 0.0032 \pm 0.0021 \) second/step, maximum of 0.1358.
Numerical example (TD)

Evolution of the states

Control inputs

Time steps [k]
Conclusions & perspectives

Conclusions:

- Developed new interval estimators for LTI, LPV and TD systems, as well as their respective state feedback controllers;
- Proposed new robust output feedback MPC algorithms;
- Illustrated the methodologies with numerical experiments;
- Advantages: low fixed complexity, ease of design, low conservativeness.

Perspectives:

- Enhance the interval estimators and the proposed MPC algorithms aiming to reduce conservativeness;
- Test their efficiency in practical scenarios.
Thank you for your attention

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