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HOCHSCHULE WISMAR UNIVERSITY OF APPLIED SCIENCES TECHNOLOGY, BUSINESS AND DESIGN

Interval-Based Design of Sliding Mode Control and State Estimation Procedures

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Motivation: Excursion in Control and Estimation

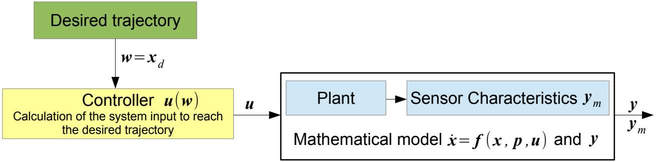

Figure : Open-loop control $\mathbf{u}(\mathbf{w})$.

Nonlinear continuous-time state equations $\mathbf{f}$, reference signal $\mathbf{w}$, state vector $\mathbf{x}$, control vector $\mathbf{u}$, parameter vector $\mathbf{p}$, measurements $\mathbf{y}_m$, corresponding system output $\mathbf{y}$

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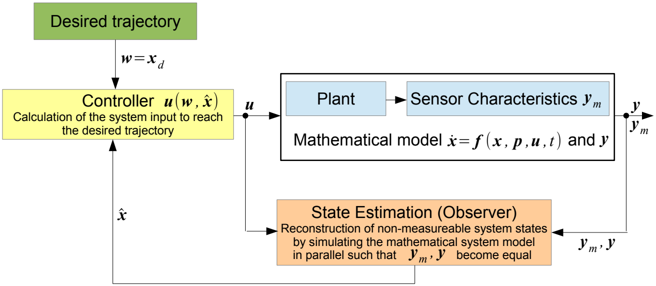


Figure : Closed-loop control $\mathbf{u}(\mathbf{w}, \hat{\mathbf{x}})$.

Nonlinear continuous-time state equations $\mathbf{f}$, reference signal $\mathbf{w}$, state vector $\mathbf{x}$, control vector $\mathbf{u}$, parameter vector $\mathbf{p}$, measurements $\mathbf{y}_m$, corresponding system output $\mathbf{y}$, estimated state vector $\hat{\mathbf{x}}$

Motivation: Excursion in Control and Estimation

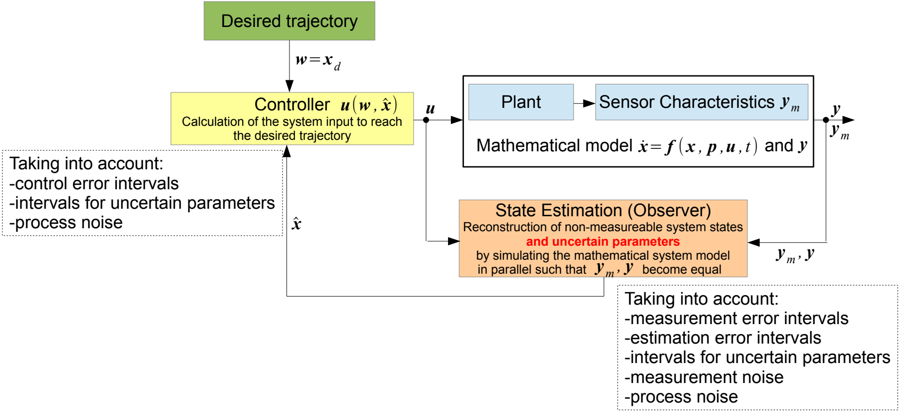


Figure : Closed-loop control with bounded and stochastic uncertainty.

Motivation: Excursion in Control and Estimation

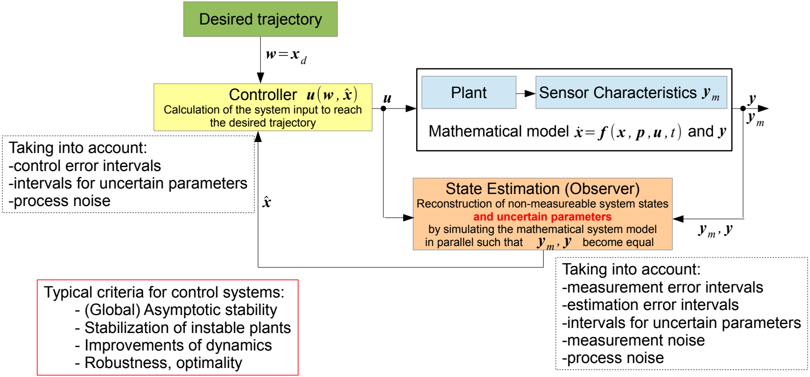
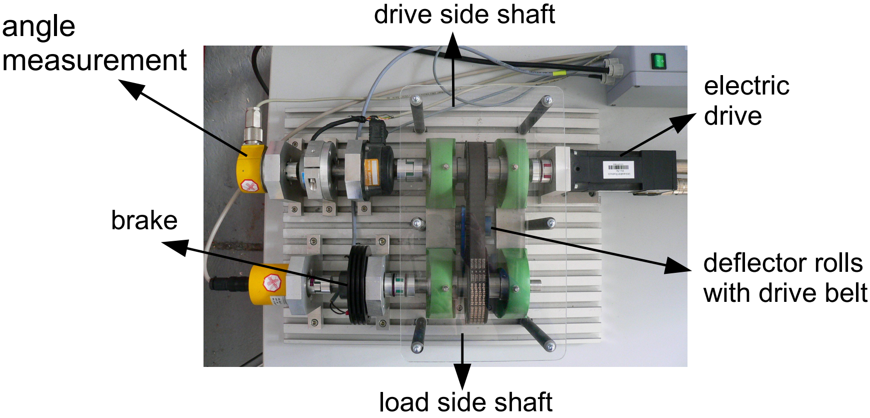


Figure : Closed-loop control with bounded and stochastic uncertainty.

Goal: Minimization of control errors (difference between reference signal and system output)

Experimental Test-Rig



Experimental Test-Rig

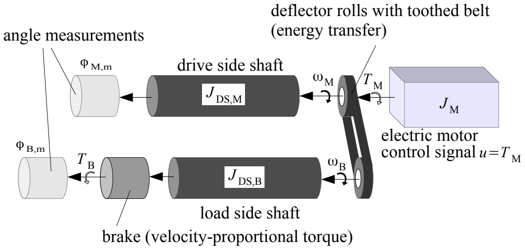
Hardware and Programming

- Real-time environment: Bachmann PLC system
- Communication via serial interface and Ethernet
- Compilation of Matlab/Simulink model for use on process unit
- Interface with C-XSC does not work in experiment, because switching of rounding mode is not fully supported (loss of accuracy can be neglected)

  $\Rightarrow$ Own structure for calculating with intervals

  $\Rightarrow$ New implementation of all algebraic operators

Drive Train Test-Rig: Modeling



- Motor torque T_M, braking torque T_B
- Angular velocity of the motor ω_M
- Measured angles $\varphi_{M,m}$
- J_{rot} contains all mass moments of inertia $J_{DS,M}$, $J_{DS,B}$, J_M with respect to the driving shaft
- Braking represents a disturbance

Drive Train Test-Rig: Modeling

ODE of System model

- Torque balance $J_{rot} \cdot \dot{\omega}_M = T_M - T_B - T_S \cdot \mathrm{sign}(\omega_M)$
- Compensation of static friction $T_S \cdot \mathrm{sign}(\omega_{M,d})$
- Transmission ratio $k = \frac{\omega_M}{\omega_B}$
- Braking torque $T_B = k_{D_2} \cdot \omega_B = \frac{k_{D_2}}{k} \cdot \omega_M = d \cdot \omega_M$

System Model (φ_M angle of rotation of the motor shaft)

$\mathbf{f} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{\varphi}_M \\ \dot{\omega}_M \end{bmatrix} = \begin{bmatrix} \omega_M \\ \alpha \cdot \omega_M + \beta \cdot \tilde{u} \end{bmatrix}$ with $\tilde{u} = T_M + T_S \cdot \mathrm{sign}(\omega_M)$

Drive Train Test-Rig: Modeling

(In the following control input $u(t)$ instead of $\tilde{u}(t)$)

System Model: State-Space Representation

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{b} \cdot u(t) = \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u(t) \text{ with}$$

$$y(t) = \mathbf{c}^T \cdot \mathbf{x}(t) = x_1(t)$$

Drive Train Test-Rig: Modeling

System Model: State-Space Representation with Intervals

$$\dot{\mathbf{x}}(t) = \mathbb{A} \cdot \mathbf{x}(t) + \mathbb{B} \cdot u(t) = \begin{bmatrix} 0 & 1 \\ 0 & [\alpha] \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ [\beta] \end{bmatrix} u(t) \text{ with}$$

$$y(t) = \mathbf{c}^T \cdot \mathbf{x}(t) = x_1(t)$$

Task for Interval-Based Sliding Mode Approaches

- Trajectory tracking $\varphi_M - \varphi_{M,d} \overset{!}{=} 0$ and $\omega_M - \omega_{M,d} \overset{!}{=} 0$ in terms of motor torque control
- Reconstruction of system states and parameter identification

despite

- Uncertain parameters $\alpha = -\frac{d}{J} \in [\alpha]$ and $\beta = \frac{1}{J} \in [\beta]$
- Friction, measurement noise

Sliding Mode Approaches and Lyapunov Functions

Stability

Control a system such that it does not become unstable despite disturbances or external influences

Asymptotic Stability

System dynamics converge over time to the system's equilibrium

Lyapunov Functions

Basis for Sliding Mode Approaches guaranteeing the system's stability for

- Control tasks: calculate a gain in the control law
- Estimation tasks: calculate a gain used for the reconstruction of system states and parameters such that $\mathbf{y} - \mathbf{y}_m \stackrel{!}{=} 0$

Sliding Mode Approaches without Intervals

Principle (Common Way)

- Assume a system in nonlinear controller normal form, $y = x_1$
- Definition of a sliding surface $s = s(\mathbf{x}, \mathbf{x}_d) = \sum_{i=0}^{n-1} \kappa_i \cdot (x_1^{(i)}(t) - x_{1,\mathrm{d}}^{(i)}(t))$ such that the system states tend to this stable mode, ideal case $s = 0$
- Suitable candidate of a Lyapunov function $V = \frac{1}{2} \cdot s^2 > 0$
- Stabilization of tracking error (ensure that system states follow the corresponding desired ones) $\dot{V} = s \cdot \dot{s} \overset{!}{\leq} 0$
- Asymptotic stability if $\dot{V} < 0$ for all $s \neq 0$

Sliding Mode Approaches without Intervals

Calculation of the Switching Amplitude

Enforcing proportionality between absolute value of the sliding surface and the time derivative of the Lyapunov function

$$\dot{V}(t) \overset{!}{\leq} -\eta \cdot |s(t)| \text{ with } \eta > 0 \text{ and } |s(t)| = s(t) \cdot \mathrm{sign}(s(t))$$

$$s(t) \cdot \dot{s}(t) \overset{!}{\leq} -\eta \cdot s(t) \cdot \mathrm{sign}(s(t))$$

$$\dot{s}(t) + \eta \cdot \mathrm{sign}(s(t)) \overset{!}{<} 0 \text{ (instead of 0, obtain a convergence rate)} \quad (1)$$

$$\mathrm{sign}(s(t)) = \begin{cases} 1, & \text{if} \quad s(t) > 0 \\ -1, & \text{if} \quad s(t) < 0 \\ 0, & \text{else} . \end{cases}$$

Calculate η such that (1) becomes true

Sliding Mode Approaches without Intervals

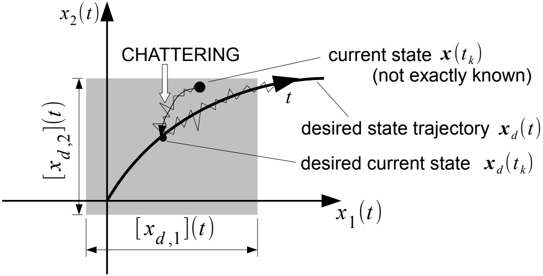
Resulting Control Law for a System with $x_1^{(n)}(t) = u(t)$

$$u(t) = x_{1,\mathrm{d}}^{(n)}(t) - \sum_{i=0}^{n-1} \kappa_i \cdot (x_1^{(i)}(t) - x_{1,\mathrm{d}}^{(i)}(t)) - \eta \cdot \mathsf{sign}(s(t))$$

Note

- Control law depends on all system states (estimation necessary)
- Switching amplitude η has to be defined by the user and is constant over all time (either too large or too small) $\Rightarrow$ Problem in case of input constraints $\Rightarrow$ Control law realizable?
- Coefficients κ_i have to be defined by the user

Sliding Mode Approaches with Intervals



Deviation of the current state from desired state included in:

$$[x_{d,1}](t) = x_{d,1}(t) + [\Delta x_{c,1}]$$
$$[x_{d,2}](t) = x_{d,2}(t) + [\Delta x_{c,2}]$$

Interval for the tracking error:

$$[\tilde{x}_1](t) = x_1(t) - [x_{d,1}](t)$$
$$[\tilde{x}_2](t) = x_2(t) - [x_{d,2}](t)$$

Figure : Control error interval for sliding mode control design.

Sliding Mode Approaches with Intervals

Note

- Intervals for control error and parameter uncertainty
- Vector-valued sliding surface depending on system order
- Calculation of the switching amplitude vector $\eta \in \mathbb{R}^{n\times 1}$
- Consideration of stochastic processes by using a Lyapunov function and the Itô differential operator

Stochastic processes (Standard Deviations & Brownian motions for probability distributions)

- Process noise: approximation error between two subsequent time steps
- Measurement noise: uncertainty in sensors
- Over-approximation of external disturbances and unknown effects on system dynamics

Itô Differential Operator to Consider Noise Processes

Example

- Dynamic system $\dot{x} = -x + p \cdot x \cdot w$
- State x
- Parameter p
- Normally distributed noise w with expectation $\mu = 0$, standard deviation $G = 1$
- Asymptotic stability of the deterministic part $\dot{x} = -x$ for all $x \in \mathbb{R}$

Formulation as Stochastic Differential Equation

$$\mathrm{d}x = \underbrace{-x}_{f(x)}\,\mathrm{d}t + \underbrace{p \cdot x}_{g(x)}\,\mathrm{d}w$$

Itô Differential Operator to Consider Noise Processes

Formulation as Stochastic Differential Equation

$$\mathrm{d}x = \underbrace{-x}_{f(x)}\,\mathrm{d}t + \underbrace{p \cdot x}_{g(x)}\,\mathrm{d}w$$

Lyapunov function

$$V(x) = \frac{1}{2}x^2 > 0$$

Itô Differential Operator to Consider Noise Processes

Formulation as Stochastic Differential Equation

$$\mathrm{d}x = \underbrace{-x}_{f(x)}\,\mathrm{d}t + \underbrace{p \cdot x}_{g(x)}\,\mathrm{d}w$$

Lyapunov function

$$V(x) = \frac{1}{2}x^2 > 0$$

How to calculate the time derivative of the Lyapunov function for a system that is affected by stochastic processes?

Itô Differential Operator

$$L(V(x)) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(x) + \frac{1}{2}\mathrm{trace}\left\{g(x) \cdot \frac{\partial^2 V}{\partial x^2} \cdot g(x)\right\}$$

Itô Differential Operator to Consider Noise Processes

$$\mathrm{d}x = \underbrace{-x}_{f(x)}\,\mathrm{d}t + \underbrace{p \cdot x}_{g(x)}\,\mathrm{d}w \quad \text{with} \quad V(x) = \frac{1}{2}x^2 > 0$$

Itô Differential Operator

$$L(V(x)) = \underbrace{\frac{\partial V}{\partial t}}_{0} + \underbrace{\frac{\partial V}{\partial x}}_{x} \cdot f(x) + \frac{1}{2}\text{trace}\left\{ g(x) \cdot \underbrace{\frac{\partial^2 V}{\partial x^2}}_{1} \cdot g(x) \right\}$$

$$= x \cdot (-x) + \frac{1}{2}\text{trace}\left\{g^2(x)\right\}$$

$$= -x^2 + \frac{1}{2} \cdot (p \cdot x)^2 = -x^2 \cdot (1 - \frac{1}{2} \cdot p^2)$$

Itô Differential Operator to Consider Noise Processes

$$\mathrm{d}x = \underbrace{-x}_{f(x)}\,\mathrm{d}t + \underbrace{p \cdot x}_{g(x)}\,\mathrm{d}w \quad \text{with} \quad V(x) = \frac{1}{2}x^2 > 0$$

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Itô Differential Operator to Consider Noise Processes

Itô Differential Operator

$$L(V(x)) = \underbrace{\frac{\partial V}{\partial t}}_{0} + \underbrace{\frac{\partial V}{\partial x}}_{x} \cdot f(x) + \frac{1}{2}\text{trace}\left\{ g(x) \cdot \underbrace{\frac{\partial^2 V}{\partial x^2}}_{1} \cdot g(x) \right\}$$

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Consequence

Stochastic system is only asymptotically stable, if

$$\left(1 - \frac{1}{1} \cdot p^2\right) < 0 \;\Rightarrow p^2 < 2 \;\Rightarrow p \in \left[-\sqrt{2}; \sqrt{2}\right]$$

Itô differential operator to consider noise processes

Itô differential operator

$$L(V(x)) = \underbrace{\frac{\partial V}{\partial t}}_{0} + \underbrace{\frac{\partial V}{\partial x}}_{x} \cdot f(x) + \frac{1}{2}\text{trace}\left\{ g(x) \cdot \underbrace{\frac{\partial^2 V}{\partial x^2}}_{1} \cdot g(x) \right\}$$

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Consequence

Stochastic system is only asymptotically stable, if

$(1 - \frac{1}{1} \cdot p^2) < 0 \;\Rightarrow p^2 < 2 \;\Rightarrow -\sqrt{2} < p < \sqrt{2}$

$\Rightarrow$ System can nevertheless become unstable due to the stochastic noise w

$\Rightarrow$ Control necessary

Sliding Mode Control: Scheme ($\mathbf{g}(x) = \mathbf{G}_p$)

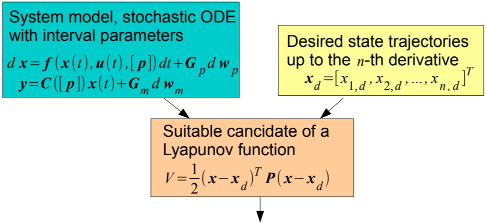
System model, stochastic ODE with interval parameters

$$d\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), [\boldsymbol{p}])\,dt + \boldsymbol{G}_p\,d\boldsymbol{w}_p$$
$$\boldsymbol{y} = \boldsymbol{C}([\boldsymbol{p}])\,\boldsymbol{x}(t) + \boldsymbol{G}_m\,d\boldsymbol{w}_m$$

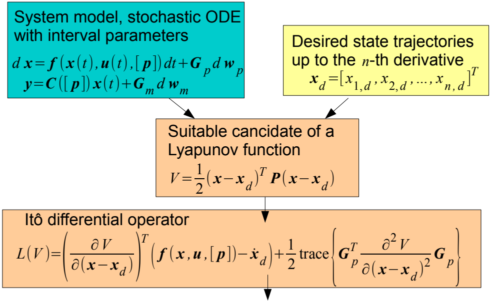
Desired state trajectories up to the n-th derivative

$$\boldsymbol{x}_d = [x_{1,d}, x_{2,d}, \ldots, x_{n,d}]^T$$

Sliding Mode Control: Scheme $(\mathbf{g}(x) = \mathbf{G}_p)$



Sliding Mode Control: Scheme ($\mathbf{g}(x) = \mathbf{G}_p$)



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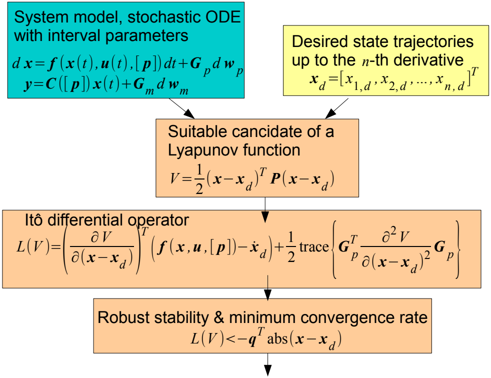
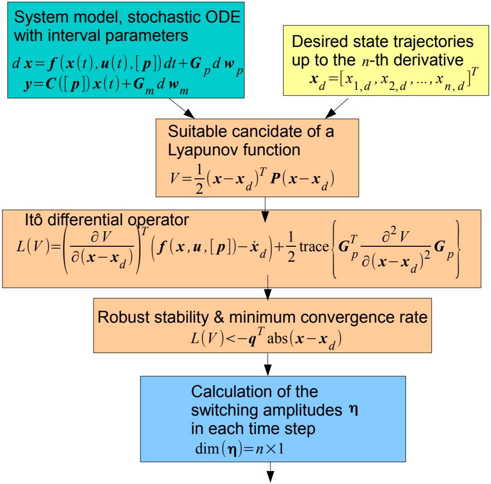
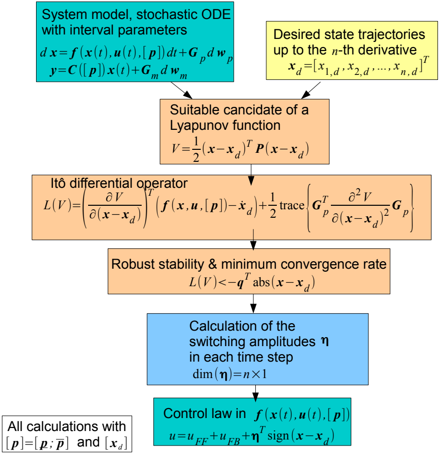


Sliding Mode Control: Scheme ($\mathbf{g}(x) = \mathbf{G}_p$)



Sliding Mode Control: Scheme ($\mathbf{g}(x) = \mathbf{G}_p$)



Derivation of the Switching Amplitude

Itô Differential Operator

$$L(V(t)) = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \tilde{\mathbf{x}}}\right)^T \cdot \left(\mathbf{f}\left([\mathbf{x}]\,(t)\,, [\mathbf{p}], \mathbf{u}(t)\right) - \dot{\mathbf{x}}_{\mathrm{d}}(t)\right) + \frac{1}{2}\text{trace}\left\{\mathbf{g}^T \frac{\partial^2 V}{\partial \tilde{\mathbf{x}}^2}\mathbf{g}\right\}$$

- Lyapunov function $V = \frac{1}{2}\tilde{\mathbf{x}}^T\mathbf{P}\tilde{\mathbf{x}}$
- Tracking error / sliding surface

$$\mathbf{s}(t) = \tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{x}_{\mathrm{d}}(t) = \begin{bmatrix} x_1(t) - x_{1,\mathrm{d}} \\ x_2(t) - x_{2,\mathrm{d}} \\ \vdots \\ x_n(t) - x_{n,\mathrm{d}} \end{bmatrix} \in \mathbb{R}^{n\times 1}$$

Derivation of the Switching Amplitude

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$$L(V(t)) = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \tilde{\mathbf{x}}}\right)^T \cdot \left(\mathbf{f}\left([\mathbf{x}]\,(t)\,,[\mathbf{p}],\mathbf{u}(t)\right) - \dot{\mathbf{x}}_{\mathrm{d}}(t)\right) + \frac{1}{2}\text{trace}\left\{\mathbf{g}^T \frac{\partial^2 V}{\partial \tilde{\mathbf{x}}^2}\mathbf{g}\right\}$$

- Stochastic dynamic system $\mathrm{d}\mathbf{x} = \mathbf{f}(\mathbf{x},\mathbf{p},\mathbf{u})\mathrm{d}t + \mathbf{g}\mathrm{d}\mathbf{w}$
- Deterministic system with parameter uncertainty and control error intervals $\mathbf{f}([\mathbf{x}](t),[\mathbf{p}],u(t)) = \mathbb{A}\cdot[\mathbf{x}](t) + \mathbb{B}\cdot u(t) + \mathbf{d}(t)$, $[\mathbf{x}](t) = \mathbf{x}(t) + [\Delta\mathbf{x}_c]$
- Interval system matrix $\mathbb{A}$
- Interval input vector $\mathbb{B}$
- External non-modeled effects $\mathbf{d}(t)$ (e.g. friction)
- Control error interval $[\Delta\mathbf{x}_c]$

Derivation of the Switching Amplitude

Itô Differential Operator

$$L(V(t)) = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \tilde{\mathbf{x}}}\right)^T \cdot \left(\mathbf{f}\left([\mathbf{x}]\,(t)\,, [\mathbf{p}], \mathbf{u}(t)\right) - \dot{\mathbf{x}}_{\mathrm{d}}(t)\right) + \frac{1}{2}\mathrm{trace}\left\{\mathbf{g}^T \frac{\partial^2 V}{\partial \tilde{\mathbf{x}}^2}\mathbf{g}\right\}$$

Extended Control Law

$$u(t) = u_{\mathrm{FF}}(t) + u_{\mathrm{FB}}(t) + u_{\mathrm{S}}(t)$$

- Feedforward control: $u_{\mathrm{FF}}(t) = S \cdot w$, $w = x_1$
- State feedback control: $u_{\mathrm{FB}}(t) = -\mathbf{k}^T \cdot \hat{\mathbf{x}}(t)$
- Switching control: $u_{\mathrm{S}}(t) = \boldsymbol{\eta}^T \cdot \mathrm{sign}(\mathbf{s}(t))$

Standard deviation of process noise $\mathbf{g} = \mathbf{G}_{\mathrm{p}}$

Derivation of the Switching Amplitude

Itô Differential Operator

$$L(V(t)) = \underbrace{\frac{\partial V}{\partial t}}_{0} + \underbrace{\left(\frac{\partial V}{\partial \tilde{\mathbf{x}}}\right)^T}_{[\tilde{\mathbf{x}}]^T(t)\cdot \mathbf{P}} \cdot \left(\mathbf{f}\left([\mathbf{x}]\,(t)\,, [\mathbf{p}], \mathbf{u}(t)\right) - \dot{\mathbf{x}}_{\mathrm{d}}(t)\right) +$$

$$\frac{1}{2}\text{trace}\left\{\mathbf{g}^T \underbrace{\frac{\partial^2 V}{\partial \tilde{\mathbf{x}}^2}}_{\mathbf{P}} \mathbf{g}\right\}$$

Calculation of the Switching Amplitude

Inserting $\mathbf{f}\left([\mathbf{x}]\,(t)\,, [\mathbf{p}], \mathbf{u}(t)\right)$ and employing a condition for a minimum convergence rate $L(V(t)) \overset{!}{<} -\mathbf{q}^T \cdot \text{abs}\left([\tilde{\mathbf{x}}](t)\right) \in \mathbb{R}^{n\times 1}$, $q_i > 0$

Derivation of the switching amplitude

3 Cases

Inserting $\mathbf{f}([\mathbf{x}](t), [\mathbf{p}], \mathbf{u}(t)$ and employing a condition for a minimum convergence rate $L(V(t)) \overset{!}{<} -\mathbf{q}^T \cdot \text{abs}([\tilde{\mathbf{x}}](t)) \in \mathbb{R}^{n\times 1}$

$$\eta_i = \begin{cases} \sup\left([\mathbf{M}]_i^+ \cdot \left(-[\dot{V}_\mathrm{a}](t) - \mathbf{q}^T \cdot \text{abs}([\tilde{\mathbf{x}}](t)) - T_\mathrm{S}\right)\right) + \epsilon, & \text{if } \sup([\mathbf{M}]_i) < 0 \\ \inf\left([\mathbf{M}]_i^+ \cdot \left(-[\dot{V}_\mathrm{a}](t) - \mathbf{q}^T \cdot \text{abs}([\tilde{\mathbf{x}}](t)) - T_\mathrm{S}\right)\right) - \epsilon, & \text{if } \inf([\mathbf{M}]_i) > 0 \\ 0, & \text{else} \end{cases}$$

- $[\mathbf{M}]^T := \mathbb{B}^T \cdot \mathbf{P} \cdot |[\tilde{\mathbf{x}}]|(t)$, left pseudo inverse
  $[\mathbf{M}]^+ = \left([\mathbf{M}]^T \cdot [\mathbf{M}]\right)^{-1} \cdot [\mathbf{M}]^T$ ($\dim([\mathbf{M}]) = n \times 1$)
- Small value guaranteeing the strict inequality ϵ

Derivation of the switching amplitude

3 Cases

Inserting $\mathbf{f}([\mathbf{x}](t), [\mathbf{p}], \mathbf{u}(t)$ and employing a condition for a minimum convergence rate $L(V(t)) \overset{!}{<} -\mathbf{q}^T \cdot \text{abs}([\tilde{\mathbf{x}}](t)) \in \mathbb{R}^{n\times 1}$

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$$|[\tilde{\mathbf{x}}]| = \begin{bmatrix} [\tilde{x}_1]\cdot\text{sign}([\tilde{x}_1]) & [\tilde{x}_1]\cdot\text{sign}([\tilde{x}_2]) & \dots & [\tilde{x}_1]\cdot\text{sign}([\tilde{x}_n]) \\ [\tilde{x}_2]\cdot\text{sign}([\tilde{x}_1]) & [\tilde{x}_2]\cdot\text{sign}([\tilde{x}_2]) & \dots & [\tilde{x}_2]\cdot\text{sign}([\tilde{x}_n]) \\ \vdots & \vdots & \ddots & \vdots \\ [\tilde{x}_n]\cdot\text{sign}([\tilde{x}_1]) & [\tilde{x}_n]\cdot\text{sign}([\tilde{x}_2]) & \dots & [\tilde{x}_n]\cdot\text{sign}([\tilde{x}_n]) \end{bmatrix}$$

Derivation of the switching amplitude

3 Cases

Inserting $\mathbf{f}([\mathbf{x}](t), [\mathbf{p}], \mathbf{u}(t)$ and employing a condition for a minimum convergence rate $L(V(t)) \overset{!}{<} -\mathbf{q}^T \cdot \text{abs}([\tilde{\mathbf{x}}](t)) \in \mathbb{R}^{n\times 1}$

$$\eta_i = \begin{cases} \sup\left([\mathbf{M}]_i^+ \cdot \left(-[\dot{V}_\mathrm{a}](t) - \mathbf{q}^T \cdot \text{abs}([\tilde{\mathbf{x}}](t)) - T_\mathrm{S}\right)\right) + \epsilon, & \text{if } \sup([\mathbf{M}]_i) < 0 \\ \inf\left([\mathbf{M}]_i^+ \cdot \left(-[\dot{V}_\mathrm{a}](t) - \mathbf{q}^T \cdot \text{abs}([\tilde{\mathbf{x}}](t)) - T_\mathrm{S}\right)\right) - \epsilon, & \text{if } \inf([\mathbf{M}]_i) > 0 \\ 0, & \text{else} \end{cases}$$

$$[\dot{V}_a] = [\tilde{\mathbf{x}}]^T \cdot \mathbf{P} \cdot (\mathbb{A} - \mathbb{B}\mathbf{k}^T) \cdot \mathbf{x} + [\tilde{\mathbf{x}}]^T \cdot \mathbf{P} \cdot \mathbb{B} \cdot u_\mathrm{FF} - [\tilde{\mathbf{x}}]^T \cdot \mathbf{P} \cdot \dot{\mathbf{x}}_d$$

Sign function $\text{sign}([\tilde{x}_i]) = \begin{cases} 1 & \text{if } \inf([\tilde{x}_i]) > 0 \\ -1 & \text{if } \sup([\tilde{x}_i]) < 0 \\ 0 & \text{else} \end{cases}$

Derivation of the switching amplitude

3 Cases

Inserting $\mathbf{f}([\mathbf{x}](t), [\mathbf{p}], \mathbf{u}(t)$ and employing a condition for a minimum convergence rate $L(V(t)) \stackrel{!}{<} -\mathbf{q}^T \cdot \text{abs}([\tilde{\mathbf{x}}](t)) \in \mathbb{R}^{n\times 1}$

$$\eta_i = \begin{cases} \sup\left([\mathbf{M}]_i^+ \cdot \left(-[\dot{V}_\text{a}](t) - \mathbf{q}^T \cdot \text{abs}([\tilde{\mathbf{x}}](t)) - T_\text{S}\right)\right) + \epsilon, & \text{if } \sup([\mathbf{M}]_i) < 0 \\ \inf\left([\mathbf{M}]_i^+ \cdot \left(-[\dot{V}_\text{a}](t) - \mathbf{q}^T \cdot \text{abs}([\tilde{\mathbf{x}}](t)) - T_\text{S}\right)\right) - \epsilon, & \text{if } \inf([\mathbf{M}]_i) > 0 \\ 0, & \text{else} \end{cases}$$

$$\text{abs}([\tilde{\mathbf{x}}](t)) = \begin{bmatrix} |[x_1](t) - x_{1,\text{d}}(t)| \\ |[x_2](t) - x_{2,\text{d}}(t)| \\ \vdots \\ |[x_n](t) - x_{n,\text{d}}(t)| \end{bmatrix}$$

Sliding Mode Approaches with Intervals

Note

- Control law depends on all system states (estimation necessary)
- One switching amplitude for each state enables smaller switching amplitudes $\Rightarrow$ as small as possible in each time step $\Rightarrow$ less violation of input range constraints
- Intervals included to consider parameter uncertainty (not exactly known / varying over time)
- Consideration of noisy processes
- Implemented using C-XSC in simulation

Sliding Mode Approaches with Intervals

For detailed information, see
Experimental and Numerical Validation of a Reliable Sliding Mode Control Strategy Considering Uncertainty with Interval Arithmetic to be published in the Mathematical Engineering Series "Variable-Structure Approaches: Analysis, Simulation, Robust Control and Estimation of Uncertain Dynamic Processes"

Simulative Results - Angle

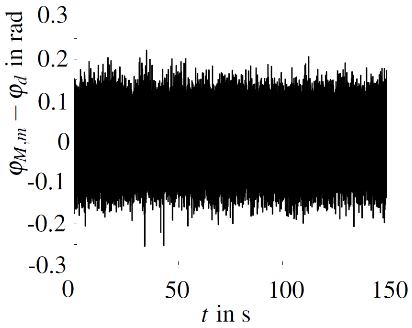


Figure : Common sliding mode control

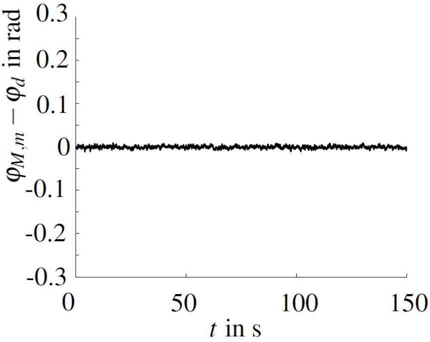


Figure : Interval sliding mode control

Simulative Results - Angular velocity

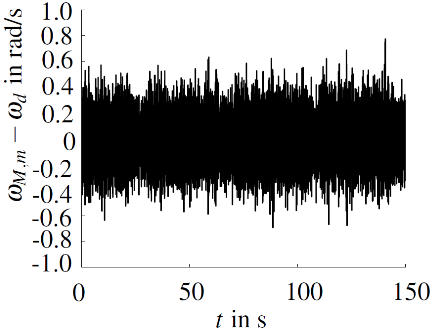


Figure : Common sliding mode control

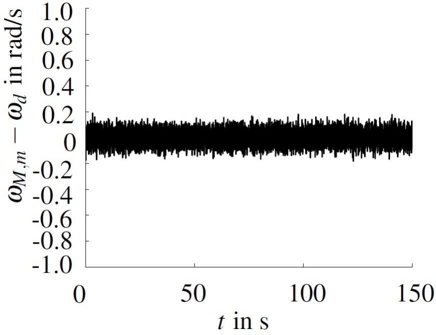


Figure : Interval sliding mode control

Experimental Results of the ISMC

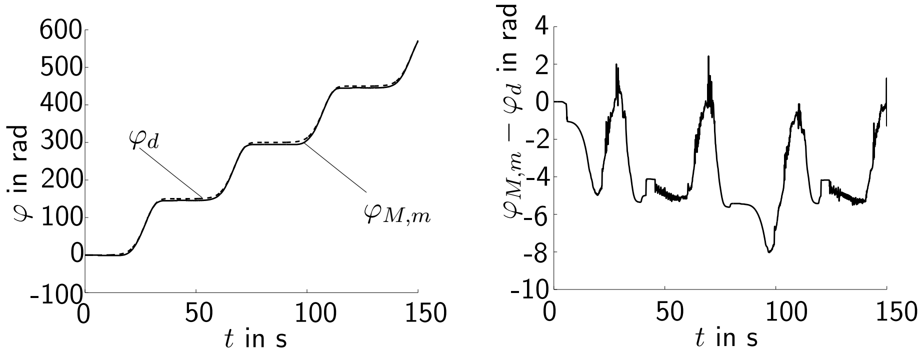
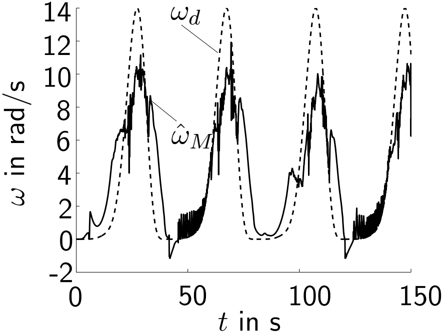


Figure : Experimental results for the desired and measured angle (left) and the corresponding deviation of both (right), Improvement 55% compared to pure state-feedback control.

Experimental Results of the ISMC



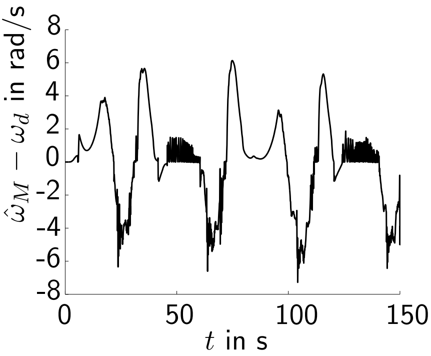


Figure : Experimental results for the desired and estimated angular velocity (left) and the corresponding deviation of both (right).

State and Parameter Estimation

Aim

- Reconstruction of non-measurable system states and uncertain parameters (specified by range bounds) such that the system is guaranteed to be stable
- Use these estimations in the controller

How?

- Simultaneous evaluation of a parallel model additionally to the system/test-rig and taking into account the deviation between the measured data and the output of the parallel model

Sliding Mode Techniques for State and Parameter Estimation

ODEs of a Dynamic System

$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{p}, \mathbf{u}(t)\right) = \mathbf{A} \cdot \mathbf{x}\left(t\right) + \mathbf{B} \cdot \mathbf{u}(t) + \mathbf{S} \cdot \xi\left(\mathbf{x}(t), \mathbf{u}(t)\right)$
$\mathbf{y}(t) = \mathbf{C} \cdot \mathbf{x}\left(t\right)$

- Nominal expressions of system, input and output matrices $\mathbf{A} := \mathbf{A}(\mathbf{x}(t), \mathbf{p}) \in [\mathbb{A}]$, $\mathbf{B} := \mathbf{B}(\mathbf{x}(t), \mathbf{p}) \in [\mathbb{B}]$ and $\mathbf{C} := \mathbf{C}(\mathbf{x}(t), \mathbf{p}) \in [\mathbb{C}]$ (included in interval expressions)
- State vector $\mathbf{x}(t)$
- Uncertain/bounded parameters $\mathbf{p}(t) \in [\mathbf{p}]$
- Input vector $\mathbf{u}(t)$
- Representation of a-priori unknown and nonlinear terms $\mathbf{S} \cdot \xi\left(\mathbf{x}(t), \mathbf{u}(t)\right)$ with $\mathbf{S} \in \mathbb{R}^{n \times q}$ and $\|\xi\left(\mathbf{x}, \mathbf{u}\right)\| \leq \overline{\xi}$ (fixed upper bound of the vector norm $\overline{\xi}$)

Sliding Mode Techniques for State and Parameter Estimation

Sliding Mode Observer ODEs Considering Uncertainty $\hat{\tilde{\mathbf{f}}} := \hat{\tilde{\mathbf{f}}}(\hat{\mathbf{x}}(t), [\mathbf{p}], \mathbf{u}(t))$

$$\hat{\tilde{\mathbf{f}}} = \hat{\mathbf{f}}(\hat{\mathbf{x}}(t), [\mathbf{p}], \mathbf{u}(t)) + \mathbf{P}^{+}[\hat{\mathbb{C}}]^{T} \cdot \mathbf{H}_{s} \cdot \mathrm{sign}(\mathbf{e}_{m} + [\Delta \mathbf{y}_{m}])$$

$$:= [\hat{\mathbb{A}}] \cdot \hat{\mathbf{x}}(t) + [\hat{\mathbb{B}}] \cdot \mathbf{u}(t) + \mathbf{H}_{p} \cdot [\mathbf{e}_{m}] + \mathbf{P}^{+}[\hat{\mathbb{C}}]^{T} \cdot \mathbf{H}_{s}\, \mathrm{sign}(\mathbf{e}_{m} + [\Delta \mathbf{y}_{m}])$$

$$\hat{\mathbf{y}}_{m} := [\hat{\mathbb{C}}] \cdot \hat{\mathbf{x}}(t)$$

- Combination of locally valid linear system model and variable structure part that handles uncertainty and nonlinearities to stabilize the error dynamics with certainty
- Uncertainty in parameters and measurements → interval arithmetic

L. Senkel et al: *Interval-Based Sliding Mode Observer Design for Nonlinear Systems with Bounded Measurement and Parameter Uncertainty*, IEEE Intl. Conference on Methods and Models in Automation and Robotics MMAR 2013, Miedzyzdroje, Poland, 2013.

Sliding Mode Techniques for State and Parameter Estimation

Sliding Mode Observer ODEs Considering Uncertainty

$$\hat{\tilde{\mathbf{f}}} := \hat{\tilde{\mathbf{f}}}(\hat{\mathbf{x}}(t), [\mathbf{p}], \mathbf{u}(t))$$

$$\begin{aligned}\hat{\tilde{\mathbf{f}}} =& \hat{\mathbf{f}}(\hat{\mathbf{x}}(t), [\mathbf{p}], \mathbf{u}(t)) + \mathbf{P}^{+}[\hat{\mathbb{C}}]^{T} \cdot \mathbf{H}_s \cdot \mathrm{sign}(\mathbf{e}_m + [\Delta \mathbf{y}_m]) \\ :=& [\hat{\mathbb{A}}] \cdot \hat{\mathbf{x}}(t) + [\hat{\mathbb{B}}] \cdot \mathbf{u}(t) + \mathbf{H}_p \cdot [\mathbf{e}_m] + \mathbf{P}^{+}[\hat{\mathbb{C}}]^{T} \cdot \mathbf{H}_s \ \mathrm{sign}(\mathbf{e}_m + [\Delta \mathbf{y}_m])\end{aligned}$$

$$\hat{\mathbf{y}}_m := [\hat{\mathbb{C}}] \cdot \hat{\mathbf{x}}(t)$$

- Instead of nominal system, input and output matrices $\rightarrow$ interval matrices $[\hat{\mathbb{A}}]$, $[\hat{\mathbb{B}}]$ and $[\hat{\mathbb{C}}]$ denoting the interval evaluations of $\hat{\mathbf{A}}(\hat{\mathbf{x}}(t), [\mathbf{p}]) \in [\hat{\mathbb{A}}]$, $\hat{\mathbf{B}}(\hat{\mathbf{x}}(t), [\mathbf{p}]) \in [\hat{\mathbb{B}}]$ and $\hat{\mathbf{C}}(\hat{\mathbf{x}}(t), [\mathbf{p}]) \in [\hat{\mathbb{C}}]$
- Measurement error vector $\mathbf{e}_m(t) \in [\mathbf{e}_m] = \mathbf{y}_m - \hat{\mathbf{y}}_m + [\Delta \mathbf{y}_m]$

Sliding Mode Techniques for State and Parameter Estimation

Sliding Mode Observer ODEs Considering Uncertainty

$\hat{\tilde{\mathbf{f}}} := \hat{\tilde{\mathbf{f}}}(\hat{\mathbf{x}}(t), [\mathbf{p}], \mathbf{u}(t))$

$$\begin{aligned}\hat{\tilde{\mathbf{f}}} =& \hat{\mathbf{f}}(\mathbf{x}(t), [\mathbf{p}], \mathbf{u}(t)) + \mathbf{P}^{+}[\hat{\mathbb{C}}]^{T} \cdot \mathbf{H}_s \cdot \mathrm{sign}(\mathbf{e}_m + [\Delta \mathbf{y}_m]) \\ :=& [\hat{\mathbb{A}}] \cdot \hat{\mathbf{x}}(t) + [\hat{\mathbb{B}}] \cdot \mathbf{u}(t) + \mathbf{H}_p \cdot [\mathbf{e}_m] + \mathbf{P}^{+}[\hat{\mathbb{C}}]^{T} \cdot \mathbf{H}_s \ \mathrm{sign}(\mathbf{e}_m + [\Delta \mathbf{y}_m])\end{aligned}$$

$$\hat{\mathbf{y}}_m := [\hat{\mathbb{C}}] \cdot \hat{\mathbf{x}}(t)$$

- Underlying stabilization of the error dynamics observer gain matrix $\mathbf{H}_p$
- Matrix $\mathbf{P}$ results from solving the Lyapunov equation $\tilde{\mathbf{A}} \cdot \mathbf{P} + \mathbf{P} \cdot \tilde{\mathbf{A}}^T + \mathbf{Q} = \mathbf{0}$ with $\tilde{\mathbf{A}} = \hat{\mathbf{A}} - \mathbf{H}_p \cdot \hat{\mathbf{C}}$ and $\mathbf{Q} > 0$
- Online evaluation of the switching amplitudes $\mathbf{H}_s = \mathrm{diag}(\mathbf{h}_s)$ in each time step to handle uncertainty

Sliding Mode Techniques for State and Parameter Estimation

Itô Differential Operator for Consideration of Stochastic Disturbances

- Suitable candidate of a Lyapunov function $V(t) = \frac{1}{2}(\mathbf{x} - \hat{\mathbf{x}})^T \mathbf{P} (\mathbf{x} - \hat{\mathbf{x}})$
- Ensure the system's stability for reliable estimations with $\dot{V}$
- System affected by stochastic processes $\Rightarrow \dot{V}$ by using Itô differential operator

$$L(V(t)) = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \mathbf{e}}\right)^T \cdot \Big(\mathbf{f}(\mathbf{x}, [\mathbf{p}], \mathbf{u}) - \hat{\tilde{\mathbf{f}}}(\hat{\mathbf{x}}, [\mathbf{p}], \mathbf{u})\Big) + \frac{1}{2}\text{trace}\left\{\mathbf{G}^T \frac{\partial^2 V}{\partial \mathbf{e}^2} \mathbf{G}\right\}$$

Sliding Mode Techniques for State and Parameter Estimation

Itô Differential Operator for Consideration of Stochastic Disturbances

$$L(V(t)) = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \mathbf{e}}\right)^T \cdot \Big(\mathbf{f}(\mathbf{x}, [\mathbf{p}], \mathbf{u}) - \hat{\tilde{\mathbf{f}}}(\hat{\mathbf{x}}, [\mathbf{p}], \mathbf{u})\Big) + \frac{1}{2}\text{trace}\left\{\mathbf{G}^T \frac{\partial^2 V}{\partial \mathbf{e}^2}\mathbf{G}\right\}$$

- System $\mathbf{f}(\mathbf{x}, [\mathbf{p}], \mathbf{u})$ and observer parallel model $\hat{\tilde{\mathbf{f}}}(\hat{\mathbf{x}}, [\mathbf{p}], \mathbf{u})$
- Estimation error $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$
- Standard deviation of process and measurement noise $\mathbf{G} = [\mathbf{G}_p \quad -\mathbf{H}_p\mathbf{G}_m]$ to simulate neglected nonlinear phenomena or inaccurate sensor measurements

Sliding Mode Techniques for State and Parameter Estimation

Calculation of the Switching Amplitude

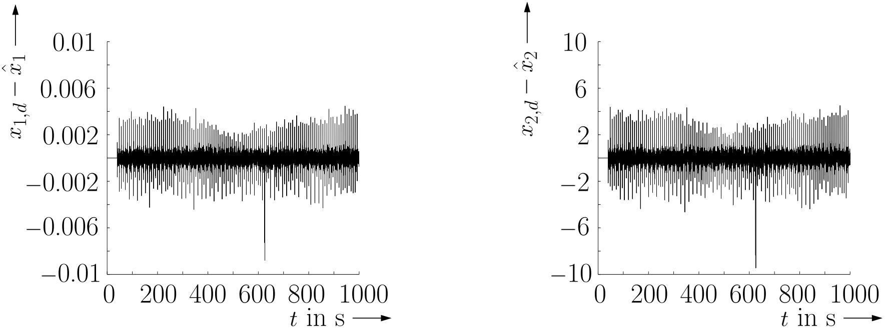
- Solving $L(V(t)) \stackrel{!}{<} -\mathbf{q}^T \,|[\mathbf{e}_m]|$ with respect to $\mathbf{H}_s$, $q_i > 0$

Cascaded structure due to multiplicative coupling of states and parameters to be estimated

For further details see
Sliding Mode Approaches Considering Uncertainty for Reliable Control and Computation of Confidence Regions in State and Parameter Estimation
presented at 16th GAMM-IMACS International Symposium on Scientifc Computing, Computer Arithmetic and Validated Numerics, SCAN 2014, Würzburg Germany

Simulative and Experimental Validation (ISMO)

State Estimation



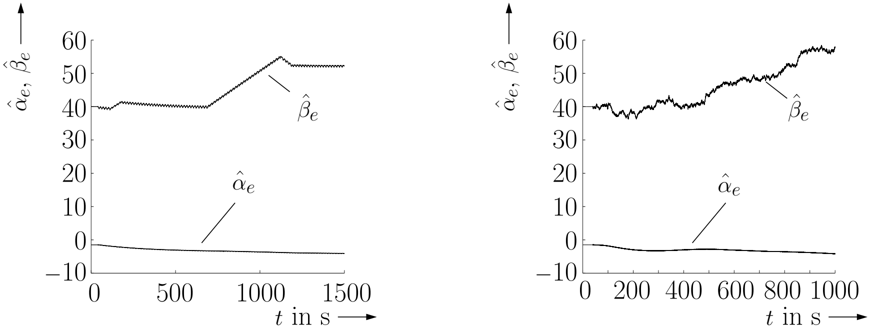
(a) Estimation error x_1.

(b) Estimation error x_2.

Figure : Experimental results

Simulative and Experimental Validation (ISMO)

Parameter Estimation $\alpha = [-1.5, 4.5]$, $\beta = [30, 90]$



(a) Estimates $\hat{\alpha}_e$, $\hat{\beta}_e$.

(b) Estimates $\hat{\alpha}_e$, $\hat{\beta}_e$.

Figure : Simulative (a) and experimental results (b)

Simulative and Experimental Validation

Comparison

- One driving cycle (acceleration, deceleration) takes 8 sec, repeated periodically
- ISMO: adaptation of parameters and states takes place at each discretization step
- Least Squares Estimation (LSE): adaptation of parameters once per driving cycle
- Comparison of root-mean square errors for angle and velocity

	LSE	ISMO	Improvement
x_1	$\Delta_{x_1,LS} = 2730$	$\Delta_{x_1,ISMO} = 261.36$	90.43%
x_2	$\Delta_{x_2,LS} = 4.79$	$\Delta_{x_2,ISMO} = 4.51$	5.85%

Comparison of Control and Observer

	ISMC	ISMO
Sliding surface	Tracking error $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_d$	Estimation error $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$
Lyapunov function	$V = \frac{1}{2}\tilde{\mathbf{x}}^T \mathbf{P}_C \tilde{\mathbf{x}}$	$V = \frac{1}{2}\mathbf{e}^T \mathbf{P}_O \mathbf{e}$
Switching amplitude	$\boldsymbol{\eta} \in \mathbb{R}^{n \times 1}$	$\boldsymbol{h}_s \in \mathbb{R}^{n_y \times 1}$
Bounded uncertainty	$[\mathbf{p}], [\Delta \mathbf{x}_c]$	$[\mathbf{p}], [\Delta \mathbf{x}_e], [\Delta \mathbf{x}_m]$
Stochastic uncertainty	process noise	process noise measurement noise
Condition	$L(V) \overset{!}{<} \mathbf{q}_C \cdot \text{abs}([\tilde{\mathbf{x}}])$	$L(V) \overset{!}{<} \mathbf{q}_O \cdot \lvert[\mathbf{e}_m]\rvert$
Aim	$\mathbf{x} \overset{!}{=} \mathbf{x}_d$	$\hat{\mathbf{x}} \overset{!}{=} \mathbf{x}$

Conclusions and Outlook

Conclusions

- Interval sliding mode control and observer
- Simultaneous identification of unknown system parameters and estimation of system state
- Consideration of bounded and stochastic disturbances
- Validation in simulation and in experiment
- Systematic computation of variable structure gains

Outlook on Further Work

- Simultaneous implementation of control and observer on the test-rig
- Experimental validations of these Sliding Mode Approaches for other real-time applications

Thank you for your attention!