Current Possibilities for Simulating Uncertain Non-Smooth Dynamic Systems

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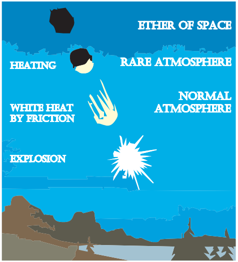
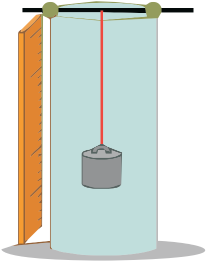
Non-smooth Models: Phenomena and Areas

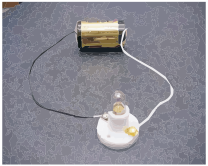
Friction | Impacts | Hysteresis

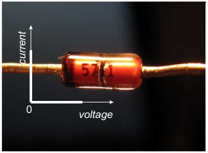
Mechanics

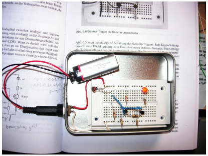




Electrics



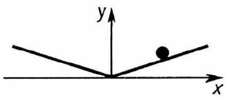




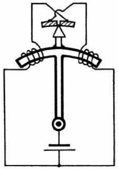
V. Acary, B. Brogliato, *Numerical Methods for Nonsmooth Dynamical Systems*, 2008.

Non-smooth Models: Further Details

Areas:	Control, biology, economics, material science, ...
Phenomena:	Saturation, “good” numerical behavior, switchings...
Formalisms:	differential inclusions, Moreau’s sweeping process, ...
Similarities:	$m\ddot{x} + h \cdot \text{sign}(x) = 0$



sliding pendulum



relay

Impact-like Problems

Impact

A high force applied over a short time period when two or more bodies collide

Mechanics: Collision of rigid bodies

Goal: model and simulate changes in the motion of two solid bodies following collision

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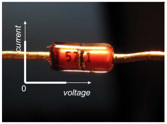

The current $i(t)$ and the voltage $v(t)$ satisfy the complementarity conditions: $0 \leq i(t) \perp v(t) \geq 0$

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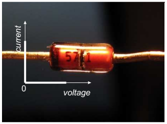

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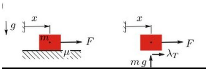
Models: E.g., differential inclusions

$$\psi_K(x) = \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{if } x \notin K \end{cases}, \; \partial\psi_{\mathbb{R}+}(x) = \begin{cases} \{0\} & \text{if } x > 0 \\ (-\infty, 0] & \text{if } x = 0 \end{cases}, \; i(t) \in -\partial\psi_{\mathbb{R}+}(v(t))$$

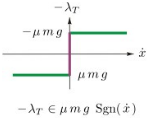
Friction

The force resisting the relative motion of e.g. solid surfaces sliding against each other ($\rightsquigarrow$ dry friction resists relative lateral motion of two solid surfaces in contact).

Mechanics: A block which slides or sticks on the table



Non-smooth approach:



slip

$$m\,\ddot{x} = F + \lambda_T$$
$$\lambda_T = \pm\mu\,m\,g$$

stick

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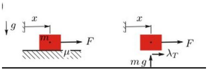

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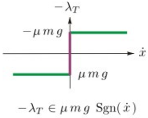

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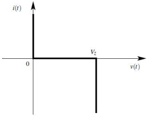
Electrical engineering: Ideal Zener diode



Allows current to flow in the forward direction, but also permits it to flow in the reverse direction when the voltage is above a certain value known as the breakdown voltage

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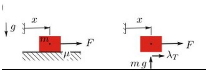

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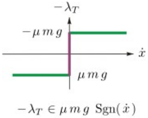

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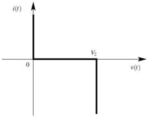

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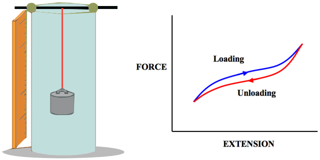
Models: E.g., Coulomb friction with $\mathrm{Sgn}(x) = \partial |x| = \begin{cases} 1 & \text{if } x > 0 \\ [-1, 1] & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

Hysteresis

The time-based dependence of a system's output on current and past inputs ($\rightsquigarrow$ "loop").

Mechanics: Rubber band

The behavior as a load is removed is not the same as that when the load is being increased



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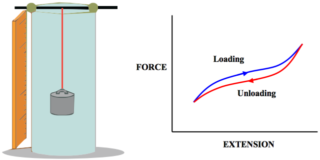
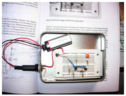
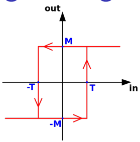


Electrical engineering: Schmitt trigger





The output retains its value until the input changes sufficiently to trigger a change

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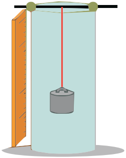

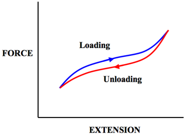

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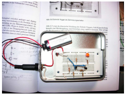

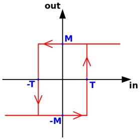

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Models: Specific to the area, e.g., Bouc-Wen model with the hysteretic displacement $\dot{z}(t) = \dot{u}(t)\,\{A - [\beta\,\mathrm{sign}(z(t)\dot{u}(t)) + \gamma]\,|z(t)|^n\}$

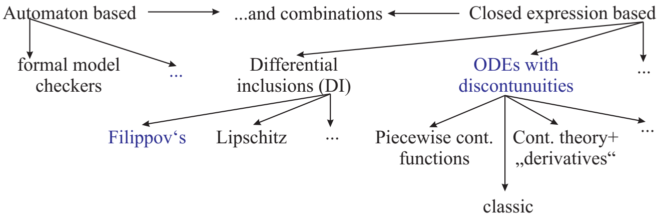
Non-smooth Models: Code Angle

Construct	Example
`IF-THEN-ELSE`	Force: $F \leq 0$
`SWITCH`	Muscle activation function: $0 \leq a(t) = A_1 e^{-c_1(t-t_1)} + A_2 e^{-c_2(t-t_2)} \leq 1$
$\lvert x \rvert$	Hysteresis: $\dot{\omega}(t) = \rho \cdot \Big(v(t) - \sigma \cdot \lvert v(t) \rvert \cdot \lvert \omega(t) \rvert^{\nu-1} \cdot \omega(t) + (\sigma - 1) \cdot v(t) \cdot \lvert \omega(t) \rvert^{\nu} \Big)$
sign$x$	Friction: $F(v) =$sign$(v) \cdot F + \mu \cdot v$

Mathematical Formalisms

Automaton (causal): a graph or an automaton with different ODEs as vertices and logical conditions for jumps as edges

Closed-expression: (non-causal): a single system of ordinary or implicit D(A)Es or inequalities that changes its right side in dependence on zeros of a certain g



Necessary: A study of relationships between modeling concepts

Example: a system with Coloumb and viscous friction

Model: $m\ddot{x}(t) + c\dot{x}(t) + kx(t) \in -\partial(\mu|\dot{x}|)$

Formalisms: DI, variational inequality, interval, ...

Possible Interval Reformulations: Example

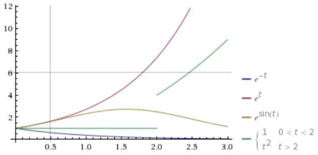
$$\dot{x}(t) = x(t)u(t), x(0) = 0, u(t) \in [-1, 1] \text{ unknown, smooth}$$

Possibility	Formulation	Solution
1. Solve the IVP	see above	$x(t) = x_0 \cdot e^{(\int_0^t u(s)ds)}$ for $x_0 \neq 0$ or $x(t) \equiv 0$
2. Consider a DI	$\dot{x}(t) \in [-x(t), x(t)]$	$x(t) = 0$ and e.g. $x(t) = \begin{cases} 0, & 0 \leq t \leq 2 \\ t^2, & t \geq 2 \end{cases}$
3. Use intervals	$\dot{x}(t) = x(t) \cdot \underbrace{[-1, 1]}_{\text{treated as constant}}$	$x(t) = x_0 \cdot e^{[-1,1]t}$ for $x_0 \neq 0$ or $x(t) \equiv 0$

Intervals offer help in solving DE with convex and closed set-valued right sides

Reformulations: Illustration

Let $\dot{x}(t) = x(t)u(t), x(0) = 1, u(t) = \cos t \in [-1, 1]$



"Normal": $x(t) = 1 \cdot e^{(\int_0^t \cos s ds)} = e^{\sin t}$

DI $x(t) = \begin{cases} 1, & 0 \leq t \leq 2 \\ t^2, & t \geq 2 \end{cases}$

"Interval": $x(t) = x_0 \cdot e^{[-1,1]t} = [e^{-t}, e^t]$, because e^t is monotone

Interval solution is wide but encloses both of the other solutions!

Systematize formalisms/applications, assign a verified method, introduce a simple way of analysis

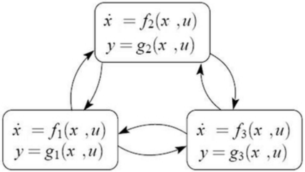
Verified Methods for Non-smooth Systems

Description of a non-smooth IVP

↙ ↘

Closed expressions | Automata

$$x' = \begin{cases} f^-(x), & h(x(t),t) < 0 \\ f^+(x), & h(x(t),t) > 0 \end{cases}$$



Rihm (1993), *Mahmoud and Chen (2008)*

Rauh (2011), *Eggers (2009), Ratschan (2012)* Nedialkov and Mohrenschildt (2002)

Verified non-smooth optimization: Slopes, generalized gradients ...

Ratz (1995), Kearfott (1996), Schnurr (2007), ...

Task: Solve the Non-smooth IVP

$\dot{x} = f(x,t),\ x(0) = x_0$, where $f(x,t)$ is non-smooth in x (or in t).

Situation 1: f is discontinuous only in t

$\downarrow$

Lebesque integration, $x(t) = x_0 + \int\limits_0^t f(x(s),s)ds$

Situation 2: f is discontinuous in x: more difficult

$\downarrow$

- Problem reformulation
- Solution definition (allowed to be discontinuous?)
- Existence (uniqueness) of the solution
- Application areas

Rihm's method

Reformulation $\dot{x}(t) = f(x(t), t) = \begin{cases} f_1(t, x(t)), g(t, x(t)) < 0 \\ f_2(t, x(t)), g(t, x(t)) > 0 \end{cases}$

Application systems with friction, switchings

Solution
- (a) a continuous function for which IVP holds except at isolated switching points
- (b) Filippov's convex definition for DIs if switching points are not isolated

Existence
- (a) unique if transversality conditions hold
- (b) exists if the function $f_0 = \alpha \cdot f_1 + (1 - \alpha) f_2$ is cont. in x; unique if f_0 cont. differentiable

Rihm's method: Enclosure over the Switching Point

Transversality

$$\dot{g}_1(t,x) := \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} f_1(t,x) > 0 (< 0)$$

$$\dot{g}_2(t,x) := \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} f_2(t,x) > 0 (< 0)$$

Endpoints $\dot{g}_1(t,x) < 0$ and $\dot{g}_2(t,x) > 0$

Sliding $\dot{g}_1(t,x) > 0$ and $\dot{g}_2(t,x) < 0$

Let t^* be the switching point, $x^* := x(t^*)$, $f^- := f_1(t^-, x(t^-))$, $f^+ := f_2(t^*, x^*)$, $h^- := t^* - t^-$, $h^+ := t^+ - t^*$, $s = t^+ - t^-$, $z^- \in \mathbf{z}^-$ and $z^+ \in \mathbf{z}^+$ local errors, then

$$\begin{aligned} x(t^+) &= x(t^-) + h^+(f^+ - f^-) + sf^- + z^- + z^+ \\ &\in \mathbf{x}^- + s\mathbf{f}^- + [0,s](\mathbf{f}^+ - \mathbf{f}^-) + \mathbf{z}^- + \mathbf{z}^+ =: \mathbf{x}^+ \end{aligned}$$

Rihm's method: Algorithm

1. Prepare: $0 \in g(t_0, [x_0])$? (Y) Check transversality, reformulate (N) Proceed
2. Enclose in the area of continuity: Define a grid $t_0, t_1, \ldots, t_{r_s}$

   Compute a rough enclosure $\mathbf{x}_{j-1,j}$ over $[t_{j-1}, t_j]$

   Refine into $\mathbf{x}_j$ of $x_1(t_j)$ until $j = r_s$

   or (t^*, x^*) is reached (check with $[t_{j-1}, t_j]$, $\mathbf{x}_{j-1,j}$)
3. Enclose t^*: Use the interval Newton method: $t^* \in I^* := [t^-, t^+]$, $h = \mathrm{wid}(I^*)$
4. Enclose x^*: Compute $\mathbf{x}(t)$ over $[t_{j-1}, t_j]$ (e.g. from Taylor coeff.)

   Compute a rough enclosure

   $$\mathbf{x}^* = \mathbf{x}_{j-1,j} \cap (\mathbf{x}(t^-) + [0,h] f_1(I^*, \mathbf{x}_{j-1,j})) \cap (\mathbf{x}(t^+) - [0,h] f_1(I^*, \mathbf{x}_{j-1,j}))$$

   Refine if possible, e.g. using $g(t,x)$; check if (t^*, u^*) is an end point
5. Continue into the next cont. area: Compute rough enclosure $\mathbf{x}^+$ and local errors $\mathbf{z}^-$, $\mathbf{z}^+$

   Compute the refined enclosure of $x(t^+)$ (previous slide)

   Reformulate and go to 2

An Automaton Based Method (Rauh et al.)

Problem: Smooth models $\{S_i\}_{i=1}^{l}$: $\dot{x}(t) = f_{S_i}(x(t), p, u(t), t)$
Transition $S_i \to S_j$ if the condition $T_i^j(x, u)$ holds true

Stage 1: Calculate a bounding box

$$\mathbf{b}_k^a = \bigcup_{i \in \mathcal{I}_a} (\mathbf{x}_0 + [0, h] \cdot f_{S_i}(\mathbf{x}_k, \mathbf{p}, \mathbf{u}(t_k), t_k))$$

Stage 2: Activate additional transitions $T_i^j(\mathbf{b}_k^a, \mathbf{u}([t_k, t_{k+1}]))$

$$\tilde{\mathbf{b}}_k^a := \mathbf{b}_k^a \bigcup_{i \in \tilde{\mathcal{I}}_a \setminus \mathcal{I}_a} (\mathbf{x}_0 + [0, h] \cdot f_{S_i}(\mathbf{b}_k^a, \mathbf{p}, \mathbf{u}([t_k, t_{k+1}]), [t_k, t_{k+1}]))$$

Stage 3: Calculate $\mathbf{x}_{k+1}$ at t_{k+1} ($\approx$ refinement of $\tilde{\mathbf{b}}_k^a$)
Stage 4: Deactivate transition conditions depending on $\mathbf{x}_{k+1}$

A "Continuous" Method: Problem Definition

R.B.Kearfott, *Rigorous Global Search: Continuous Problems*, 1996

Simplicity is a major advantage of treating non-smooth problems with the same techniques as smooth problems

Interval IVP: $\begin{cases} x' & = & f(x), \\ x(0) & \in & [x_0], \end{cases}$ where $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ or $\mathcal{D} \subset \mathbb{IR}^n \to \mathbb{IR}^n$.

f is given in its algorithmic representation (inductive):

$$\begin{cases} \tau_i(x) & = & g_i(x) = x_i,\ i = 1 \dots n \\ \tau_i(x) & = & g_i(\tau_1(x), \dots, \tau_{i-1}(x)),\ i = n+1 \dots l, \\ g_i & \in & S_{EO} \cup S_{PW} \end{cases} .$$

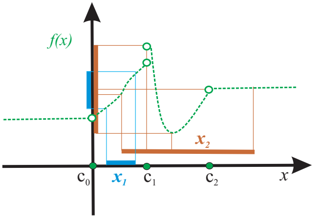
$S_{EO} = \{c, +, -, *, /, \sin, \cos, \dots\}$ and S_{PW} are piecewise cont.

Goal: Find a derivative generalization to use with the usual theory

Definition of Piecewise Functions $\phi(y)$ in S_{PW}

$y = \tau_\nu(x)$, $\phi_j(y)$, $j = 0, \ldots, L$ smooth

$$\phi(y) = \begin{cases} \phi_0(y) & \text{for } c_{-1} = -\infty < y < c_0, \\ \phi_1(y) & \text{for } c_0 < y < c_1, \\ \ldots & \ldots \\ \phi_{L-1}(y) & \text{for } c_{L-2} < y < c_{L-1}, \\ \phi_L(y) & \text{for } c_{L-1} < y < c_L = +\infty \end{cases} .$$



An interval extension of ϕ over $\mathbf{x}$ ($\phi(\mathbf{x})$):

$$\begin{cases} \phi_i(\mathbf{x}), & \text{if } \mathbf{x} \subseteq (c_{i-1}, c_i), \\ \underline{\bigcup}_{k=i+1}^{j-1} \phi_k([c_{k-1}, c_k]) \underline{\cup} \phi_i([\underline{x}, c_i]) \underline{\cup} \phi_j([c_{j-1}, \overline{x}]), & \text{if } \mathbf{x} \subseteq (c_{i-1}, c_j) \end{cases}$$

Definition of the Derivative: Top-Down Approach

An interval extension of ϕ' over $\mathbf{x}$ ($\phi'(\mathbf{x})$)

$$\begin{cases} \phi_i'(\mathbf{x}), & \text{if } \mathbf{x} \subseteq (c_{i-1}, c_i), \\ \underline{\bigcup}_{k=i+1}^{j-1} \phi_k'([c_{k-1}, c_k]) \underline{\cup} \phi_i'([\underline{x}, c_i]) \underline{\cup} \phi_j'([c_{j-1}, \overline{x}]) & \\ \underline{\cup}\ \mathrm{REST}, & \text{if } \mathbf{x} \subseteq (c_{i-1}, c_j), \end{cases}$$

where REST depends on:

- how many switching points $\mathbf{x}$ contains,
- whether ϕ is continuous,

if we want the mean value theorem to hold.

Derivative for IF–THEN–ELSE (One Switching Point)

$$\phi(x) = \begin{cases} \phi_0(x), & x < c_0, \\ \phi_1(x), & x > c_0. \end{cases}$$

- If ϕ is continuous, there is no REST
- If ϕ is discontinuous

$$\begin{cases} \phi_0'([\underline{x}, c_0]) \underline{\cup} \left(\dfrac{\phi_1(c_0) - \phi_0(c_0)}{[c_0, \overline{x}] - x_0} + \phi_0'([\underline{x}, c_0]) \underline{\cup} \phi_1'([c_0, \overline{x}]) \right) & \text{if } x_0 \in [\underline{x}, c_0), \\ \phi_1'([c_0, \overline{x}]) \underline{\cup} \left(\dfrac{\phi_0(c_0) - \phi_1(c_0)}{[\underline{x}, c_0] - x_0} + \phi_0'([\underline{x}, c_0]) \underline{\cup} \phi_1'([c_0, \overline{x}]) \right) & \text{if } x_0 \in (c_0, \overline{x}], \\ \phi_0'([\underline{x}, c_0]) \underline{\cup} \phi_1'([c_0, \overline{x}]) & \text{if } x_0 = c_0 \quad . \end{cases}$$

x_0 needed to avoid overconservative enclosures $([-\infty, +\infty])$

Generalization

This definition can be generalized for the arbitrary number of $c_i \in \mathbf{x}$, but:
$\mathbf{x}$ containing many c_i might be simply too wide.

If $\phi(\cdot)$ contains several switching points c_i, but $\mathbf{x}$ contains only one:

$$\phi'(\mathbf{x}) = \begin{cases} \phi_i'(\mathbf{x}) & \text{for } \mathbf{x} \subset (c_{i-1}, c_i), \\ \phi'_{\text{cont}}(\mathbf{x}) & \text{for } \mathbf{x} \subset (c_{i-1}, c_{i+1}),\ \text{if } \phi \text{ is cont. in } c_i, \\ \phi'_{\text{dis}}(\mathbf{x}) & \text{for } \mathbf{x} \subset (c_{i-1}, c_{i+1}),\ \text{if } \phi \text{ is discont. in } c_i \end{cases} .$$

Features

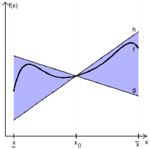
+	Right sides with several variables represented	$f(x_1, x_2) = \|x_1\| + x_1 \cdot \text{sign}\,(x_2)$, $\|\text{sign}\,(x_1)\|$
–	No PW operations like	$f(x_1, x_2) = \begin{cases} 1, & x_2 < x_1 \\ 2, & x_2 > x_1 \end{cases}.$
	Covered by Rihm	$g(x_1, x_2) = x_2 - x_1$, $f_1 = 1 \; f_2 = 2$
+	Better coverage for	$f(x) = \begin{cases} -h, & x < -x_+ \\ 0, & -x_+ < x < x_+ \\ h, & x_+ < x \end{cases}.$

Another Derivative Definitions

Slopes (implementation in Schnurr (2007), also of order 2)

$f : D \subseteq \mathbb{R} \mapsto \mathbb{R}$ continuous, $x_0 \in D$, then $f(x) = f(x_0) + \underbrace{\delta f(x, x_0)}_{\text{slope in 1D}}(x - x_0)$

Interval slope of f over $\mathbf{x} \in D$: $\delta f(\mathbf{x}, x_0) \supseteq \{\delta f(x, x_0) | x \in \mathbf{x}\}$



$$f(x) \in f(x_0) + \delta f(\mathbf{x}, x_0)(\mathbf{x} - x_0)$$

$$\delta f(\mathbf{x}, x_0) = \left[\inf_{x \in [x], x \neq x_0} \frac{f(x) - f(x_0)}{x - x_0}, \sup_{x \in [x], x \neq x_0} \frac{f(x) - f(x_0)}{x - x_0}\right]$$

A triple $(F_x, F_{x_0}, \delta F)$ with $f(x) \in F_x, f(x_0) \in F_{x_0}$, $f(x) - f(x_0) \in \delta F(x - x_0)$

($\rightsquigarrow$ suitable for a bottom-up approach)

More definitions in H. Munoz, R.B. Kearfott, *Slope Intervals, Generalized Gradients, Semigradients, Slant Derivatives, and Csets*, 2004

Usage

Approach Plug the definition into VALENCIA-IVP

VALENCIA

- an a posteriori method
- for smooth problems
- $x(t) \in \underbrace{[x(t)]}_{\text{verified enclosure}} := \underbrace{x_{app}(t)}_{\text{approximation}} + \underbrace{[R(t)]}_{\text{error bounds}}$
- uses MVT, a fixed point theorem
- Jacobians only $\rightarrow$ easy to adapt

Non-smooth

- upper semi-continuous right sides
- Kakutani's fixed point theorem
- f' satisfies MVT

+ for Lipschitz cont. right sides or isolated switching points

– overestimation for sliding solutions

Remarks on Software

⚠ Although descriptions of methods are easily available the corresponding software is not easy to obtain (not maintained, etc.)

Automaton-based
(→ all rely on a kind of meta-language for defining the system)

- SpaceEx `http://spaceex.imag.fr/` (linear systems),
- dReach `http://dreal.github.io/` (the satisfiability modulo theories solver for the nonlinear theories of the reals),
- Ariadne `http://trac.parades.rm.cnr.it/ariadne/` (extendable)
- Flow* `http://systems.cs.colorado.edu/research/cyberphysical/taylormodels/` (open source, Taylor models)

Several Mentioned Articles

Eggers (2009) A. Eggers et al., *Application of Constraint Solving and ODE-Enclosure Methods to the Analysis of Hybrid Systems*, 2009

Mahmoud and Chen (2008) S. Mahmoud and X. Chen, *A Verified Inexact Implicit Runge-Kutta Method for Nonsmooth ODEs*, 2008

Nedialkov and Mohrenschildt (2002) N.S. Nedialkov and M. von Mohrenschildt, *Rigorous Simulation of Hybrid Dynamic Systems with Symbolic and Interval Methods*, 2002

Ratschan (2012) S. Ratschan, *An Algorithm for Formal Safety Verification of Complex Heterogeneous Systems*, 2012

Rauh (2011) A. Rauh et al.,. *Verified Simulation and Optimization of Dynamic Systems with Friction and Hysteresis*, 2011

Rihm (1993) R. Rihm, *Enclosing Solutions with Switching Points in Ordinary Differential Equations*, 1992

Schnurr (2007),Ratz (1995) M. Schnurr, *Steigungen höherer Ordnung zur verifizierten globalen Optimierung*, in German, 2007

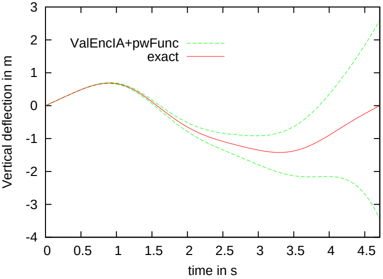
Other interesting papers: N. Ramdani, N. S. Nedialkov, *Computing Reachable Sets for Uncertain Nonlinear Hybrid Systems Using Interval Constraint-Propagation Techniques*, 2011
Z. Galias, *Rigorous Study of the Chua's Circuit Spiral Attractor*, 2012
Th. A. Henzinger et al., *Beyond HYTECH: Hybrid Systems Analysis Using Interval Numerical Methods*, 2000
D. Ishii, *Simulation and Verification of Hybrid Systems Based on Interval Analysis...*, 2010

Known Exact Solution: Tacoma Narrows Suspension Bridge

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}\left(\sin(4t) - q(x_1)\right)$$

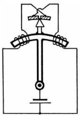
$$x_1(0) = 0$$
$$x_2(0) = 1$$

$$q(x_1) = \begin{cases} x_1, & x_1 < 0 \\ 4x_1, & x_1 > 0 \end{cases}$$

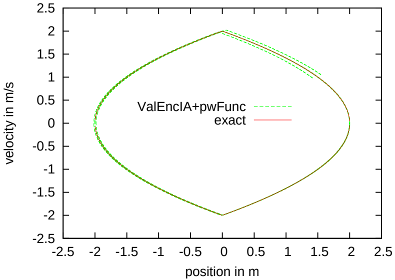


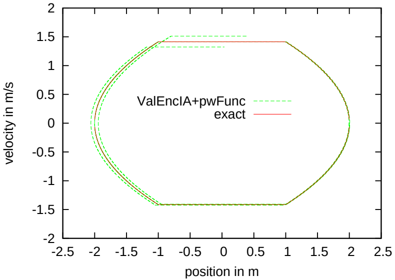
Known Exact Solution: Oscillator

$$m\ddot{(x)} = -f_i(x),\ h = m = x_t = 1,\ x(0) = 2,\ v(0) = 0$$



$$f_1(x_1) = \begin{cases} -h, & x < 0 \\ +h, & x > 0 \end{cases} \qquad f_2(x_1) = \begin{cases} -h, & x < -x_t \\ 0, & -x_t < x < x_t \\ +h, & x > x_t \end{cases}$$



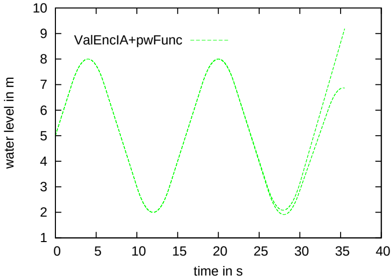


Comparisons with Other Methods: Water Level

$$\dot{x}_1 = x_2 \qquad x_1(0) = 5$$
$$\dot{x}_2 = 0.5u(x_1) \qquad x_2(0) = 1$$

$$u(x_1) = \begin{cases} 1, & x_1 < 3 \\ -1, & x_1 > 7 \\ 0, & otherwise \end{cases}$$

Width at $t = 35$: $\text{wid}(\mathbf{x}_1) = 0.28$ as opposed to Nedilkov/von Mohrenschildt $\text{wid}(\mathbf{x}_1) = 10^{-7}$



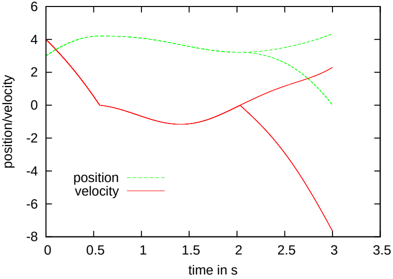
Example without a Classical Solution

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -0.2x_2 - x_1 + 2\cos(\pi t) - u(x_2)$$

$$u(x_1) = \begin{cases} -4, & x_1 < 0 \\ +4, & x_1 > 0 \end{cases}$$

$$x_1(0) = 3, x_2(0) = 4$$

The first switching point at $t \approx 0.5$, the second at $t \approx 2.03$, the solution leaves the switching surface $v = 0$ after $t \approx 2.6$



Summary

Considered:

→ Different formulations and reformulations of non-smooth problems
→ Several approaches with result verification:
  → ODEs with switchings
  → Automaton-based
  → "Continuous"

Interesting topics:

- Systematize reformulations, find equivalencies, supply interval (etc.) reformulation
- Software-related issues, e.g. how to implement in oder to easily integrate into available simulation packages?
- Reduction of overestimation (→ slopes? combination with non-verified methods?)

Properties of $\phi'(\mathbf{x})$

1. If the derivative of ϕ exists for $x \in \mathbf{x}$, then $\phi'(x) \in \phi'(\mathbf{x})$
2. The slope $\delta\phi(\mathbf{x}, x_0) \subseteq \phi'(\mathbf{x})$
3. The mean value theorem holds:

$$\phi(x) = \phi(x_0) + \phi'(\xi)(x - x_0) \in \phi(x_0) + \phi'(\mathbf{x})(\mathbf{x} - x_0)$$

4. If ϕ is continuous ($\phi_j(c_j) = \phi_{j+1}(c_j)$, $0 \le j < L$), then $f(x)$ is continuous if all operations in S_{EO} are continuous.

VALENCIA-IVP[1] For Non-smooth IVPs

General approach in VALENCIA: A posteriori

$$x(t) \in \underbrace{[x(t)]}_{\text{verified state enclosure}} := \underbrace{x_{app}(t)}_{\text{non-verified approximation}} + \underbrace{[R(t)]}_{\text{error bounds}}$$

Conditions for the right side:

1. continuous
2. Lipschitz

[1] *VAL*idation of state *ENC*losures using *I*nterval *A*rithmetic for *I*nitial *V*alue *P*roblems

VALENCIA-IVP For Non-smooth IVPs (Cont.)

The algorithm for $0 \leq t \leq T$:

1. Start with $[x^{(0)}]$, $x_{app}(t)$, $[R(0)]$
2. $k = 1 \dots k_{max}$ or while $[\dot{R}^{(k+1)}([0,T])] != [\dot{R}^{(k)}([0,T])]$

   Compute $[\dot{R}^{(k+1)}([0,T])] := \dot{x}_{app} + f([x^{(k)}])$, (MVT)
   where $[x^{(k)}] := [x^{(k)}([0,T])]$

   If $[\dot{R}^{(k+1)}([0,T])] \subseteq [\dot{R}^{(k)}([0,T])]$ then

$$\begin{aligned}
[R^{(k+1)}([0,T])] &:= [R(0)] + [\dot{R}^{(k+1)}([0,T])][0,T] \\
[x^{(k+1)}([0,T])] &:= x_{app} + [R^{(k+1)}([0,T])]
\end{aligned}$$

Differences ((non-)smooth): Derivative definition, the fixed point theorem

To-do-list: Discontinuities in x for the right side

Implementation Issues: Class pwFunc

Remarks on $f(x)$

- $f'(\mathbf{x})$ is obtained with pwFunc
- pwFunc uses FADBAD++ and overloads hull, d()
- $f'(\mathbf{x})$ encloses both left and right derivatives
- pwFunc is plugged into ValEncIA

Class declaration

```
template<class T>
class pwFunc{
public:
typedef T (*ptrFct)(const T& x);
pwFunc(const vector<interval>& p,
    const vector<ptrFct>& f);
T operator()(const T& x)
    { return getValueAtX(x);}
private:
vector< ptrFct > functions;
vector<interval> points;
vector<T> subintervals;
T getValueAtX(const T& x);
void generateSubintervals(const T& x);
};
```

Implementation Example: A Discontinuous Function

```
template <class T> T f1(const T& x){ return -1+x;}
template <class T> T f2(const T& x){ return 1+x;}
template<class T> T ff(const T& a){
    vector<INTERVAL> p; p.push_back(0);
    vector<pwFunc<T>::ptrFct> functions;
    functions.push_back(&f1<T>);functions.push_back(&f2<T>);
    pwFunc<T> fp(p, functions); return fp(a); }
ff([-1,2]);
```

Equation:

$$F_f(v) = \begin{cases} -1.0 + x & x < 0 \\ +1.0 + x & x > 0 \end{cases}$$

Result:

[-2,3]([1,6])

System for the SOFC temperature

$$\dot{\vartheta}_{FC} = \frac{1}{c_{FC}\ m_{FC}} \Bigg[\frac{1}{R_A}(\vartheta_A - \vartheta_{FC}) - (c_{N_2}\zeta_{N_2,C} + c_{O_2}\zeta_{O_2})u(t)$$

$$- (c_{H_2}\dot{m}_{H_2} + c_{H_2O}\dot{m}_{H_2O} + c_{N_2}\dot{m}_{N_2,A})(\vartheta_{FC} - \vartheta_{AG,in}) + \frac{\Delta H_m(\vartheta_{FC})\dot{m}}{M_{H_2}}$$

$$= a(\vartheta_{FC}(t), p) + b(\vartheta_{FC}(t), p, d) \cdot u(t)$$