Interval Reachability Analysis

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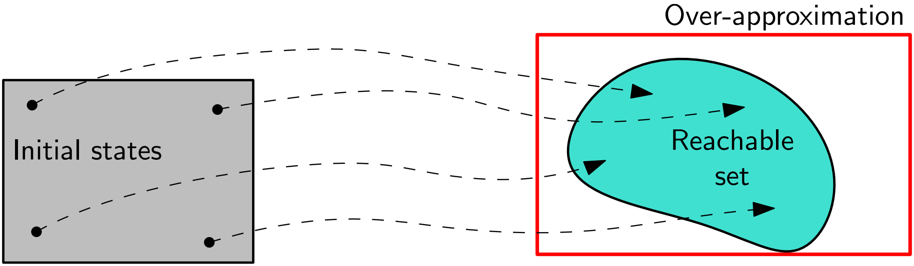
4^{th} of June 2021

Reachability analysis

Discrete-time $x^+ = f(x)$ or continuous-time system $\dot{x} = f(x)$



Exact computation of the reachable set: impossible
$\rightarrow$ **over-approximation** by a **multi-dimensional interval**

Motivations for reachability analysis

- **Verification of safety and reachability specifications**
  - Does the over-approximation intersect the unsafe set ?
  - Is the over-approximation contained in the target set ?
- **Abstraction-based control synthesis**
  - Partition of the state space
  - Over-approximations to abstract the continuous system into a finite transition system
  - High-level control synthesis on the discrete abstraction
- **Measuring the robustness of control policies**
  - Volume of the over-approximation under bounded disturbances

Objectives of the presentation

- **Overview of several methods and how to use them**
  - Tutorial-like presentation
    - Intuition
    - Requirements and limitations
    - Over-approximation computation

  - Monotonicity (DT + CT)
  - Mixed monotonicity (DT + CT)
  - Sampled-data mixed monotonicity (CT)
  - Growth bounds/Contraction analysis (CT)
  - Quasi-Monte Carlo (DT + CT)
  - Monte Carlo (DT + CT)

- **Overview of Matlab toolbox TIRA**

Monotonicity - Intuition

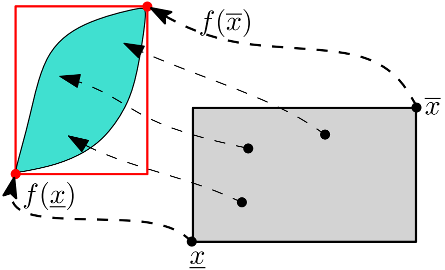
Discrete-time system: $x^+ = f(x)$

Definition (Cooperative system)

A system is cooperative if f preserves the inequality $\geq$:

$$x \geq \hat{x} \Rightarrow f(x) \geq f(\hat{x})$$

Over-approximation of the reachable set after one time step



Monotonicity

Discrete-time system: $x^+ = f(x)$

Assumption

There exist $\varepsilon = [\varepsilon_1; \dots; \varepsilon_n] \in \{0,1\}^n$ *such that for all* $x \in [\underline{x}, \overline{x}]$ *and* $i, j \in \{1, \dots, n\}$ *we have:*

$$(-1)^{\varepsilon_i + \varepsilon_j} \frac{\partial f_i(x)}{\partial x_j} \geq 0$$

Proposition

$$f\left([\underline{x}, \overline{x}]\right) \subseteq \left[f\left(\underline{x}\left(1_n - \varepsilon\right) + \overline{x}\varepsilon\right), f\left(\underline{x}\varepsilon + \overline{x}\left(1_n - \varepsilon\right)\right)\right]$$

- Over-approximation from only 2 evaluations of f
- Tightest interval over-approximation

Monotonicity

Proposition

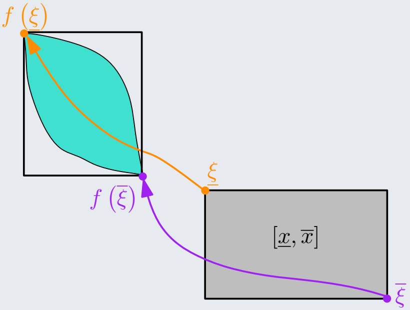
$$f\left([\underline{x},\overline{x}]\right) \subseteq \left[f\left(\underline{x}\left(1_n-\varepsilon\right)+\overline{x}\varepsilon\right), f\left(\underline{x}\varepsilon+\overline{x}\left(1_n-\varepsilon\right)\right)\right]$$

Example

$$sign\left(\frac{\partial f(x)}{\partial x}\right) = \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

$$\varepsilon = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{\xi} = \begin{pmatrix} \underline{x}_1 \\ \overline{x}_2 \end{pmatrix} \quad \overline{\xi} = \begin{pmatrix} \overline{x}_1 \\ \underline{x}_2 \end{pmatrix}$$



Monotonicity - Continuous-time

Continuous-time system: $\dot{x} = f(x)$
Trajectories: $x(t, x_0)$

Assumption

There exist an invariant set $X \subseteq \mathbb{R}^n$ and $\varepsilon = [\varepsilon_1; \dots; \varepsilon_n] \in \{0,1\}^n$ such that for all $x \in X$ and $i, j \in \{1, \dots, n\}$ with $i \neq j$ we have:

$$(-1)^{\varepsilon_i + \varepsilon_j} \frac{\partial f_i(x)}{\partial x_j} \geq 0$$

Proposition

$$x(T, [\underline{x}, \overline{x}]) \subseteq [x(T, \underline{x}(1_n - \varepsilon) + \overline{x}\varepsilon), x(T, \underline{x}\varepsilon + \overline{x}(1_n - \varepsilon))]$$

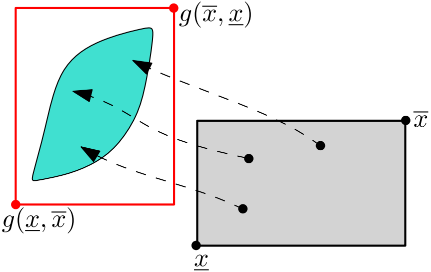
Mixed monotonicity - Intuition

Discrete-time system: $x^+ = f(x)$, $x \in X$

Definition (Mixed monotonicity)

A system is mixed-monotone is there exists $g : X \times X \to X$ such that

- $g(x, y)$ is increasing with x
- $g(x, y)$ is decreasing with y
- $g(x, x) = f(x)$



Mixed monotonicity

Assumption

There exists $L \in \mathbb{R}^{n \times n}$ such that for all $i, j \in \{1, \dots, n\}$ we have:

- *either* $\frac{\partial f_i(x)}{\partial x_j} + L_{ij} \geq 0, \quad \forall x \in [\underline{x}, \overline{x}]$
- *or* $\frac{\partial f_i(x)}{\partial x_j} + L_{ij} \leq 0, \quad \forall x \in [\underline{x}, \overline{x}]$

Constructive definition of g

For all $i \in \{1, \dots, n\}$,

$$g_i(x, \hat{x}) = f_i(\xi^i) + [|L_{i1}|, \dots, |L_{in}|] * (x - \hat{x})$$

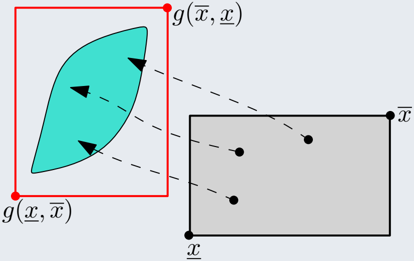
with $\xi^i \in \mathbb{R}^n$ defined for all $j \in \{1, \dots, n\}$ such that

$$\xi^i_j = \begin{cases} x_j & \text{if } \frac{\partial f_i(x)}{\partial x_j} + L_{ij} \geq 0 \\ \hat{x}_j & \text{if } \frac{\partial f_i(x)}{\partial x_j} + L_{ij} < 0 \end{cases}$$

Mixed monotonicity

Proposition

$$f([\underline{x}, \overline{x}]) \subseteq [g(\underline{x}, \overline{x}), g(\overline{x}, \underline{x})]$$



- Over-approximation from only 2 evaluations of g
- Tightness guaranteed when the Jacobian is sign-stable ($L = 0_{n\times n}$)
- Monotonicity result is a particular case

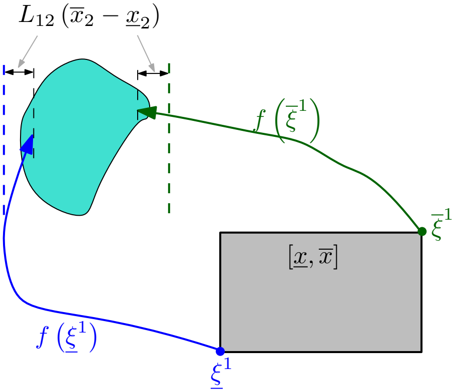
Mixed monotonicity - Illustration

$$sign\left(\frac{\partial f(x)}{\partial x}+L\right)=\begin{pmatrix}+ & +\\ - & +\end{pmatrix} \qquad L=\begin{pmatrix}0 & L_{12}\\ 0 & 0\end{pmatrix}$$

Dimension 1:

$$g_1(\underline{x},\overline{x}) = f_1(\underline{\xi}^1) - L_{12}(\overline{x}_2-\underline{x}_2)$$
$$g_1(\overline{x},\underline{x}) = f_1(\overline{\xi}^1) + L_{12}(\overline{x}_2-\underline{x}_2)$$

$$\underline{\xi}^1=\begin{pmatrix}\underline{x}_1\\ \underline{x}_2\end{pmatrix} \quad \overline{\xi}^1=\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$$



Mixed monotonicity - Illustration

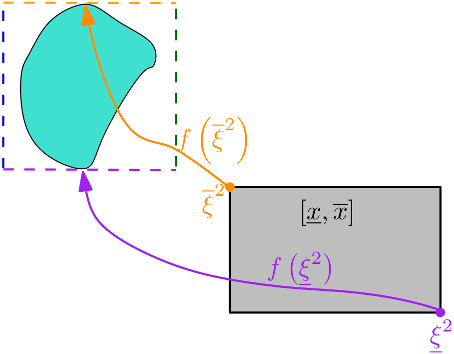
$$sign\left(\frac{\partial f(x)}{\partial x} + L\right) = \begin{pmatrix} + & + \\ - & + \end{pmatrix} \qquad L = \begin{pmatrix} 0 & L_{12} \\ 0 & 0 \end{pmatrix}$$

Dimension 2:

$g_2(\underline{x}, \overline{x}) = f_2(\underline{\xi}^2) + 0$

$g_2(\overline{x}, \underline{x}) = f_2(\overline{\xi}^2) + 0$

$$\underline{\xi}^2 = \begin{pmatrix} \overline{x}_1 \\ \underline{x}_2 \end{pmatrix} \quad \overline{\xi}^2 = \begin{pmatrix} \underline{x}_1 \\ \overline{x}_2 \end{pmatrix}$$



Mixed monotonicity - Continuous-time

Continuous-time system $\dot{x} = f(x)$, trajectories $x(t, x_0)$

Assumption

$\exists L \in \mathbb{R}^{n \times n}$ *such that for all* $i, j \in \{1, \dots, n\}$ *with* $i \neq j$ *we have:*

- *either* $\frac{\partial f_i(x)}{\partial x_j} + L_{ij} \geq 0, \quad \forall x \in X$
- *or* $\frac{\partial f_i(x)}{\partial x_j} + L_{ij} \leq 0, \quad \forall x \in X$

Decomposition function g: same definition

Embedding system: $\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = h(x, \hat{x}) = \begin{pmatrix} g(x, \hat{x}) \\ g(\hat{x}, x) \end{pmatrix}$

Proposition

$$x(T, [\underline{x}, \overline{x}]) \subseteq \left[\Phi^h_{1\dots n}(T, \underline{x}, \overline{x}), \Phi^h_{n+1\dots 2n}(T, \underline{x}, \overline{x})\right]$$

Sampled-data mixed monotonicity

Motivation

Sign-stable Jacobian matrix ($L = 0_{n\times n}$)
$\rightarrow$ **tighteness only guaranteed in discrete time**

	Discrete-time mixed-monotonicity	**Sampled-data mixed-monotonicity**
System	$x^+ = F(x)$	$\dot{x} = f(x)$ $x^+ = x(T; x_0)$
Requirement	Bounded Jacobian $\frac{\partial F(x)}{\partial x} \in [\underline{J}, \overline{J}]$	Bounded sensitivity $\frac{\partial x(T; x_0)}{\partial x_0} \in [\underline{S}, \overline{S}]$

Main challenge: bounding the sensitivity matrix

Sensitivity bounds - Interval Analysis

Continuous-time system: $\dot{x} = f(x)$

Jacobian matrix: $J^x(x) = \frac{\partial f(x)}{\partial x}$

Sensitivity matrix: $S^x(t; x_0) = \frac{\partial x(t;x_0)}{\partial x_0}$

Assumption

Bounded Jacobian: $\forall x \in X,\ J^x(x) \in [\underline{J^x}, \overline{J^x}]$

Interval analysis method

$$\dot{S}^x = J^x * S^x, \qquad S^x(0; x_0) = I_n$$

$S^x(T) \in e^{[\underline{J^x}, \overline{J^x}]*T} \rightarrow$ Taylor OA of the interval matrix exponential [a]

[a] M. Althoff, O. Stursberg, and M. Buss. *Reachability analysis of linear systems with uncertain parameters and inputs*. CDC07.

Fast and sound, but usually very conservative

Sensitivity bounds - Sampling and Falsification

No assumption

Sampling

- Sample initial interval $[\underline{x}, \overline{x}]$
- Numerical evaluation of $S^x(T; x_0)$ for each sample x_0
- Interval hull: $[\min_{x_0} S^x(T; x_0), \max_{x_0} S^x(T; x_0)]$

Falsification

- Optimization problem: find $x_0^* \in [\underline{x}, \overline{x}]$ such that $S^x(T; x_0^*) \notin [\underline{S^x}, \overline{S^x}]$
- Update sensitivity bounds:

$$\underline{S^x} \leftarrow \min\left(\underline{S^x}, S^x(T; x_0^*)\right)$$
$$\overline{S^x} \leftarrow \max\left(\overline{S^x}, S^x(T; x_0^*)\right)$$

No guarantee that the final $[\underline{S^x}, \overline{S^x}]$ is an over-approximation

Sensitivity bounds - Tunable method

Second-order Jacobian: $J^{xx}(x) = \frac{\partial J^x(x)}{\partial x}$

Second-order sensitivity: $S^{xx}(t; x_0) = \frac{\partial S^x(t; x_0)}{\partial x_0}$

$$\dot{S}^{xx} = J^x * S^{xx} + J^{xx} * (S^x \otimes S^x) \qquad S^{xx}(0; x_0) = 0$$

Assumption

Bounded Jacobian: $\forall x \in X,\ J^x(x) \in \left[\underline{J^x}, \overline{J^x}\right]$ *and* $J^{xx}(x) \in \left[\underline{J^{xx}}, \overline{J^{xx}}\right]$

3-step method

- Interval analysis on $\dot{S}^x = J^x * S^x$
  $\rightarrow$ over-approximation of $S^x([0, T]; [\underline{x}, \overline{x}])$
- Interval analysis on $\dot{S}^{xx} = J^x * S^{xx} + J^{xx} * (S^x \otimes S^x)$
  $\rightarrow$ over-approximation of $S^{xx}(T; [\underline{x}, \overline{x}])$
- Evaluation of $S^x(T; x_0)$ on a sampling grid of $[\underline{x}, \overline{x}]$
  $\rightarrow$ over-approximation of $S^x(T; [\underline{x}, \overline{x}])$

Sensitivity bounds - Comparison

	Interval analysis	Sampling	3-step approach
Soundness	**yes**	no	**yes**
Conservativeness	large	**small**	**tunable**
Complexity	**low**	high	**tunable**
Requirements	$[\underline{J^x}, \overline{J^x}]$	**none**	$[\underline{J^x}, \overline{J^x}]$, $[\underline{J^{xx}}, \overline{J^{xx}}]$

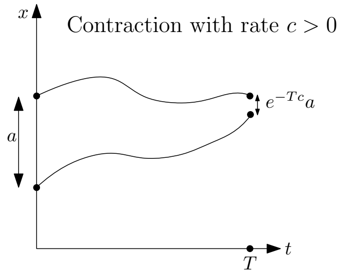
Growth bounds - Intuition

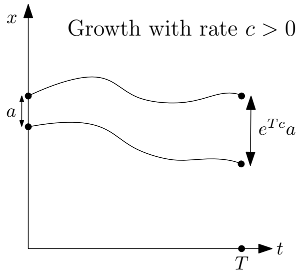
Continuous-time system: $\dot{x} = f(x)$

Various terms

- Growth bounds
- Contraction
- Incremental stability
- Discrepancy functions

Exponential convergence/divergence of any pair of trajectories





Growth bounds

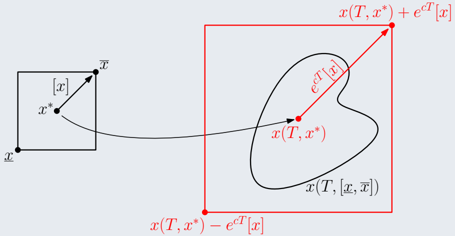
Assumption (Scalar growth bound)

There exist an invariant set $X \subseteq \mathbb{R}^n$ and a scalar $c \in \mathbb{R}$ such that the infinity matrix measure of the Jacobian is bounded:

$$\forall x \in X, \quad \max_i \left(\frac{\partial f_i(x)}{\partial x_i} + \sum_{i \neq j} \left| \frac{\partial f_i(x)}{\partial x_j} \right| \right) \leq c$$

Proposition

$$x(T, [\underline{x}, \overline{x}]) \subseteq$$
$$\{x(T, x^*)\} + \left[-e^{cT}[x], e^{cT}[x] \right]$$



Growth bounds

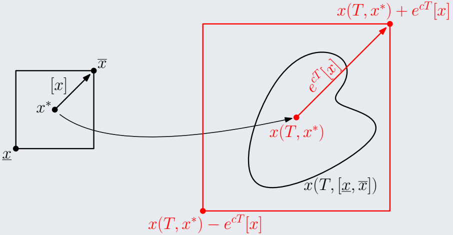
Assumption (Matrix growth bound)

There exist an invariant set $X \subseteq \mathbb{R}^n$ and a matrix $C \in \mathbb{R}^{n \times n}$ such that

$$\forall x \in X, \begin{cases} \dfrac{\partial f_i(x)}{\partial x_i} \leq C_{ii} \\ \left| \dfrac{\partial f_i(x)}{\partial x_j} \right| \leq C_{ij}, \quad i \neq j \end{cases}$$

Proposition

$$x(T, [\underline{x}, \overline{x}]) \subseteq$$
$$\{x(T, x^*)\} + \left[-e^{CT}[x], e^{CT}[x] \right]$$

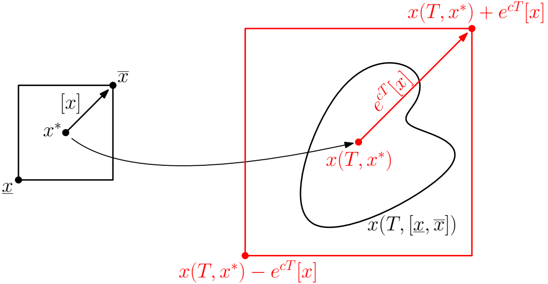


Quasi Monte Carlo - Intuition

Generalization of the growth-bound method to a grid sampling

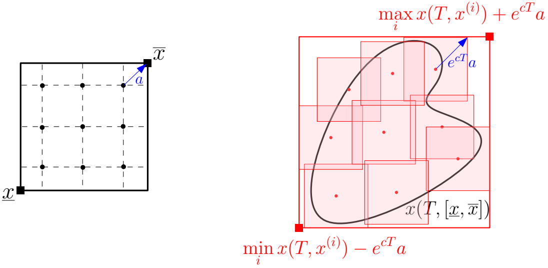
Growth bound

- Unique sample x^*
- Growth for the interval half width $e^{cT}[x]$



Quasi Monte Carlo

- Uniform grid $\{x^{(i)}\}$
- Growth for half the grid size $e^{cT}a$
- Interval hull



Quasi Monte Carlo

Continuous-time system: $\dot{x} = f(x)$

Assumption

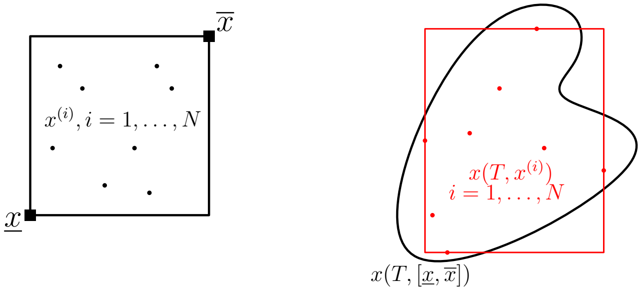
- *There exists a scalar growth bound* $c \in \mathbb{R}$
- *The initial interval* $[\underline{x}, \overline{x}]$ *has a uniform sampling grid* $\{x^{(i)}\}$ *of size* $2a$

Proposition

$$x(T, [\underline{x}, \overline{x}]) \subseteq \left[\min_i x(T, x^{(i)}), \max_i x(T, x^{(i)})\right] + \left[-e^{cT}a, e^{cT}a\right]$$

- Grid granularity $\rightarrow$ tune the tradeoff precision/complexity
- Also works for discrete-time systems

Monte Carlo - Intuition



Random sampling and nested probabilistic objectives

- **Accuracy** $\varepsilon \in (0,1)$: a random initial state yields a successor in the interval approximation, with probability $> 1 - \varepsilon$
- **Confidence** $\delta \in (0,1)$: probability $> 1 - \delta$ that the interval approximation has the above accuracy

→ Want **high confidence** that the approximation has **high accuracy**

- How many sample points *N* are needed to achieve this ?

Monte Carlo

- No assumption
- Works for both continuous-time and discrete-time systems
- $\{x^{(i)}, i \in \{1, \dots, N\}\}$: random sampling of $[\underline{x}, \overline{x}]$

Proposition

If $N = \left\lceil \frac{1}{\varepsilon} \left(\frac{e}{e-1} \right) \left(\log \frac{1}{\delta} + 2n \right) \right\rceil$, *we have the nested probabilities:*

$$P\left(P\left(\left[\min_i x(T, x^{(i)}), \max_i x(T, x^{(i)}) \right] \right) \geq 1 - \varepsilon \right) \geq 1 - \delta$$

$1 - \delta$ *confidence that the interval approximation is at least* $(1 - \varepsilon)$*-accurate*

- Sampling complexity only linear in n
- No over-approximation guarantees

Monte Carlo - Example

Proposition

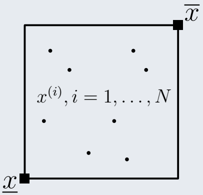
If $N = \left\lceil \frac{1}{\varepsilon}\left(\frac{e}{e-1}\right)\left(\log\frac{1}{\delta} + 2n\right)\right\rceil$, *we have the nested probabilities:*

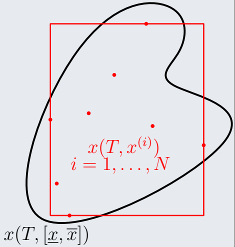
$$P\left(P\left(\left[\min_i x(T,x^{(i)}), \max_i x(T,x^{(i)})\right]\right) \geq 1-\varepsilon\right) \geq 1-\delta$$

Example

$n = 5$, $\varepsilon = 0.05$, $\delta = 0.01$

$N = 463$ random samples give 99% chances that the interval approximation contains 95% of the reachable set





Inputs and time-varying systems

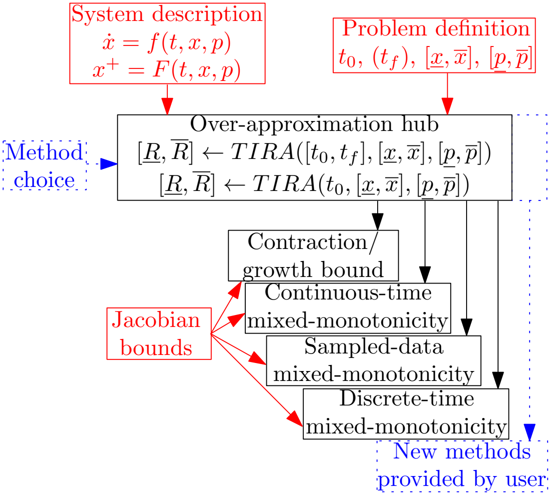
Discrete-time system: $x^+ = f(t, x, p)$
Continuous-time system: $\dot{x} = f(t, x, p)$

Time-varying vector field $\rightarrow$ all methods

Input limitations

- Continuous-time systems: need constant input signals over $[0, T]$ for
  - Sampled-data mixed-monotonicity
  - Quasi Monte Carlo
  - Monte Carlo
- Growth-bound method: need additive input $\dot{x} = f(t, x) + p$

TIRA: Toolbox for Interval Reachability Analysis



Design objectives

- Library of interval reachability methods in a unified framework
- Easily extensible with new methods
- Usable by non-experts

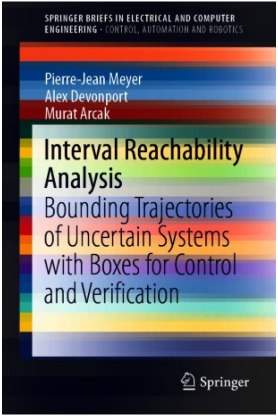
TIRA: Toolbox for Interval Reachability Analysis

Implemented methods and those coming soon

- Monotonicity (DT + CT)
- Mixed monotonicity (DT + CT)
- Sampled-data mixed monotonicity (CT)
  - Interval analysis
  - Sampling and falsification
  - Tunable method with second-order sensitivity
- Growth bounds (CT)
- Quasi-Monte Carlo (DT + CT)
- Monte Carlo (DT + CT)

Conclusions

- Overview of several interval reachability methods
- With various performance objectives
  $\rightarrow$ generality, complexity, tightness, accuracy, soundness



- **Book**: *Interval Reachability Analysis* (Meyer, Devonport, Arcak)
- **Matlab toolbox**: *TIRA: Toolbox for Interval Reachability Analysis*
  `https://gitlab.com/pj_meyer/TIRA`

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