

# Reliable Sliding Mode Approaches for the Temperature Control of Solid Oxide Fuel Cells with Input and Input Rate Constraints

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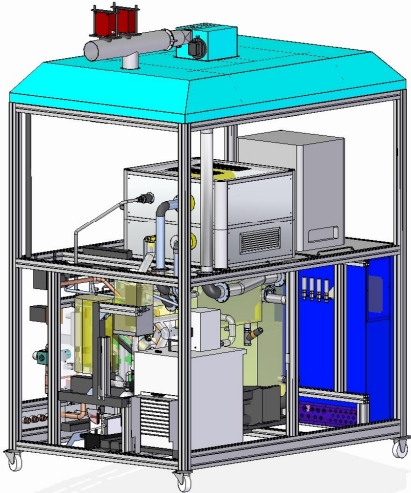
# Contents

- Control-oriented modeling of high-temperature Solid Oxide Fuel Cell Systems (SOFC systems)
- Focus on the non-stationary thermal behavior of SOFC stack modules
- Verified (global) parameter identification
- Structural analysis of the control-oriented model: Transformation into nonlinear controller canonical form
- Interval-based sliding mode control for a dynamic system model with bounded uncertainty and disturbances
- Handling of state and actuator constraints
- Numerical validation
- Conclusions and outlook on future work

# Control-Oriented Modeling of SOFC Systems (1)

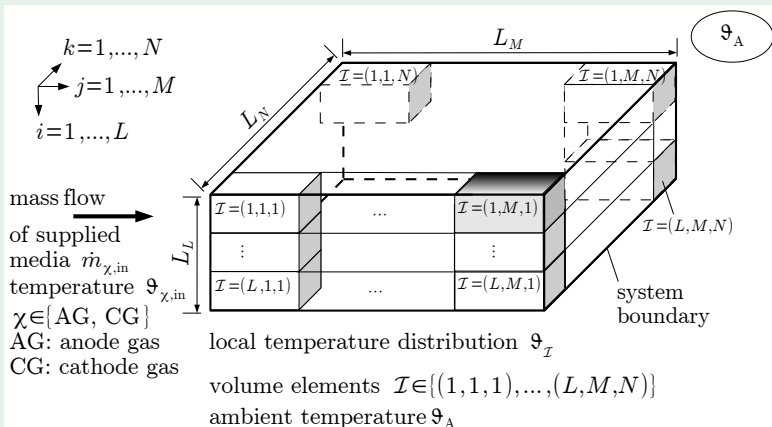
## Configuration of the SOFC test rig at the Chair of Mechatronics

- Supply of fuel gas (hydrogen and/or mixture of methane, carbon monoxide, water vapor)
- Supply of air
- Independent preheaters for fuel gas and air
- Stack module containing fuel cells in electric series connection
- Electric load as disturbance



## Control-Oriented Modeling of SOFC Systems (2)

### Spatial semi-discretization of the fuel cell stack module

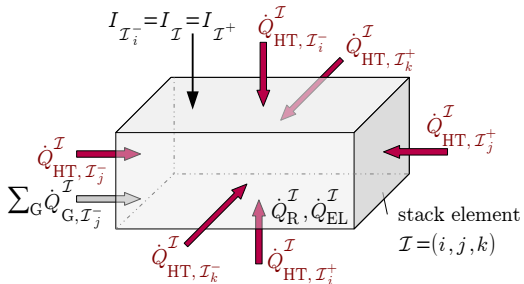


## Control-Oriented Modeling of SOFC Systems (3)

Mathematical representation of the piecewise homogeneous temperature distribution  $\implies$  spatial semi-discretization

$$\dot{\vartheta}_{\mathcal{I}}(t) = \frac{1}{c_{\mathcal{I}}m_{\mathcal{I}}} \left( \dot{Q}_{\text{HT}}^{\mathcal{I}}(t) + \sum_{G \in \{\text{AG}, \text{CG}\}} \dot{Q}_{G, \mathcal{I}_j^-}^{\mathcal{I}}(t) + \dot{Q}_{\text{R}}^{\mathcal{I}}(t) + \dot{Q}_{\text{EL}}^{\mathcal{I}}(t) \right)$$

- 1 HT: Heat transfer (heat conduction and convection)
- 2 G: Enthalpy flows of supplied gases
- 3 R: Exothermic reaction enthalpy
- 4 EL: Ohmic losses

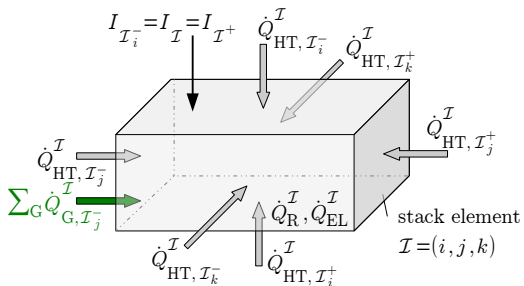


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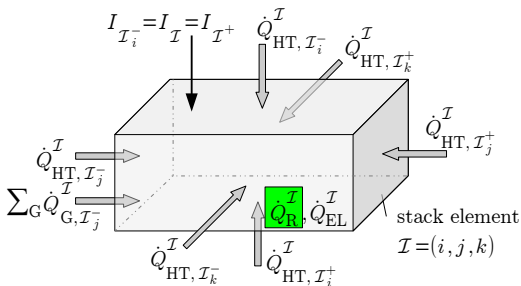


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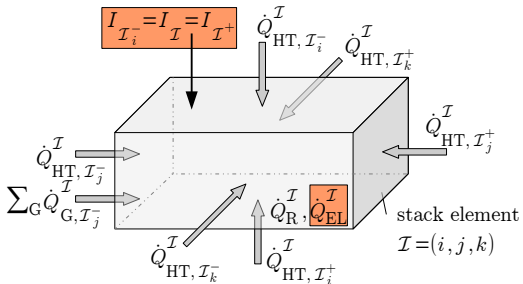


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## Control-Oriented Modeling of SOFC Systems (4)

### Heat transfer due to heat conduction and convection

$$\begin{aligned}\dot{Q}_{\text{HT}}^{\mathcal{I}}(t) &= \dot{Q}_{\text{HT},\mathcal{I}_i^-}^{\mathcal{I}}(t) + \dot{Q}_{\text{HT},\mathcal{I}_i^+}^{\mathcal{I}}(t) + \dot{Q}_{\text{HT},\mathcal{I}_j^-}^{\mathcal{I}}(t) \\ &\quad + \dot{Q}_{\text{HT},\mathcal{I}_j^+}^{\mathcal{I}}(t) + \dot{Q}_{\text{HT},\mathcal{I}_k^-}^{\mathcal{I}}(t) + \dot{Q}_{\text{HT},\mathcal{I}_k^+}^{\mathcal{I}}(t)\end{aligned}$$

### Unified modeling of heat conduction and convection

- Heat flows over each finite volume element boundary

$$\dot{Q}_{\text{HT},\mathcal{J}}^{\mathcal{I}}(t) = \beta_{\mathcal{J}}^{\mathcal{I}} \cdot (\vartheta_{\mathcal{J}}(t) - \vartheta_{\mathcal{I}}(t)) \quad , \quad \mathcal{J} \in \{\mathcal{I}_i^-, \mathcal{I}_i^+, \mathcal{I}_j^-, \mathcal{I}_j^+, \mathcal{I}_k^-, \mathcal{I}_k^+\}$$

- Set of indices (neighboring finite volume elements):

$$\begin{aligned}\mathcal{I}_i^- &:= (i-1, j, k), \quad \mathcal{I}_i^+ := (i+1, j, k), \quad \mathcal{I}_j^- := (i, j-1, k), \\ \mathcal{I}_j^+ &:= (i, j+1, k), \quad \mathcal{I}_k^- := (i, j, k-1), \quad \mathcal{I}_k^+ := (i, j, k+1)\end{aligned}$$

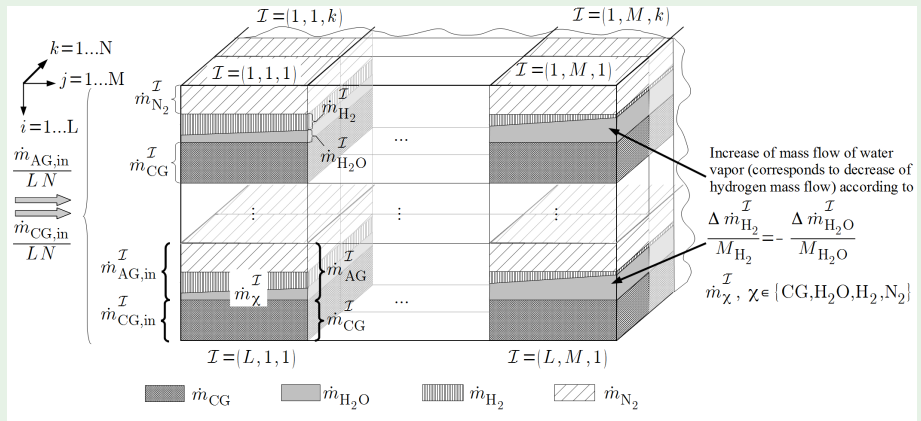
## Control-Oriented Modeling of SOFC Systems (5)

Definition of the experimentally identified parameters  $\lambda^{(\cdot)}$ ,  $\alpha^{(\cdot)}$ ;  
 Boundary elements represent radiation in linearized form

$$\beta_{\mathcal{J}}^{\mathcal{I}} = \begin{cases} \text{heat conduction} \\ \lambda^{(i)} \cdot \frac{l_N l_M}{l_L} & \text{for } \mathcal{J} = \mathcal{I}_i^-, i \geq 2 \text{ or } \mathcal{J} = \mathcal{I}_i^+, i \leq L - 1 \\ \lambda^{(j)} \cdot \frac{l_L l_N}{l_M} & \text{for } \mathcal{J} = \mathcal{I}_j^-, j \geq 2 \text{ or } \mathcal{J} = \mathcal{I}_j^+, j \leq M - 1 \\ \lambda^{(k)} \cdot \frac{l_L l_M}{l_N} & \text{for } \mathcal{J} = \mathcal{I}_k^-, k \geq 2 \text{ or } \mathcal{J} = \mathcal{I}_k^+, k \leq N - 1 \\ \text{convection} \\ \alpha^{(i)} \cdot l_N l_M & \text{for } \mathcal{J} = \mathcal{I}_i^-, i = 1 \text{ or } \mathcal{J} = \mathcal{I}_i^+, i = L \\ \alpha^{(j)} \cdot l_L l_N & \text{for } \mathcal{J} = \mathcal{I}_j^-, j = 1 \text{ or } \mathcal{J} = \mathcal{I}_j^+, j = M \\ \alpha^{(k)} \cdot l_L l_M & \text{for } \mathcal{J} = \mathcal{I}_k^-, k = 1 \text{ or } \mathcal{J} = \mathcal{I}_k^+, k = N \end{cases}$$

# Control-Oriented Modeling of SOFC Systems (6)

## Local mass flow balances in the semi-discretized fuel cell stack module



Anode gas composition:  $\dot{m}_{AG,in}(t) = \dot{m}_{H_2,in}(t) + \dot{m}_{N_2,in}(t) + \dot{m}_{H_2O,in}(t)$

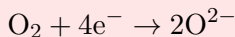
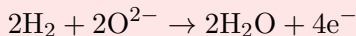
## Control-Oriented Modeling of SOFC Systems (7)

### Modeling of local gas mass flows at the element outlets

- Local mass flows for each gas fraction  $\chi \in \{\text{H}_2, \text{N}_2, \text{H}_2\text{O}, \text{CG}\}$

$$\dot{m}_{\chi}^{\mathcal{I}}(t) = \begin{cases} \frac{\dot{m}_{\chi, \text{in}}(t)}{L \cdot N} + \Delta \dot{m}_{\chi}^{\mathcal{I}}(t) & \text{for } j = 1 \\ \dot{m}_{\chi}^{\mathcal{I}^-}(t) + \Delta \dot{m}_{\chi}^{\mathcal{I}}(t) & \text{for } 2 \leq j \leq M \end{cases}$$

- Electrochemical reaction with the Faraday constant  $F$  and the number of electrons  $z = 4$



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- Variation of the anode gas mass flow

$$\Delta \dot{m}_{\text{H}_2}^{\mathcal{I}}(t) = -\frac{I_{\mathcal{I}}(t) \cdot M_{\text{H}_2}}{z \cdot F}, \quad \Delta \dot{m}_{\text{H}_2\text{O}}^{\mathcal{I}}(t) = +\frac{I_{\mathcal{I}}(t) \cdot M_{\text{H}_2\text{O}}}{z \cdot F}, \quad \Delta \dot{m}_{\text{N}_2}^{\mathcal{I}}(t) = 0$$

- Variation of the cathode gas mass flow

$$\Delta \dot{m}_{\text{CG}}^{\mathcal{I}}(t) = -\frac{I_{\mathcal{I}}(t) \cdot M_{\text{O}_2}}{z \cdot F}$$

## Control-Oriented Modeling of SOFC Systems (8)

### Heat flows due to the exothermic electrochemical reaction

- Reaction enthalpy (local molar flow of hydrogen  $\Delta\dot{m}_{\text{H}_2}^{\mathcal{I}}(t)/M_{\text{H}_2}$ )

$$\dot{Q}_{\text{R}}^{\mathcal{I}}(t) = -\frac{H_{\text{R}}(\vartheta_{\mathcal{I}}(t)) \cdot \Delta\dot{m}_{\text{H}_2}^{\mathcal{I}}(t)}{M_{\text{H}_2}} = \frac{H_{\text{R}}(\vartheta_{\mathcal{I}}(t)) \cdot I_{\mathcal{I}}(t)}{z \cdot F}$$

- Approximation of the temperature-dependent molar reaction enthalpy by a second-order polynomial

$$H_{\text{R}}(\vartheta_{\mathcal{I}}(t)) = \sum_{\nu=0}^2 \gamma_{\text{H}_{\text{R}},\nu} \cdot \vartheta_{\mathcal{I}}^{\nu}(t) \quad , \quad \gamma_{\text{H}_{\text{R}},\nu} : \text{experimentally identified}$$

- Heat flow due to Ohmic losses  $\dot{Q}_{\text{EL}}^{\mathcal{I}}(t) = R_{\text{EL},\mathcal{I}} \cdot I_{\mathcal{I}}^2(t)$

## Control-Oriented Modeling of SOFC Systems (9)

### Enthalpy flows of the supplied gases

- General expression for enthalpy flows

$$\sum_G \dot{Q}_{G, \mathcal{I}_j^-}^{\mathcal{I}}(t) := \sum_{G \in \{AG, CG\}} \dot{Q}_{G, \mathcal{I}_j^-}^{\mathcal{I}}(t) = \dot{Q}_{AG, \mathcal{I}_j^-}^{\mathcal{I}}(t) + \dot{Q}_{CG, \mathcal{I}_j^-}^{\mathcal{I}}(t)$$

- Separate notation of anode and cathode gas

$$\dot{Q}_{AG, \mathcal{I}_j^-}^{\mathcal{I}}(t) = C_{AG, \mathcal{I}}(\vartheta_{\mathcal{I}}, t) \cdot \Delta \vartheta_{AG, \mathcal{I}_j^-}^{\mathcal{I}}(t) \quad \text{and}$$

$$\dot{Q}_{CG, \mathcal{I}_j^-}^{\mathcal{I}}(t) = C_{CG, \mathcal{I}}(\vartheta_{\mathcal{I}}, t) \cdot \Delta \vartheta_{CG, \mathcal{I}_j^-}^{\mathcal{I}}(t)$$

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$$\dot{Q}_{CG, \mathcal{I}_j^-}^{\mathcal{I}}(t) = C_{CG, \mathcal{I}}(\vartheta_{\mathcal{I}}, t) \cdot \Delta\vartheta_{CG, \mathcal{I}_j^-}^{\mathcal{I}}(t)$$

- Temperature differences

$$\Delta\vartheta_{AG, \mathcal{I}_j^-}^{\mathcal{I}}(t) = \begin{cases} \vartheta_{AG, \text{in}}(t) - \vartheta_{(i,1,k)}(t) & \text{for } j = 1 \\ \vartheta_{\mathcal{I}_j^-}(t) - \vartheta_{\mathcal{I}}(t) & \text{for } j \in \{2, \dots, M\} \end{cases}$$

and

$$\Delta\vartheta_{CG, \mathcal{I}_j^-}^{\mathcal{I}}(t) = \begin{cases} \vartheta_{CG, \text{in}}(t) - \vartheta_{(i,1,k)}(t) & \text{for } j = 1 \\ \vartheta_{\mathcal{I}_j^-}(t) - \vartheta_{\mathcal{I}}(t) & \text{for } j \in \{2, \dots, M\} \end{cases}$$



## Control-Oriented Modeling of SOFC Systems (10)

### Enthalpy flows of the supplied gases (cont'd)

- Heat capacity of the anode gas mixture

$$C_{AG,\mathcal{I}}(\vartheta_{\mathcal{I}}, t) = c_{\text{H}_2}(\vartheta_{\mathcal{I}}(t)) \cdot \dot{m}_{\text{H}_2}^{\mathcal{I}_j^-}(t) + c_{\text{N}_2}(\vartheta_{\mathcal{I}}(t)) \cdot \dot{m}_{\text{N}_2}^{\mathcal{I}_j^-}(t) \\ + c_{\text{H}_2\text{O}}(\vartheta_{\mathcal{I}}(t)) \cdot \dot{m}_{\text{H}_2\text{O}}^{\mathcal{I}_j^-}(t)$$

with  $\dot{m}_{\text{H}_2}^{\mathcal{I}_j^-}(t)$ ,  $\dot{m}_{\text{N}_2}^{\mathcal{I}_j^-}(t)$ , and  $\dot{m}_{\text{H}_2\text{O}}^{\mathcal{I}_j^-}(t)$  as the inflows of hydrogen, nitrogen, and water vapor in the volume element  $\mathcal{I}$

- Heat capacity of cathode gas

$$C_{CG,\mathcal{I}}(\vartheta_{\mathcal{I}}, t) = c_{CG}(\vartheta_{\mathcal{I}}(t)) \cdot \dot{m}_{CG}^{\mathcal{I}_j^-}(t)$$

with the cathode gas inflow  $\dot{m}_{CG}^{\mathcal{I}_j^-}(t)$  into the volume element  $\mathcal{I}$

# Control-Oriented Modeling of SOFC Systems (11)

## Enthalpy flows of the supplied gases (cont'd)

- Heat capacities

$$C_{AG,I}(\vartheta_I, t) = c_{H_2}(\vartheta_I(t)) \cdot \dot{m}_{H_2}^{I_j^-}(t) + c_{N_2}(\vartheta_I(t)) \cdot \dot{m}_{N_2}^{I_j^-}(t) \\ + c_{H_2O}(\vartheta_I(t)) \cdot \dot{m}_{H_2O}^{I_j^-}(t) \quad \text{and}$$

$$C_{CG,I}(\vartheta_I, t) = c_{CG}(\vartheta_I(t)) \cdot \dot{m}_{CG}^{I_j^-}(t)$$

- Approximation of specific heat capacities  $c_\chi(\vartheta_I(t))$  for each gas fraction  $\chi \in \{H_2, N_2, H_2O, CG\}$  by second-order polynomials

$$c_\chi(\vartheta_I(t)) = \sum_{\nu=0}^2 \gamma_{\chi,\nu} \cdot \vartheta_I^\nu(t) \quad , \quad \gamma_{\chi,\nu} : \text{experimentally identified}$$

# Control-Oriented Modeling of SOFC Systems: Summary

## Modeling assumptions/ Model properties

- Capability to represent time-varying hotspot locations

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- Representation of the reaction enthalpy by a second-order temperature-dependent polynomial

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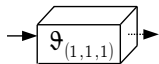
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- Possible extension: Inclusion of the preheater dynamics by linear lag elements to account for underlying control time constants

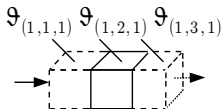
## Different Variants of the Finite Volume Model (1)

(I)



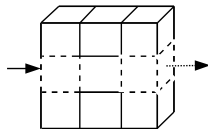
$$\mathbf{x}_{FC} = \vartheta_{(1,1,1)}$$

(II)



$$\mathbf{x}_{FC}^T = [\vartheta_{(1,1,1)}, \vartheta_{(1,2,1)}, \vartheta_{(1,3,1)}]$$

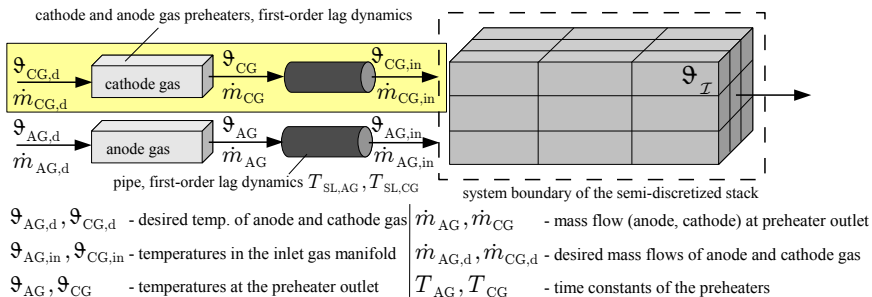
(III)



$$\mathbf{x}_{FC}^T = [\vartheta_{(1,1,1)}, \dots, \vartheta_{(3,3,1)}]$$

- Configuration (I): Typical for synthesizing a controller that is only applied during the system's heating phase
- Configuration (II): Simplest option for preventing local overtemperatures: Differentially flat or non-flat scenarios, depending on the choice of the system output  $\vartheta_{\mathcal{I}^*}$
- Configuration (III): Generally non-flat configuration

## Different Variants of the Finite Volume Model (2)

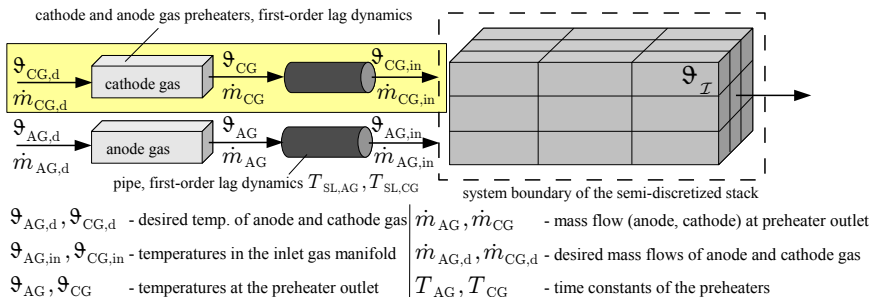


System input: Cathode gas enthalpy flow (single-input single-output formulation) in configuration (II), preheater dynamics neglected

$$v_{CG,in}(t) = \dot{m}_{CG,in}(t) \cdot (\vartheta_{CG,in}(t) - \vartheta_{(1,1,1)}(t))$$



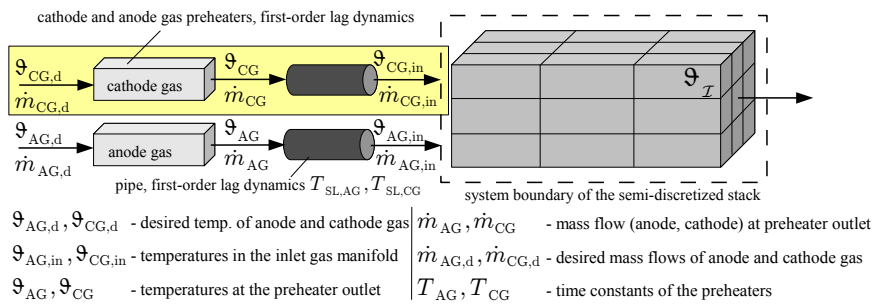
## Different Variants of the Finite Volume Model (2)



System input: Cathode gas enthalpy flow (single-input single-output formulation) in configuration (II), preheater dynamics included

$$v_{CG,d}(t) = \dot{m}_{CG,d}(t) \cdot (\vartheta_{CG,d}(t) - \vartheta_{(1,1,1)}(t)) \quad , \quad \dot{m}_{CG,d}(t) \approx \dot{m}_{CG,in}(t)$$

## Different Variants of the Finite Volume Model (2)



Vector representation of the input (multi-input single-output formulation)

$$\mathbf{u}_{CG,d}(t) = \begin{bmatrix} \dot{m}_{CG,d}(t) \\ \vartheta_{CG,d}(t) - \vartheta_{(1,1,1)}(t) \end{bmatrix} =: \begin{bmatrix} \dot{m}_{CG,d}(t) \\ \Delta\vartheta_{CG}(t) \end{bmatrix}$$

# Transformation into Nonlinear Controller Normal Form (1)

## Input-affine state-space representation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p}, \mathbf{v}_{\text{CG,d}}(t), \mathbf{v}_{\text{AG,d}}(t))$$

## Computation of Lie derivatives of the system output

$$y(t) = h(\mathbf{x}(t)) = \vartheta_{\mathcal{I}^*}, \mathbf{x}(t) \in \mathbb{R}^{\mathcal{N}}$$

$$\frac{d^r y(t)}{dt^r} = y^{(r)}(t) = L_{\mathbf{f}}^r h(\mathbf{x}(t)) = L_{\mathbf{f}}(L_{\mathbf{f}}^{r-1} h(\mathbf{x}(t))) \quad , \quad r = 1, \dots, \delta - 1$$

with the relative degree  $\delta$  defined according to

$$\frac{\partial L_{\mathbf{f}}^r h(\mathbf{x}(t))}{\partial \mathbf{v}_{\text{CG,d}}} \equiv 0 \quad \text{for } r = 0, \dots, \delta - 1 \quad \text{and} \quad \frac{\partial L_{\mathbf{f}}^{\delta} h(\mathbf{x}(t))}{\partial \mathbf{v}_{\text{CG,d}}} \neq 0$$

## Transformation into Nonlinear Controller Normal Form (2)

### Introduction of the new state vector

$$\xi = [h(\mathbf{x}), L_f h(\mathbf{x}), \dots, L_f^{\delta-1} h(\mathbf{x})]^T \in \mathbb{R}^\delta \quad \text{with} \quad \xi_1 = y = h(\mathbf{x})$$

### New set of state equations (Brunovsky canonical form)

$$\begin{aligned} \begin{bmatrix} \dot{\xi}^T \\ \dot{\zeta}^T \end{bmatrix} &= \begin{bmatrix} L_f h(\mathbf{x}), \dots, L_f^\delta h(\mathbf{x}) \\ L_f^{\delta+1} h(\mathbf{x}), \dots, L_f^N h(\mathbf{x}) \end{bmatrix}^T \\ &= \begin{bmatrix} \xi_2, \dots, \xi_\delta, \tilde{a}(\mathbf{x}, \mathbf{p}, d) \\ \mathbf{a}^\diamond(\mathbf{x}, \mathbf{p}, d)^T \end{bmatrix}^T \\ &\quad + \begin{bmatrix} 0, \dots, \tilde{b}(\mathbf{x}, \mathbf{p}) \cdot v_{CG,d} \\ \mathbf{b}^\diamond(\mathbf{x}, \mathbf{p}, d, v_{CG,d}, \dot{v}_{CG,d}, \dots)^T \end{bmatrix}^T \end{aligned}$$

with the additive bounded disturbance  $d \in [d]$ ,  $d \in \mathbb{R}$ , and the interval parameters  $\mathbf{p} \in [\mathbf{p}]$ ,  $\mathbf{p} \in \mathbb{R}^{n_p}$

## Transformation into Nonlinear Controller Normal Form (3)

Goal: Accurate trajectory tracking and stabilization of the error dynamics despite the interval uncertainties  $d \in [d]$  and  $\mathbf{p} \in [\mathbf{p}]$

$$\begin{bmatrix} \dot{\boldsymbol{\xi}}^T \\ \dot{\boldsymbol{\zeta}}^T \end{bmatrix}^T = \begin{bmatrix} \xi_2, & \dots, & \xi_\delta, & \tilde{a}(\mathbf{x}, \mathbf{p}, d) \\ \mathbf{a}^\diamond(\mathbf{x}, \mathbf{p}, d)^T \end{bmatrix}^T \\ + \begin{bmatrix} 0, & \dots, & \tilde{b}(\mathbf{x}, \mathbf{p}) \cdot v_{\text{CG},d} \\ \mathbf{b}^\diamond(\mathbf{x}, \mathbf{p}, d, v_{\text{CG},d}, \dot{v}_{\text{CG},d}, \dots)^T \end{bmatrix}^T$$

- Use of the variable  $v_{\text{CG},d}$  as the control input
- Derivation of an interval-based variable structure control law

### Requirements

- Estimation of all state variables  $\mathbf{x}$ , of the parameters  $\mathbf{p}$ , the disturbance  $d$ , and their corresponding interval bounds in real time
- Note: If  $\delta \equiv \mathcal{N}$ , the output  $y$  coincides with the flat system output
- Otherwise: The bounded states  $\boldsymbol{\zeta}$  of the non-controllable internal dynamics act as disturbances onto the system model

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### Possible estimation approaches

- Linear gain-scheduled state observer (Luenberger-like structure)
- Sensitivity-based estimation: Receding horizon approach (online minimization of quadratic error measure)
- Observer in controller canonical form (current work)
- Robustification by LMIs possible

# First-Order vs. Second-Order Sliding Mode Control (1)

Illustrative benchmark system:  $y(t) = x_1(t)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \vdots \\ x_n(t) \\ u(t) \end{bmatrix}$$

## Definition of the tracking error

$$\tilde{\xi}_1^{(r)}(t) = x_1^{(r)}(t) - x_{1,d}^{(r)}(t) \quad \text{with} \quad r \in \{0, 1, \dots, n\}$$

## First-order sliding mode (Hurwitz polynomial of order $n - 1$ )

$$s := s(t) = \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r)}(t) \quad \implies \quad s \rightarrow 0$$

## First-Order vs. Second-Order Sliding Mode Control (1)

Illustrative benchmark system:  $y(t) = x_1(t)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \vdots \\ x_n(t) \\ u(t) \end{bmatrix}$$

Definition of the tracking error

$$\tilde{\xi}_1^{(r)}(t) = x_1^{(r)}(t) - x_{1,d}^{(r)}(t) \quad \text{with} \quad r \in \{0, 1, \dots, n\}$$

Second-order sliding mode (integral component for  $\alpha_{-1} \neq 0$ )

$$\gamma_1 \dot{s} + \gamma_0 s = \alpha_{-1} \int_0^t \tilde{\xi}_1(\tau) d\tau + \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r)}(t) \quad \implies \quad s \rightarrow 0, \quad \dot{s} \rightarrow 0$$



## Derivation of the Control Law (1)

Lyapunov function candidate (first-order)

$$V^{(I)} = \frac{1}{2}s^2 > 0 \quad \text{for } s \neq 0$$

Lyapunov function candidate (second-order)

$$V^{(II)} = \frac{1}{2} \cdot (s^2 + \lambda \dot{s}^2) > 0 \quad \text{with } \lambda > 0$$

## Derivation of the Control Law (2)

### Stability requirement (first-order)

$$\dot{V}^{(I)} = s \cdot \dot{s} = \left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r)}(t) \right) \cdot \left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r+1)}(t) \right) < 0 \quad \text{for } s \neq 0$$

### Stability requirement (second-order), $\lambda = \gamma_1 > 0$

$$\begin{aligned} \dot{V}^{(II)} &= s \cdot \dot{s} + \lambda \cdot \dot{s} \cdot \ddot{s} \\ &= s \cdot \dot{s} + \dot{s} \cdot \left( -\frac{\lambda \gamma_0}{\gamma_1} \dot{s} + \frac{\lambda}{\gamma_1} \sum_{r=0}^n \alpha_{r-1} \tilde{\xi}_1^{(r)}(t) \right) < 0 \end{aligned}$$

## Derivation of the Control Law (2)

### Stability requirement (first-order)

$$\dot{V}^{(I)} = s \cdot \dot{s} = \left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r)}(t) \right) \cdot \left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r+1)}(t) \right) < -\eta \cdot |s|$$

### Stability requirement (second-order), $\lambda = \gamma_1 > 0$

$$\dot{V}^{(II)} = s \cdot \dot{s} + \dot{s} \cdot \left( -\gamma_0 \dot{s} + \sum_{r=0}^{n-1} \alpha_{r-1} \tilde{\xi}_1^{(r)}(t) + \alpha_{n-1} \cdot \left( u(t) - x_{1,d}^{(n)}(t) \right) \right) < 0$$

## Derivation of the Control Law (2)

### Stability requirement (first-order)

$$\left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r)}(t) \right) \cdot \left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r+1)}(t) \right) < -\eta \cdot \left( \sum_{r=0}^{n-1} \alpha_r \tilde{\xi}_1^{(r)}(t) \right) \cdot \text{sign}\{s\}$$

### Stability requirement (second-order), $\lambda = \gamma_1 > 0$

$$\dot{V}^{(II)} < -\eta_1 \cdot |\dot{s}| - \eta_2 \cdot |s| \cdot |\dot{s}| = -\dot{s} \cdot \text{sign}\{\dot{s}\} \cdot (\eta_1 + \eta_2 \cdot |s|)$$

## Derivation of the Control Law (3)

### Control law (first-order)

$$u(t) = u^{(I)}(t) = x_{1,d}^{(n)}(t) - \sum_{r=0}^{n-2} \alpha_r \tilde{\xi}_1^{(r+1)}(t) - \tilde{\eta} \cdot \text{sign}\{s\}$$

### Questions

- Necessary extensions for the interval case
- Implementation requirements for an *interval control signal*
- Why/ How to generalize the first-order case?

## Derivation of the Control Law (3)

### Control law (second-order)

$$u(t) = u^{(II)}(t) = x_{1,d}^{(n)}(t) + \frac{1}{\alpha_{n-1}} \cdot \left( \gamma_0 \dot{s} - s - \sum_{r=0}^{n-1} \alpha_{r-1} \tilde{\xi}_1^{(r)}(t) - \text{sign}\{\dot{s}\} \cdot (\tilde{\eta}_1 + \tilde{\eta}_2 \cdot |s|) \right)$$

### Questions

- Necessary extensions for the interval case
- Implementation requirements for an *interval control signal*
- Why/ How to generalize the first-order case?

# Transformation into Nonlinear Controller Normal Form (1)

## Input-affine state-space representation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p}, \mathbf{v}_{CG,d}(t), \mathbf{v}_{AG,d}(t))$$

## Computation of Lie derivatives of the system output

$$y(t) = h(\mathbf{x}(t)) = \vartheta_{I^*}, \mathbf{x}(t) \in \mathbb{R}^{\mathcal{N}}$$

$$\frac{d^r y(t)}{dt^r} = y^{(r)}(t) = L_{\mathbf{f}}^r h(\mathbf{x}(t)) = L_{\mathbf{f}}(L_{\mathbf{f}}^{r-1} h(\mathbf{x}(t))) \quad , \quad r = 1, \dots, \delta - 1$$

with the relative degree  $\delta$  defined according to

$$\frac{\partial L_{\mathbf{f}}^r h(\mathbf{x}(t))}{\partial \mathbf{v}_{CG,d}} \equiv 0 \quad \text{for } r = 0, \dots, \delta - 1 \quad \text{and} \quad \frac{\partial L_{\mathbf{f}}^{\delta} h(\mathbf{x}(t))}{\partial \mathbf{v}_{CG,d}} \neq 0$$

## Transformation into Nonlinear Controller Normal Form (2)

### Introduction of the new state vector

$$\xi = [h(\mathbf{x}), L_f h(\mathbf{x}), \dots, L_f^{\delta-1} h(\mathbf{x})]^T \in \mathbb{R}^\delta \quad \text{with} \quad \xi_1 = y = h(\mathbf{x})$$

### New set of state equations (Brunovsky canonical form)

$$\begin{aligned} \begin{bmatrix} \dot{\xi}^T \\ \dot{\zeta}^T \end{bmatrix}^T &= \begin{bmatrix} L_f h(\mathbf{x}), \dots, L_f^\delta h(\mathbf{x}) \\ L_f^{\delta+1} h(\mathbf{x}), \dots, L_f^N h(\mathbf{x}) \end{bmatrix}^T \\ &= \begin{bmatrix} \xi_2, \dots, \xi_\delta, \tilde{a}(\mathbf{x}, \mathbf{p}, d) \\ \mathbf{a}^\diamond(\mathbf{x}, \mathbf{p}, d)^T \end{bmatrix}^T \\ &\quad + \begin{bmatrix} 0, \dots, \tilde{b}(\mathbf{x}, \mathbf{p}) \cdot v_{CG,d} \\ \mathbf{b}^\diamond(\mathbf{x}, \mathbf{p}, d, v_{CG,d}, \dot{v}_{CG,d}, \dots)^T \end{bmatrix}^T \end{aligned}$$

with the additive bounded disturbance  $d \in [d]$ ,  $d \in \mathbb{R}$ , and the interval parameters  $\mathbf{p} \in [\mathbf{p}]$ ,  $\mathbf{p} \in \mathbb{R}^{n_p}$



# Interval-Based Sliding Mode Control (1)

## Definition of tracking error signals

- Specification of a sufficiently smooth desired output trajectory  
 $y_d = \xi_{1,d}$
- Introduction of the error vector

$$\tilde{\boldsymbol{\xi}} = \left[ (\xi_1 - \xi_{1,d}) \quad \left( \xi_1^{(1)} - \xi_{1,d}^{(1)} \right) \quad \dots \quad \left( \xi_1^{(\delta-1)} - \xi_{1,d}^{(\delta-1)} \right) \right]^T \in \mathbb{R}^\delta$$

- Desired operating points are located on the *sliding surface*

$$s := s \left( \tilde{\boldsymbol{\xi}}(t) \right) = \tilde{\xi}_1^{(\delta-1)}(t) + \sum_{r=0}^{\delta-2} \alpha_r \cdot \tilde{\xi}_1^{(r)}(t) = 0$$

- $\alpha_0, \dots, \alpha_{\delta-2}$  are coefficients of a Hurwitz polynomial of order  $\delta - 1$

# Interval-Based Sliding Mode Control (1)

## Definition of tracking error signals

- Specification of a sufficiently smooth desired output trajectory

$$y_d = \xi_{1,d}$$

- Introduction of the error vector

$$\tilde{\xi} = \left[ (\xi_1 - \xi_{1,d}) \quad \left( \xi_1^{(1)} - \xi_{1,d}^{(1)} \right) \quad \dots \quad \left( \xi_1^{(\delta-1)} - \xi_{1,d}^{(\delta-1)} \right) \right]^T \in \mathbb{R}^\delta$$

- Desired operating points are located on the *sliding surface*

$$s := s \left( \tilde{\xi}(t) \right) = \tilde{\xi}_1^{(\delta-1)}(t) + \sum_{r=0}^{\delta-2} \alpha_r \cdot \tilde{\xi}_1^{(r)}(t) = 0$$

## Guaranteed stabilizing control: Lyapunov function candidate

$$V = \frac{1}{2} s^2 > 0 \quad \text{with} \quad \dot{V} = s \cdot \dot{s} < 0 \quad \text{for} \quad s \neq 0$$

## Interval-Based Sliding Mode Control (2)

Guaranteed stabilization despite uncertainty: Interval formulation of a variable-structure control law

$$[v_{CG,d}] := \frac{-\tilde{a}(\mathbf{x}, [\mathbf{p}], [d]) + \xi_{1,d}^{(\delta)} - \sum_{r=0}^{\delta-2} \alpha_r \cdot \tilde{\xi}_1^{(r+1)} - \tilde{\eta} \cdot \text{sign}\{s\}}{\tilde{b}(\mathbf{x}, [\mathbf{p}])}$$

with a suitably chosen parameter  $\tilde{\eta} > 0$  and  $0 \notin \tilde{b}(\mathbf{x}, [\mathbf{p}])$

Guaranteed stabilizing control: Extraction of suitable point values

$$\mathcal{V}_{CG,d} := \{\underline{v}_{CG,d} - \epsilon, \underline{v}_{CG,d} + \epsilon, \bar{v}_{CG,d} - \epsilon, \bar{v}_{CG,d} + \epsilon\}$$

with  $\underline{v}_{CG,d} := \inf\{[v_{CG,d}]\}$ ,  $\bar{v}_{CG,d} := \sup\{[v_{CG,d}]\}$  and some small  $\epsilon > 0$   
 $\implies \dot{V} < 0$  needs to be satisfied with certainty

## Interval-Based Sliding Mode Control (2)

Guaranteed stabilization despite uncertainty: Interval formulation of a variable-structure control law

$$[v_{CG,d}] := \frac{-\tilde{a}(\mathbf{x}, [\mathbf{p}], [d]) + \xi_{1,d}^{(\delta)} - \sum_{r=0}^{\delta-2} \alpha_r \cdot \tilde{\xi}_1^{(r+1)} - \tilde{\eta} \cdot \text{sign}\{s\}}{\tilde{b}(\mathbf{x}, [\mathbf{p}])}$$

with a suitably chosen parameter  $\tilde{\eta} > 0$  and  $0 \notin \tilde{b}(\mathbf{x}, [\mathbf{p}])$

Guaranteed stabilizing control: Extraction of suitable point values

- Guaranteed stabilization of system dynamics
- Inclusion of preheater model for reduction of chattering (caused by neglected dynamics)
- Extension: Guaranteed state constraints in terms of strict barrier functions

## Interval-Based Sliding Mode Control (3)

Handling of one-sided state constraint: Extended Lyapunov function

$$\tilde{V} = V + \rho_V \cdot \sum_{i \in \{\mathcal{I}\}} \ln \left( \frac{\bar{\theta}_{\max}}{\bar{\theta}_{\max} - \vartheta_i} \right) > 0 \quad \text{with} \quad V = \frac{1}{2} s^2, \quad \rho_V > 0$$

Constraint  $\vartheta_{\mathcal{I}} \leq \theta_{\max}$  is expressed by the strict barrier  $\vartheta_{\mathcal{I}} < \bar{\theta}_{\max}$

Corresponding time derivative and control law

$$\dot{\tilde{V}} = \dot{V} + \rho_V \cdot \sum_{i \in \{\mathcal{I}\}} \left( \frac{\dot{\vartheta}_i}{\bar{\theta}_{\max} - \vartheta_i} \right)$$

$$[\tilde{v}_{\text{CG,d}}] := [v_{\text{CG,d}}] - \frac{s}{s^2 + \tilde{\epsilon}} \cdot \frac{\rho_V}{\tilde{b}(\mathbf{x}, [\mathbf{p}]}) \cdot \sum_{i \in \{\mathcal{I}\}} \left( \frac{\dot{\vartheta}_i}{\bar{\theta}_{\max} - \vartheta_i} \right)$$

## Interval-Based Sliding Mode Control (4)

### Corresponding time derivative and control law

$$\dot{\tilde{V}} = \dot{V} + \rho_V \cdot \sum_{i \in \{I\}} \left( \frac{\dot{\vartheta}_i}{\bar{\theta}_{\max} - \vartheta_i} \right)$$

$$[\tilde{v}_{CG,d}] := [v_{CG,d}] - \frac{s}{s^2 + \tilde{\epsilon}} \cdot \frac{\rho_V}{\tilde{b}(\mathbf{x}, [\mathbf{p}])} \cdot \sum_{i \in \{I\}} \left( \frac{\dot{\vartheta}_i}{\bar{\theta}_{\max} - \vartheta_i} \right)$$

### Remarks

- Extraction of point-valued control signals as before
- Approximation  $\frac{s}{s^2 + \tilde{\epsilon}} \approx \frac{1}{s}$  ensures regularity of the control law
- For  $s = 0$ : Control is identical to the previous case
- Barrier at  $\bar{\theta}_{\max} - \vartheta_i = 0$  represents *repelling potential*
- Approximation errors are negligible if  $\tilde{\epsilon}$  is sufficiently small

## Control Parameterization: Basic Approach (Excerpt)

Control signal feasible?		
Yes	No	
<p><b>Break</b>, apply the control for the time step <math>t_k</math>, and proceed with the subsequent discretization step</p>	<p><b>Adaption</b> of <math>\tilde{\eta}</math> (Alternative: adapt the parameters <math>\alpha_r</math> in definition of sliding surface)</p>	
	<p>Input saturation exceeded</p>	
	<p>a) <math>\tilde{v}_{CG,d}(t_k) &lt; \inf\{[v_{CG,max}]\}</math></p>	<p>b) <math>\tilde{v}_{CG,d}(t_k) &gt; \sup\{[v_{CG,max}]\}</math></p>
	<p><b>Increase</b> <math>\tilde{\eta}</math> if <math>\frac{\partial \tilde{v}_{CG,d}}{\partial \tilde{\eta}} &gt; 0</math></p> <p><b>Decrease</b> <math>\tilde{\eta}</math> if <math>\frac{\partial \tilde{v}_{CG,d}}{\partial \tilde{\eta}} &lt; 0</math></p>	<p><b>Increase</b> <math>\tilde{\eta}</math> if <math>\frac{\partial \tilde{v}_{CG,d}}{\partial \tilde{\eta}} &lt; 0</math></p> <p><b>Decrease</b> <math>\tilde{\eta}</math> if <math>\frac{\partial \tilde{v}_{CG,d}}{\partial \tilde{\eta}} &gt; 0</math></p>

## Control Parameterization: Extension for Online Gain Scheduling

- Case 1: Offline parameterization with cutoff for control signal
  - Case 2: Online parameterization
    - ① Define a desired eigenvalue  $\lambda_r$  of multiplicity  $\delta - 1$  on the sliding surface with corresponding parameters  $\alpha_r$
    - ② Initialize  $\tilde{\eta}$  with the desired value
    - ③ Adapt  $\tilde{\eta}$  in a line-search approach (fixed number of  $N_\eta = 5$  steps) to ensure compatibility of  $\tilde{v}_{CG,d}$  with the control constraints
      - ★ Stop, if admissible control is found
      - ★ If no admissible control is found within  $N_\eta$  steps, adapt the eigenvalue  $\lambda_r$  and restart with Step (2); Break after at most  $N_\lambda = 5$  repetitions
- 
- Treatment of input rate constraints: Extension of the system input by a further lag element
  - Simulation case study:  $L = N = 1, M = 3$



## Handling of Input Rate Limitations

Extension of the system input by a further lag element

$$T_r \cdot \dot{\tilde{v}}_{CG,d} + \tilde{v}_{CG,d} = \check{v}_{CG,d}$$

with the new system input  $\check{v}_{CG,d}$  and the fixed time constant  $T_r > 0$

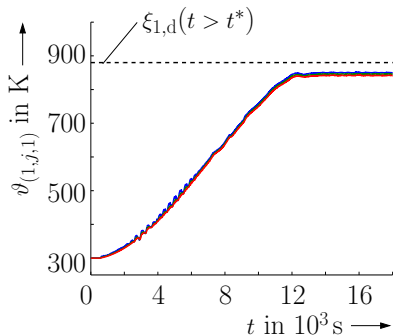
Guaranteed compatibility of the actual system input with the rate constraints

$$|\dot{\tilde{v}}_{CG,d}| \leq T_r^{-1} \cdot (\sup\{[v_{CG,max}]\} - \inf\{[v_{CG,max}]\})$$

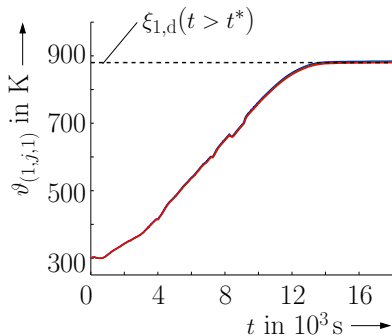
under the prerequisite

$$\begin{aligned} \inf\{[v_{CG,max}]\} &\equiv \inf\{[\tilde{v}_{CG,max}]\} \equiv \inf\{[\check{v}_{CG,max}]\}, \\ \sup\{[v_{CG,max}]\} &\equiv \sup\{[\tilde{v}_{CG,max}]\} \equiv \sup\{[\check{v}_{CG,max}]\} \end{aligned}$$

## Simulation Results: Stack Temperatures

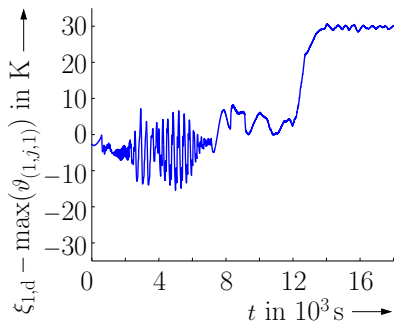


Offline parameterization

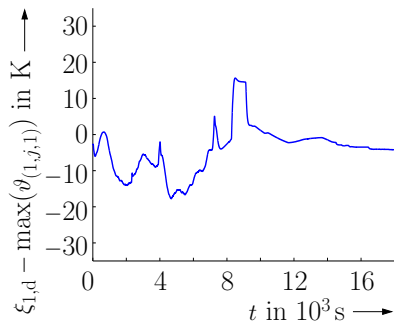


Online parameterization

## Simulation Results: Tracking Error

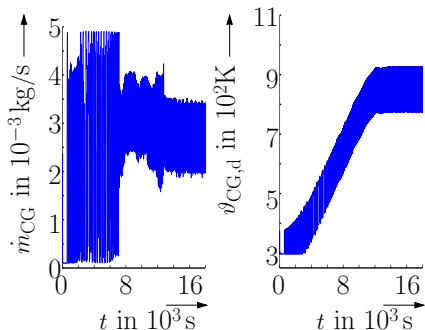


Offline parameterization

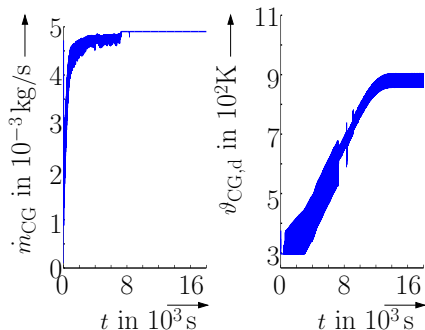


Online parameterization

## Simulation Results: CG Preheater Inputs



Offline parameterization



Online parameterization

## Conclusions and Outlook on Future Work

- Control-oriented modeling of a complex thermodynamic application
- Verified parameter identification as the basis for control design
- Stabilization of the error dynamics using interval arithmetic
- Online optimization of the control signal in the multi-input case: energy efficiency and lifetime
- Applicable if switchings of the output segment occur
- Handling of input and state constraints (guaranteed overshoot prevention)

## Conclusions and Outlook on Future Work

- Control-oriented modeling of a complex thermodynamic application
- Verified parameter identification as the basis for control design
- Stabilization of the error dynamics using interval arithmetic
- Online optimization of the control signal in the multi-input case: energy efficiency and lifetime
- Applicable if switchings of the output segment occur
- Handling of input and state constraints (guaranteed overshoot prevention)
  
- Extension by a sensitivity-based predictive controller
- Extension by a (sensitivity-based) state and disturbance observer
- Extension by a more detailed description of the preheater dynamics

Thank you for your attention!

Спасибо за Ваше внимание!

Merci beaucoup pour votre attention!

Dziękuję bardzo za uwagę!

¡Muchas gracias por su atención!

Grazie mille per la vostra attenzione!

Vielen Dank für Ihre Aufmerksamkeit!

