Reachability Analysis for a Class of Uncertain Discrete-Time Systems

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Reachability Analysis:

Consider an uncertain system described by:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{f}(\mathbf{x}_k, \mathbf{w}_k)$$

where:

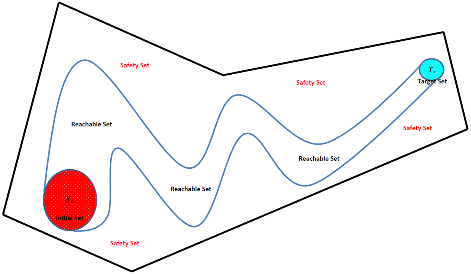
- $\mathbf{x}_k \in \mathcal{D} \subset \mathbb{R}^n$ is the state vector and $\mathbf{x}_0 \in [\mathbf{x}_0]$.
- $\mathbf{u}_k \in \mathcal{U} \subset \mathbb{R}^m$ is the input vector.
- The nonlinear term $\mathbf{f}(.,.)$ stands for the poorly-known part of this system, which is assumed to be bounded:

$$\forall \mathbf{x}_k \in \mathcal{D} \text{ and } \forall \mathbf{w}_k \in \mathcal{W} \subset \mathbb{R}^p, \quad \mathbf{f}(\mathbf{x}_k, \mathbf{w}_k) \in [\underline{\mathbf{f}}, \overline{\mathbf{f}}]$$

Reachability Analysis: Motivation

Numerical Proof

- Interval-based state estimation.
- Fault Detection and Diagnosis.
- Safety verification.
- Robust control.
- . . .



Reachability Analysis: Definitions

Definition 1:

The reachable set of the an uncertain dynamical system, denoted by

$$\mathcal{R}\big([t_0, t_k], t_0, \mathcal{X}_0\big)$$

is the set of all the possible state trajectories generated from an initial set $\mathcal{X}_0 \subset \mathcal{D}$ and solutions to the set of difference equations

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{f}(\mathbf{x}_k, \mathbf{w}_k) + \mathbf{B}\mathbf{u}_k$$

Definition 2:

An outer approximation of this reachable set, denoted by $\mathcal{Y}\big([t_0, t_k], t_0, \mathcal{Y}_0\big)$, is a set that satisfies the following inclusion:

$$\forall k, \quad \mathcal{R}\big([t_0, t_k], t_0, \mathcal{X}_0\big) \subseteq \mathcal{Y}\big([t_0, t_k], t_0, \mathcal{Y}_0\big)$$

Outline

Outline

Reachability Analysis: New Algorithm

Proposition:

The following interval predictor provides a tight outer approximation of the reachable set of the introduced uncertain system.

$$\begin{aligned} [\mathbf{x}_k] &= \mathbf{A}^k[\mathbf{x}_0] + \mathbf{b}_{k-1} + [\mathbf{f}_{k-1}] \\ [\mathbf{f}_k] &= \mathbf{A}^k[\mathbf{f}_0] + [\mathbf{f}_{k-1}] \\ \mathbf{b}_k &= \mathbf{A}\mathbf{b}_{k-1} + \mathbf{B}\mathbf{u}_k \end{aligned}$$

where $\mathbf{b}_0 = \mathbf{B}\mathbf{u}_0$ and $[\mathbf{f}_0] = [\underline{\mathbf{f}}, \overline{\mathbf{f}}]$.

$$\mathcal{Y}([t_0, t_k], t_0, [\mathbf{x}_0]) = \bigcup_0^k [\mathbf{x}_k] \supseteq \mathcal{R}([t_0, t_k], t_0, \mathcal{X}_0)$$

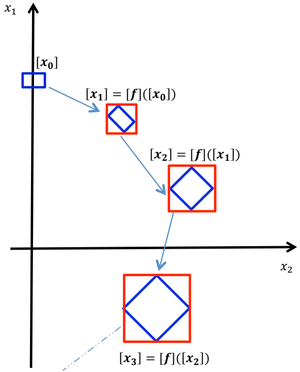
Main Idea: Avoid the wrapping effect

Consider the interval system:

$[\mathbf{x}_{k+1}] = \mathbf{A}[\mathbf{x}_k] + \mathbf{B}\mathbf{u}_k + [\mathbf{f}_0]$. An iterative formula.

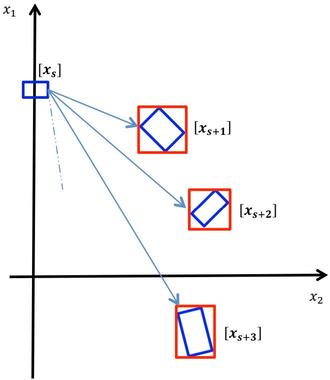
The wrapping effect:

$$
\begin{aligned}
[f]([\mathbf{x}_0]) &= \mathbf{A}[\mathbf{x}_0] + \mathbf{B}\mathbf{u}_0 + [\mathbf{f}_0] \\
[f]([\mathbf{x}_1]) &= \mathbf{A}[\mathbf{x}_1] + \mathbf{B}\mathbf{u}_1 + [\mathbf{f}_0] \\
&\vdots \\
[f]([\mathbf{x}_{k+1}]) &= \mathbf{A}[\mathbf{x}_k] + \mathbf{B}\mathbf{u}_k + [\mathbf{f}_0]
\end{aligned}
\quad \Longrightarrow
$$



Main Idea: Avoid the wrapping effect

- Based on the analytical expression of the response of linear discrete-time systems a new numerical scheme is proposed.
- In this new scheme one computes directly the upcoming state enclosures, $[\mathbf{x}_k]$ $k > 0$ from the initial state enclosure $[\mathbf{x}_0]$, no wrapping effecct.



Proof of the Proposition

Proof by induction:

In the base case we have to prove that the equations of the proposed interval predictor is true for $k = 1$. To achieve that:

- From the first iteration of the system, for all $\mathbf{x}_0 \in [\mathbf{x}_0]$ and $\mathbf{f}(\mathbf{x}_0, \mathbf{w}_0) \in [\underline{\mathbf{f}}, \ \overline{\mathbf{f}}]$ one gets:

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{A}\mathbf{x}_0 + \mathbf{B}\mathbf{u}_0 + \mathbf{f}(\mathbf{x}_0, \mathbf{w}_0) \\ &\in \mathbf{A}[\mathbf{x}_0] + \mathbf{B}\mathbf{u}_0 + [\underline{\mathbf{f}}, \ \overline{\mathbf{f}}] = \mathbf{A}[\mathbf{x}_0] + \mathbf{b}_0 + [\mathbf{f}_0] \end{aligned}$$

- From the equations of the proposed interval predictor one gets:

$$\begin{aligned} [\mathbf{x}_1] &= \mathbf{A}[\mathbf{x}_0] + \mathbf{b}_0 + [\mathbf{f}_0] \\ [\mathbf{f}_1] &= \mathbf{A}[\mathbf{f}_0] + [\mathbf{f}_0] \\ \mathbf{b}_1 &= \mathbf{A}\mathbf{b}_0 + \mathbf{B}\mathbf{u}_1 \end{aligned}$$

where: $\mathbf{b}_0 = \mathbf{B}\mathbf{u}_0$ and $[\mathbf{f}_0] = [\underline{\mathbf{f}}, \overline{\mathbf{f}}]$.

Proof of the Proposition

Now, we assume that the statement holds for some natural number k, and prove that this statement holds for $k+1$. Consider again the state equation of the system, for all $\mathbf{x}_k \in [\mathbf{x}_k]$ and $\mathbf{f}(\mathbf{x}_k, \mathbf{w}_k) \in [\mathbf{f}_0]$ one obtains:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{f}(\mathbf{x}_k, \mathbf{w}_k) \\ &\in \mathbf{A}^{k+1}[\mathbf{x}_0] + \mathbf{B}\mathbf{u}_k + \mathbf{A}\mathbf{b}_{k-1} + \mathbf{A}[\mathbf{f}_{k-1}] + [\mathbf{f}_0] \\ &\in \mathbf{A}^{k+1}[\mathbf{x}_0] + \mathbf{b}_k + \mathbf{A}[\mathbf{f}_{k-1}] + [\mathbf{f}_0] \end{aligned}$$

Thus, to complete this proof, we have to show that:

$$[\mathbf{f}_k] = \mathbf{A}[\mathbf{f}_{k-1}] + [\mathbf{f}_0] \tag{1}$$

From (1), and using the definition of $[\mathbf{f}_{k-1}]$ given in the second equation of the interval predictor one obtains:

$$\begin{aligned} [\mathbf{f}_k] &= \mathbf{A}\left(\sum_{i=k-1}^{0} \mathbf{A}^i[\mathbf{f}_0]\right) + [\mathbf{f}_0] \\ &= \left(\sum_{i=k}^{1} \mathbf{A}^i[\mathbf{f}_0]\right) + [\mathbf{f}_0] \\ &= \sum_{i=k}^{0} \mathbf{A}^i[\mathbf{f}_0] \end{aligned}$$

On the other hand, from the second equation of the proposed interval predictor, which describes $[\mathbf{f}_k]$, one has:

$$\begin{aligned} [\mathbf{f}_k] &= \mathbf{A}^k[\mathbf{f}_0] + [\mathbf{f}_{k-1}] \\ &= \mathbf{A}^k[\mathbf{f}_0] + \sum_{i=k-1}^{0} \mathbf{A}^i[\mathbf{f}_0] \\ &= \sum_{i=k}^{0} \mathbf{A}^i[\mathbf{f}_0] \end{aligned}$$

Then one can claim that:

$$[\mathbf{f}_k] = \mathbf{A}^k[\mathbf{f}_0] + [\mathbf{f}_{k-1}] = \mathbf{A}[\mathbf{f}_{k-1}] + [\mathbf{f}_0]$$

Outline

Case where the matrix **A** is Schur stable

Proposition:

If the matrix **A** of the uncertain system is Schur stable, then the width of the state enclosures,

$$W([\mathbf{x}_k]) = \frac{1}{2}(\overline{\mathbf{x}}_k - \underline{\mathbf{x}}_k), \quad k > 0$$

computed by the proposed interval predictor converges towards a constant vector when k tends to $+\infty$

Proof of the Propositipn

- By definition one has:

$$
\begin{aligned}
W([\mathbf{x}_k]) &= |\mathbf{A}^k| W([\mathbf{x}_0]) + W([\mathbf{f}_{k-1}]) \\
W([\mathbf{f}_k]) &= |\mathbf{A}^k| W([\mathbf{f}_0]) + W([\mathbf{f}_{k-1}])
\end{aligned}
$$

- As **A** is assumed Schur stable:

$$
\lim_{k \to +\infty} \mathbf{A}^k = 0
$$

<u>**Then one can claim that:**</u>

$$
\begin{aligned}
\lim_{k \to +\infty} W([\mathbf{x}_k]) &= \lim_{k \to +\infty} |\mathbf{A}^k| W([\mathbf{x}_0]) + \lim_{k \to +\infty} W([\mathbf{f}_{k-1}]) \\
&= \lim_{k \to +\infty} W([\mathbf{f}_{k-1}]) \\
&= \lim_{k \to +\infty} W([\mathbf{f}_k]) \\
&= \text{a fixed point}
\end{aligned}
$$

Outline

Example borrowed from (F. Mazenc et al., 2014)

Consider the following uncertain system borrowed from the literature.

$$\mathbf{x}_{k+1} = \frac{1}{2}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\mathbf{x}_k + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} f_k$$

where:

- The box of the initial condition:

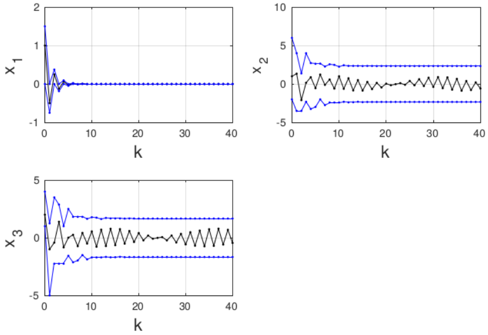
$$\mathbf{x}_0 \in [0,\ 1.5] \times [-2,\ 6] \times [1,\ 4]$$

- The poorly-known part of the system:

$$f_k \in [-1,\ 1],\ \forall k \geq 0$$

Simulation Results

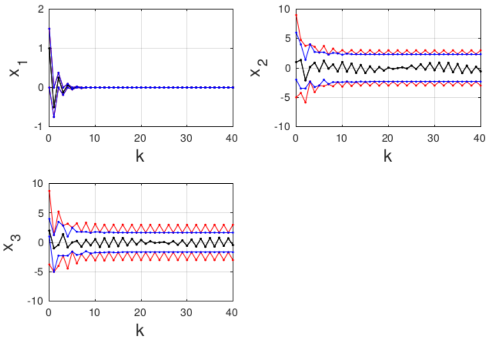
- The actual initial state of the system $\mathbf{x}_0 = (1, 1, 2)^T$.
- The considred nonlinear term $f_k = \sin\left(x_2(k) + 3k\right)$.



- Black curves show the actual state variables of the system.
- Blue curves plot the reachable set computed by the interval-based predictor.

Comparison with the similarity transformation approach (Mazenc et al., 2014)

- In the same simulation conditions $\mathbf{x}_0$ and f_k.



- Red curves show the reachable set computed by the time-varying similarity transformation approach.
- Blue curves plot the reachable set computed by the interval-based predictor.

Comparison with the similarity transformation approach (Mazenc et al., 2014)

CPU time (Intel Core i7-2620M CPU @ 2.70GHz × 4):

$$t_{Interval\ \ Predictor} \approx 0.015s \quad < \quad t_{Similarity\ \ Transformation} \approx 0.035s$$

Table: Arithmetic operations at each iteration

	Interval-based Predictor	Similarity Transformation approach
Matrix Vector Multiplication	8	14
Vector Vector Addition and subtraction	8	8

- In the case of the similarity transformation approach, the time-varying transformation matrix is updated at each iteration.

Outline

Concluding remarks

Conclusion

The proposed Interval-based Predictor has the following properties:

- Efficient against the wrapping effect.
- Simple to implement.
- Proof of the convergence of the width of the computed state enclosure.

Thank you !