On Mode Discernibility and Bounded-Error State Estimation with Hybrid Systems

Nacim Ramdani, Louise Travé-Massuyès, Carine Jauberthie

PRISME U. Orléans & LAAS CNRS U. Toulouse

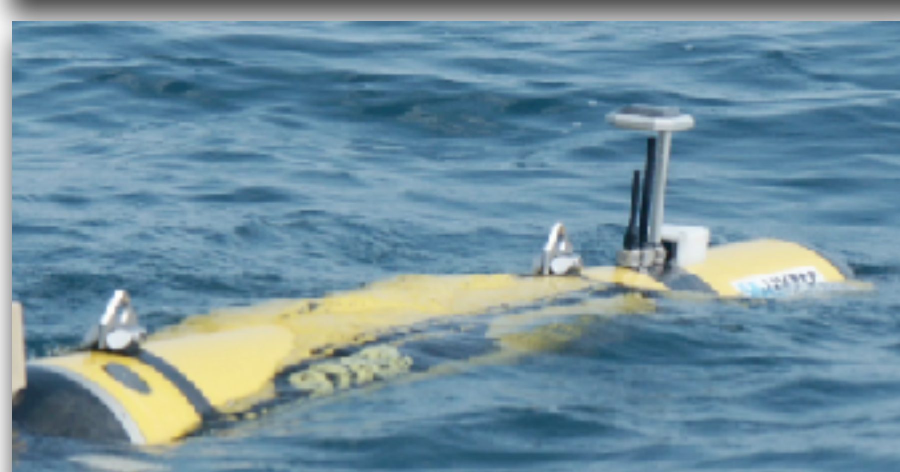
11th SWIM, 25-27 July 2018, Rostock.

- **Hybrid and Cyber-Physical Systems**

LAAS CNRS

PRISME
Laboratoire Pluridisciplinaire de Recherche
Ingénierie des Systèmes, Mécanique, Énergétique

Hybrid Cyber-Physical Systems



- Interaction discrete
  + continuous dynamics
- Safety-critical
  embedded systems
- Networked
  autonomous systems

LAAS CNRS
PRISME
Laboratoire Pluridisciplinaire de Recherche
Ingénierie des Systèmes, Mécanique, Énergétique

Hybrid Cyber-Physical Systems

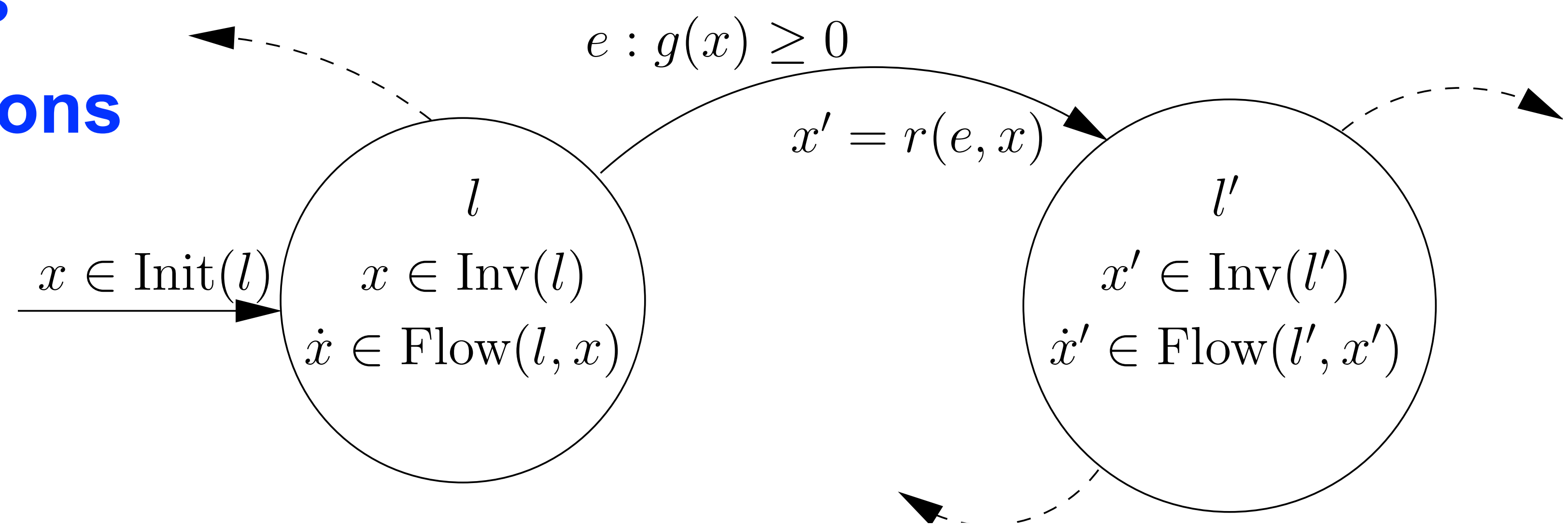
Operation in challenging environment, requires …

- **Verification**
  - Numerical proof, or
  - Falsification via counter-example
- **Synthesis**
  - « Correct by construction » …
- **Monitoring, FDI**
  - Complete state reconstruction
  - Worst-case scenario

Hybrid Cyber-Physical Systems

Modelling → hybrid automaton (Alur, et al. 1995)

- Non-linear continuous dynamics
- Nonlinear guards sets
- Nonlinear reset functions
- Bounded uncertainty



$$H = (\mathcal{Q}, \mathcal{D}, \mathcal{P}, \Sigma, \mathcal{A}, \mathsf{Inv}, \mathcal{F}),$$

Continuous dynamics

$$\begin{aligned} \mathsf{flow}(q): &\quad \dot{\mathbf{x}}(t) = f_q(\mathbf{x}, \mathbf{p}, t), \\ \mathsf{Inv}(q): &\quad \nu_q(\mathbf{x}(t), \mathbf{p}, t) < 0, \end{aligned}$$

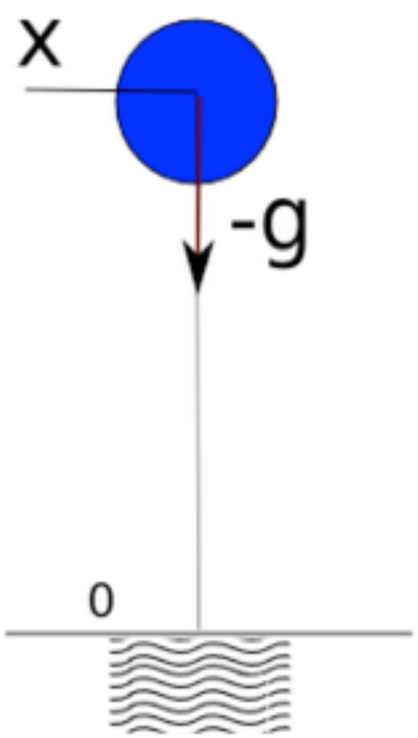
Discrete dynamics

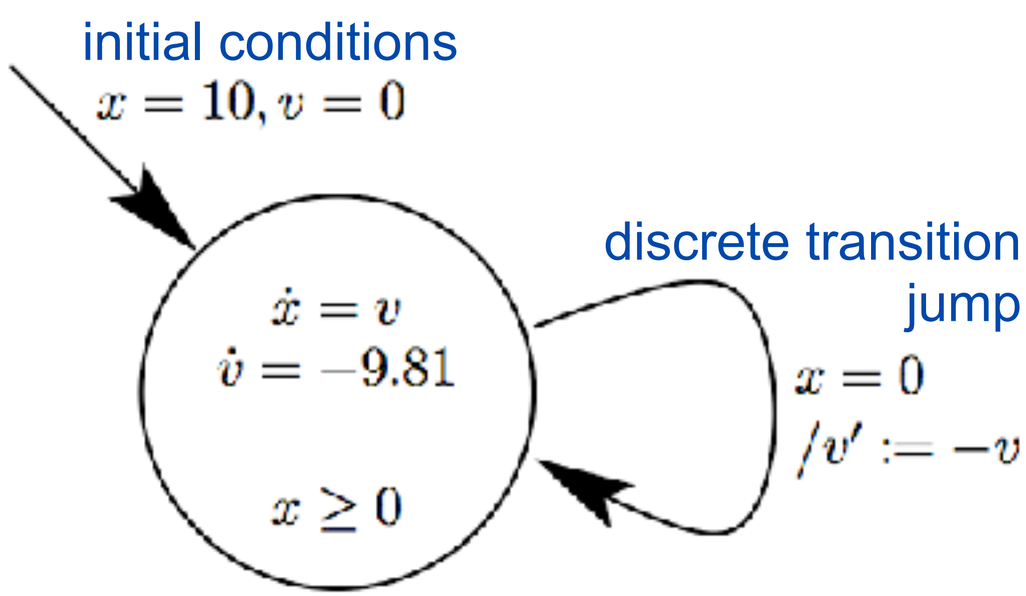
$$\begin{aligned} \mathcal{A} \ni e: &\quad (q \to q') = (q, \mathsf{guard}, \sigma, \rho, q'), \\ \mathsf{guard}(e): &\quad \gamma_e(\mathbf{x}(t), \mathbf{p}, t) = 0, \end{aligned}$$

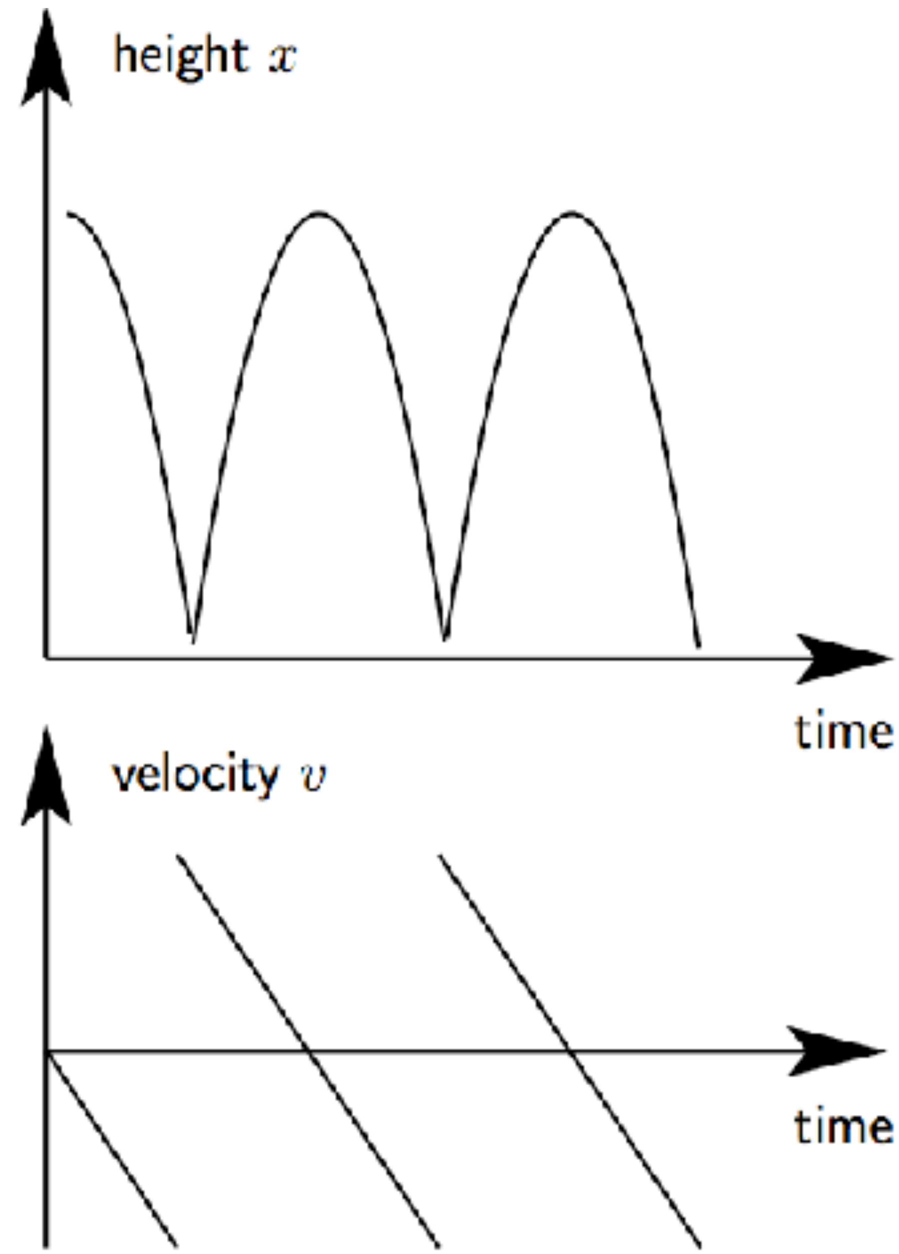
$$t_0 \leq t \leq t_N, \quad \mathbf{x}(t_0) \in \mathbb{X}_0 \subseteq \mathbb{R}^n, \quad \mathbf{p} \in \mathbb{P}$$

Hybrid Cyber-Physical Systems

- **Example :** the bouncing ball



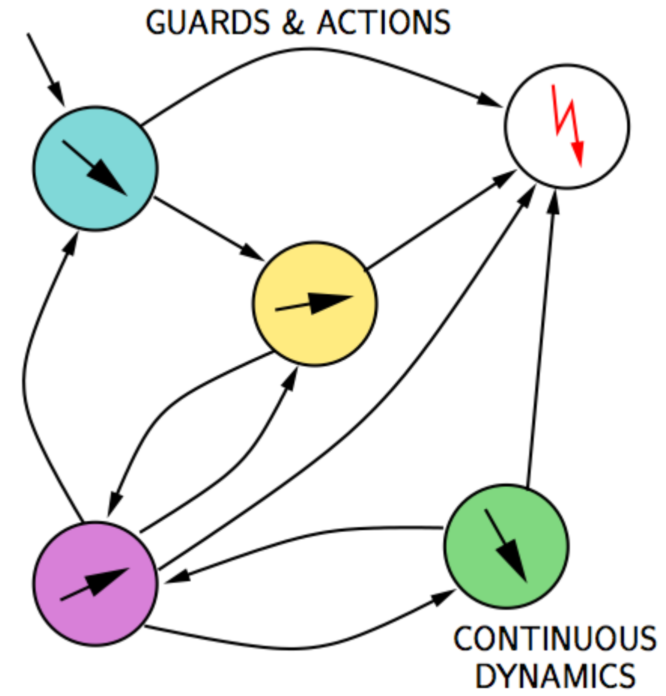




- **Monitoring, Estimation**

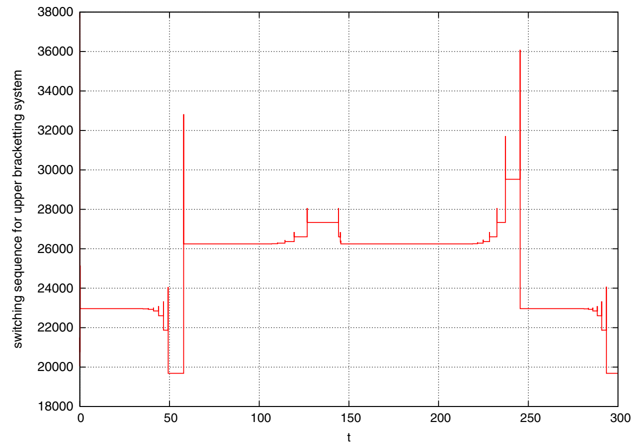
Monitoring of Hybrid Systems

- **Modelling → hybrid automaton**
  - **Non-linear continuous dynamics**
  - **Bounded uncertainty**



- **State Estimation**
  → reconstruct system state variables
  - **switching sequence**
  - **continuous variables**

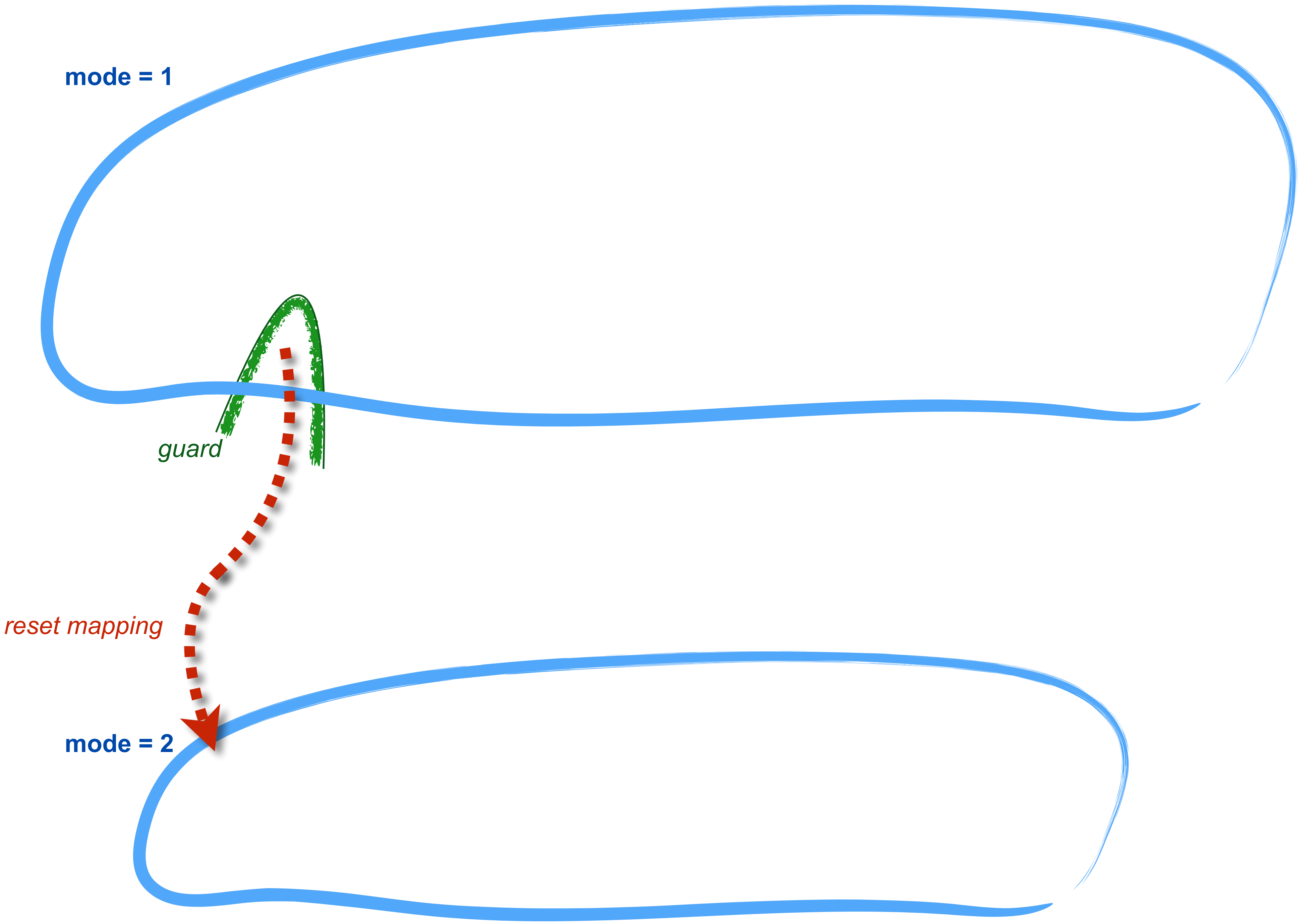
- **Important issue**
  - **Control & Diagnosis ...**



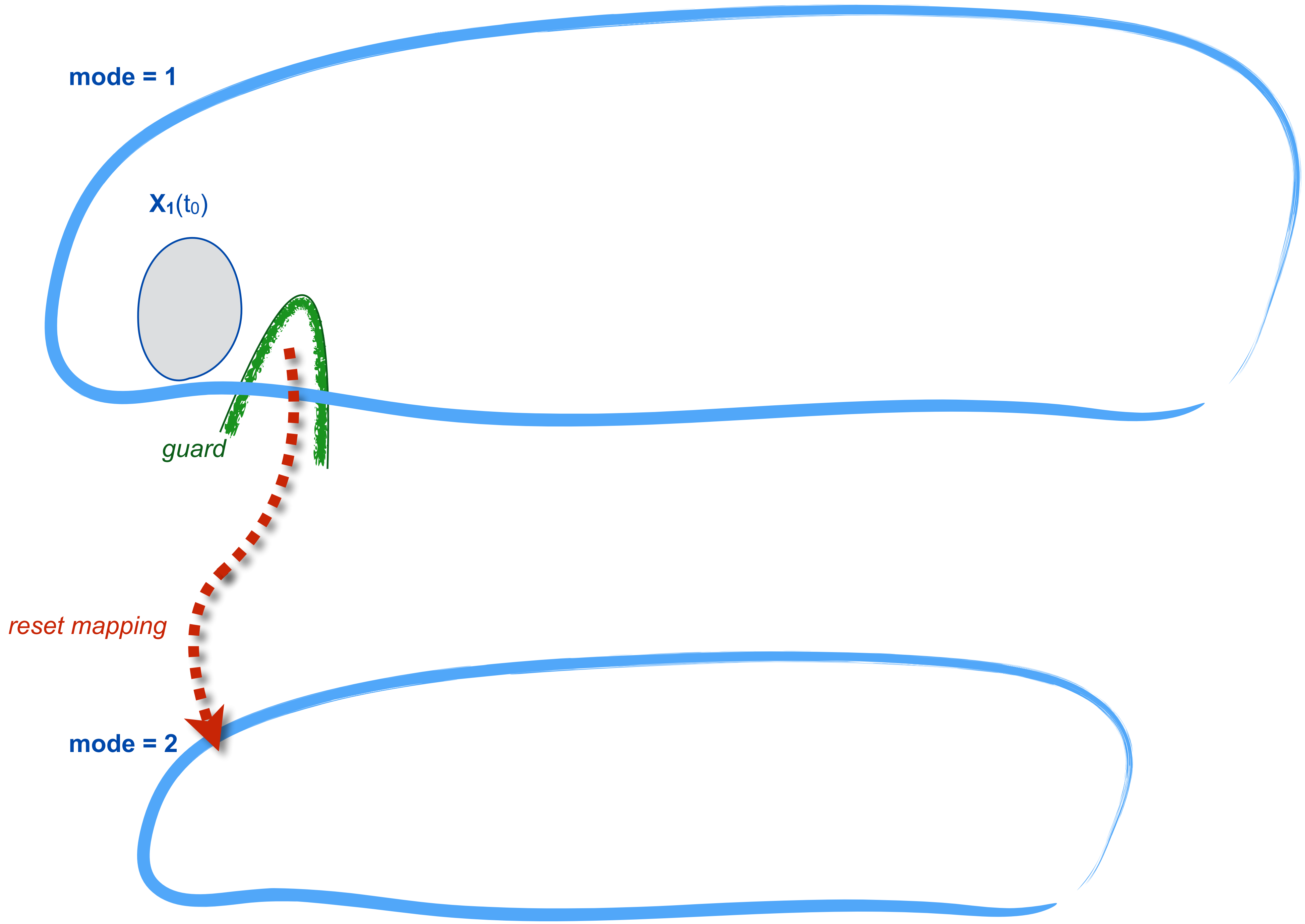
- **Complete Hybrid State Estimation**

Reachability-based approach

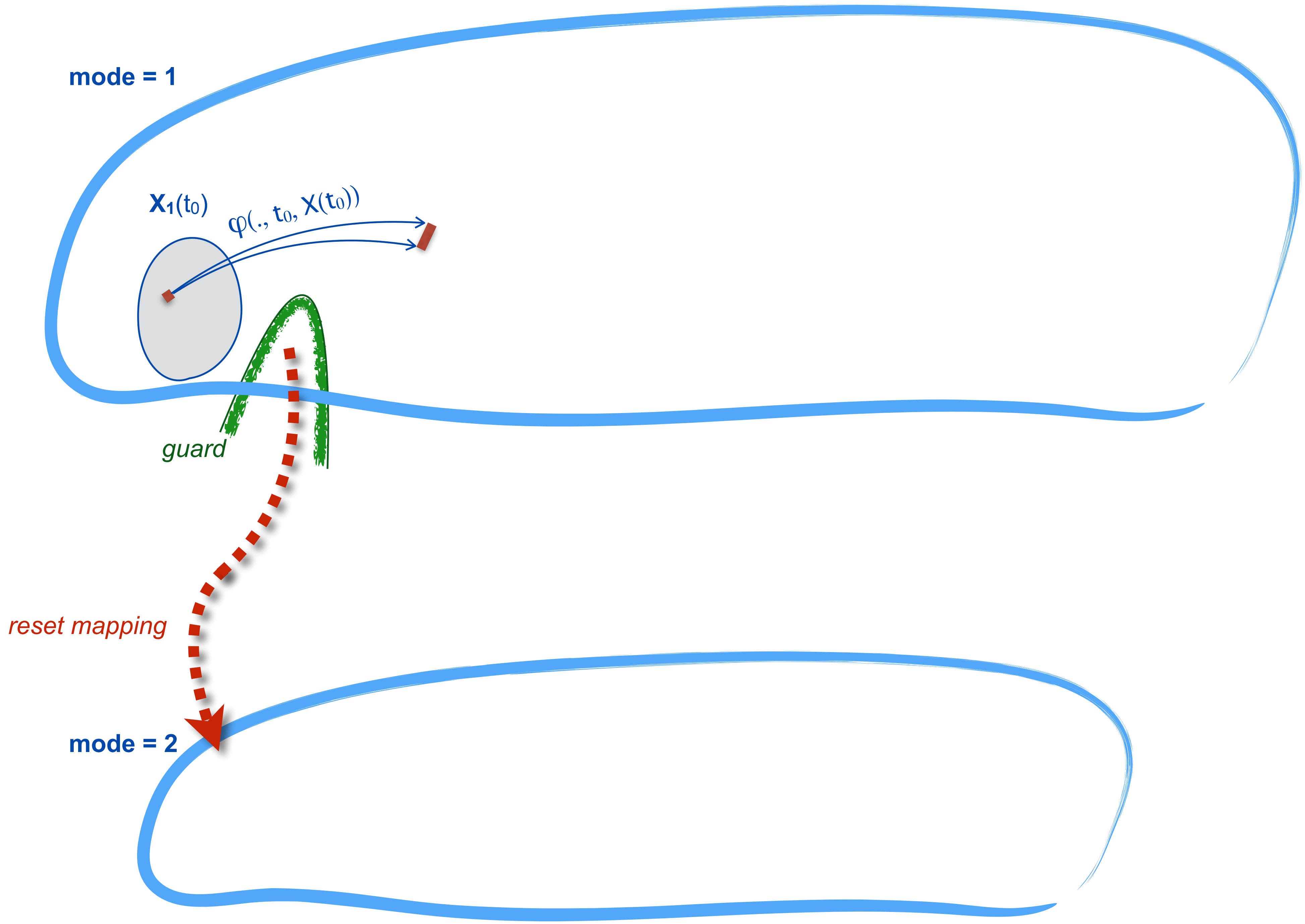
Reachability-based approach



Reachability-based approach

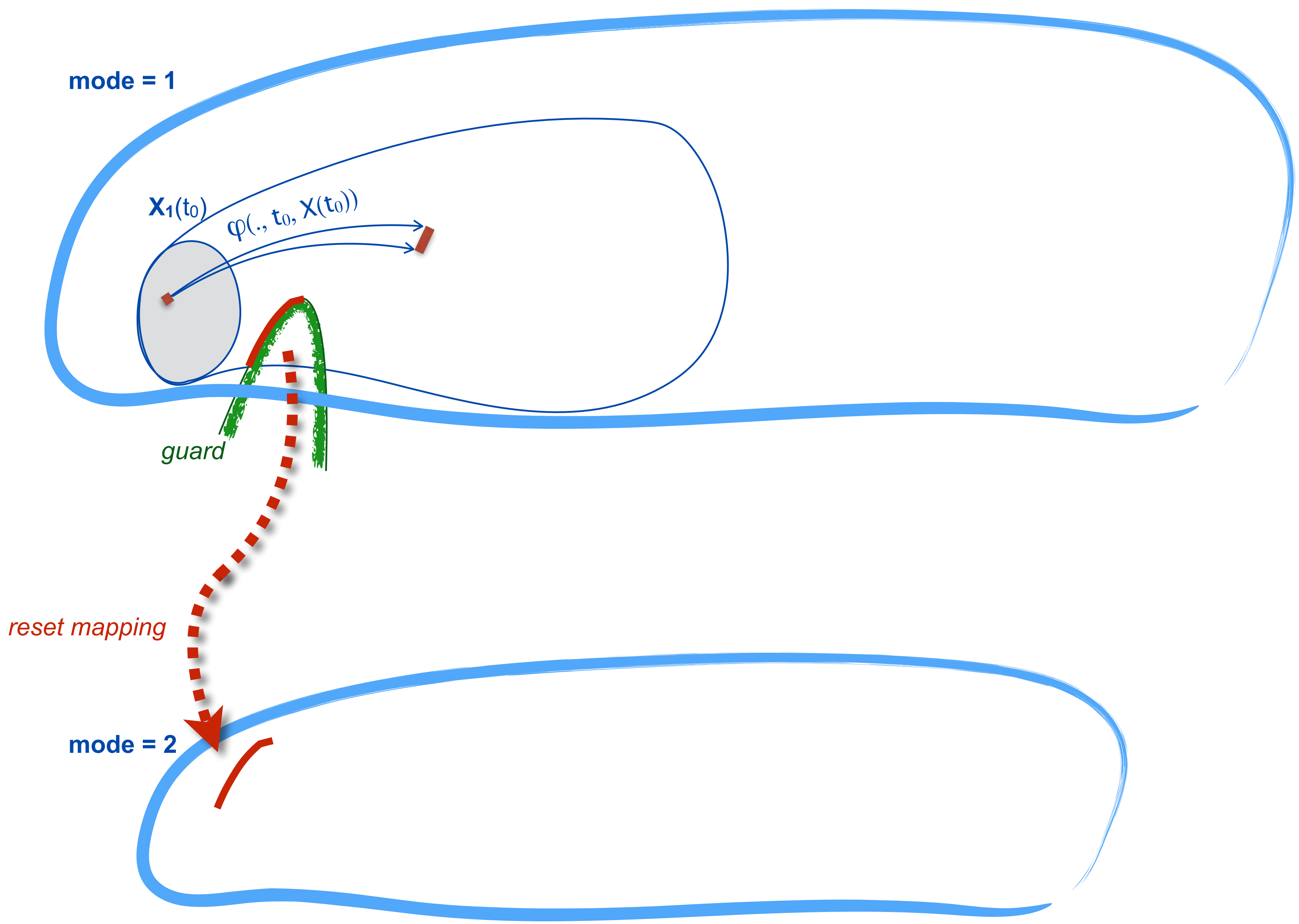


Reachability-based approach

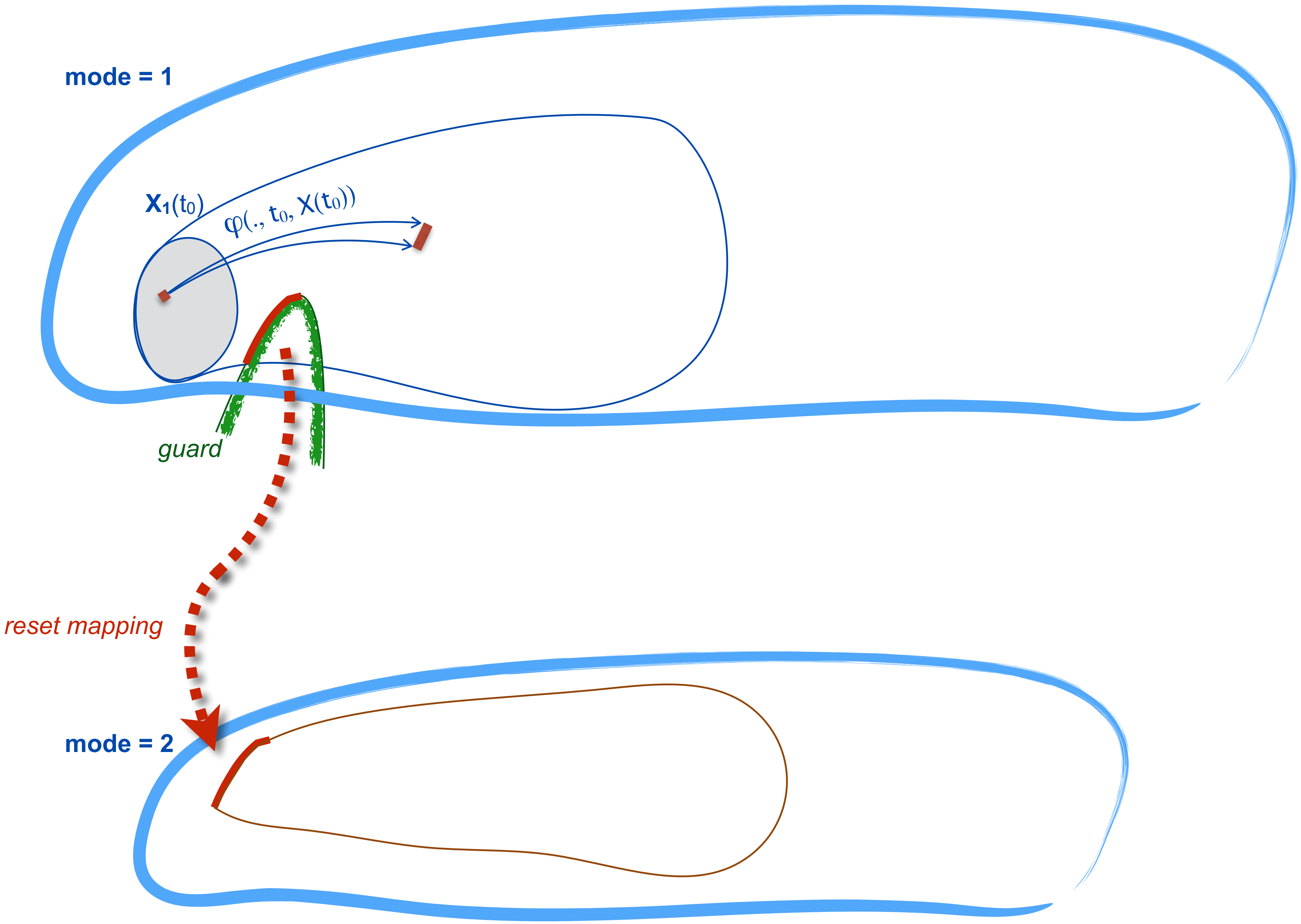


LAAS CNRS
PRISME
Laboratoire Pluridisciplinaire de Recherche
Ingénierie des Systèmes, Mécanique, Énergétique

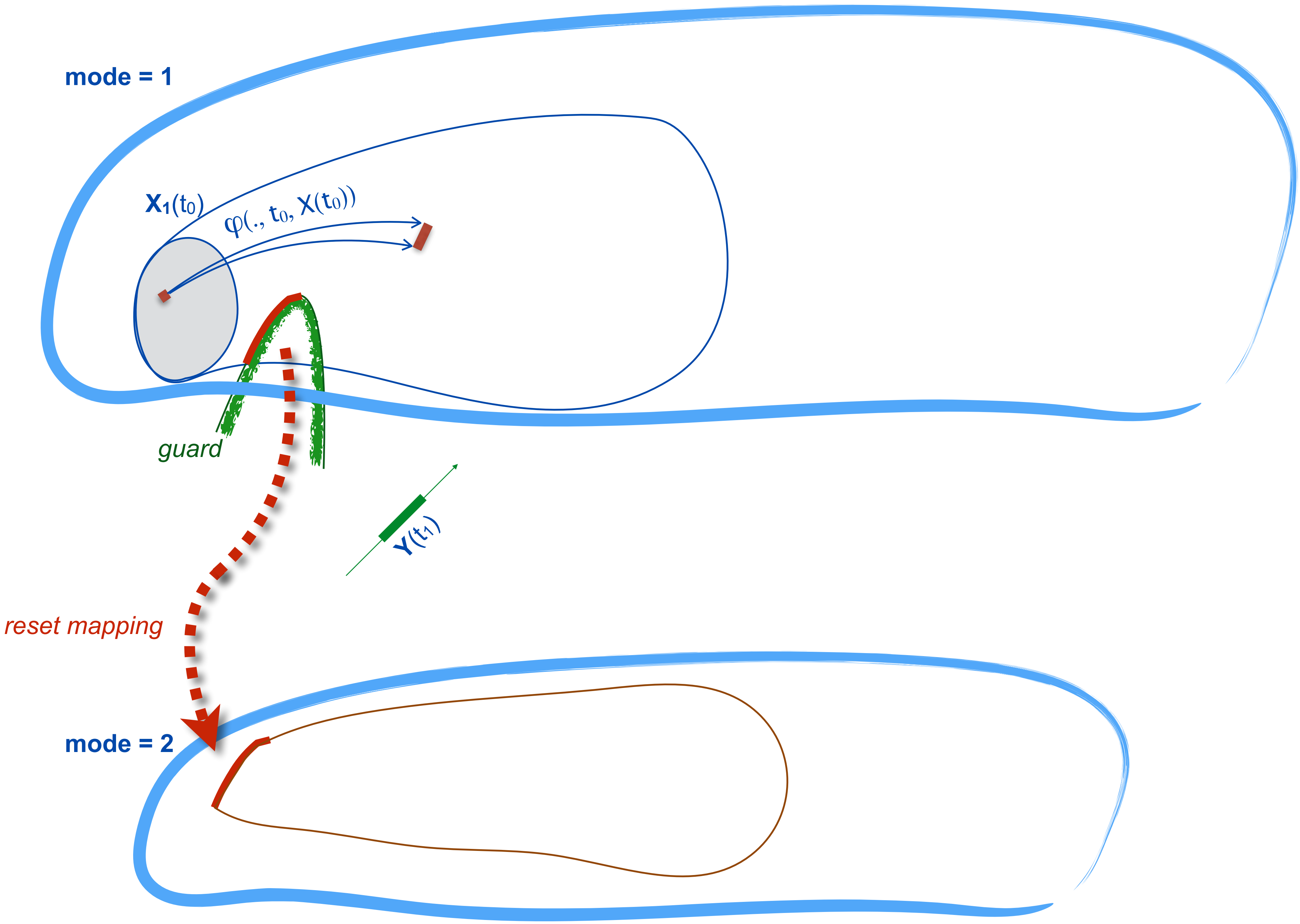
Reachability-based approach



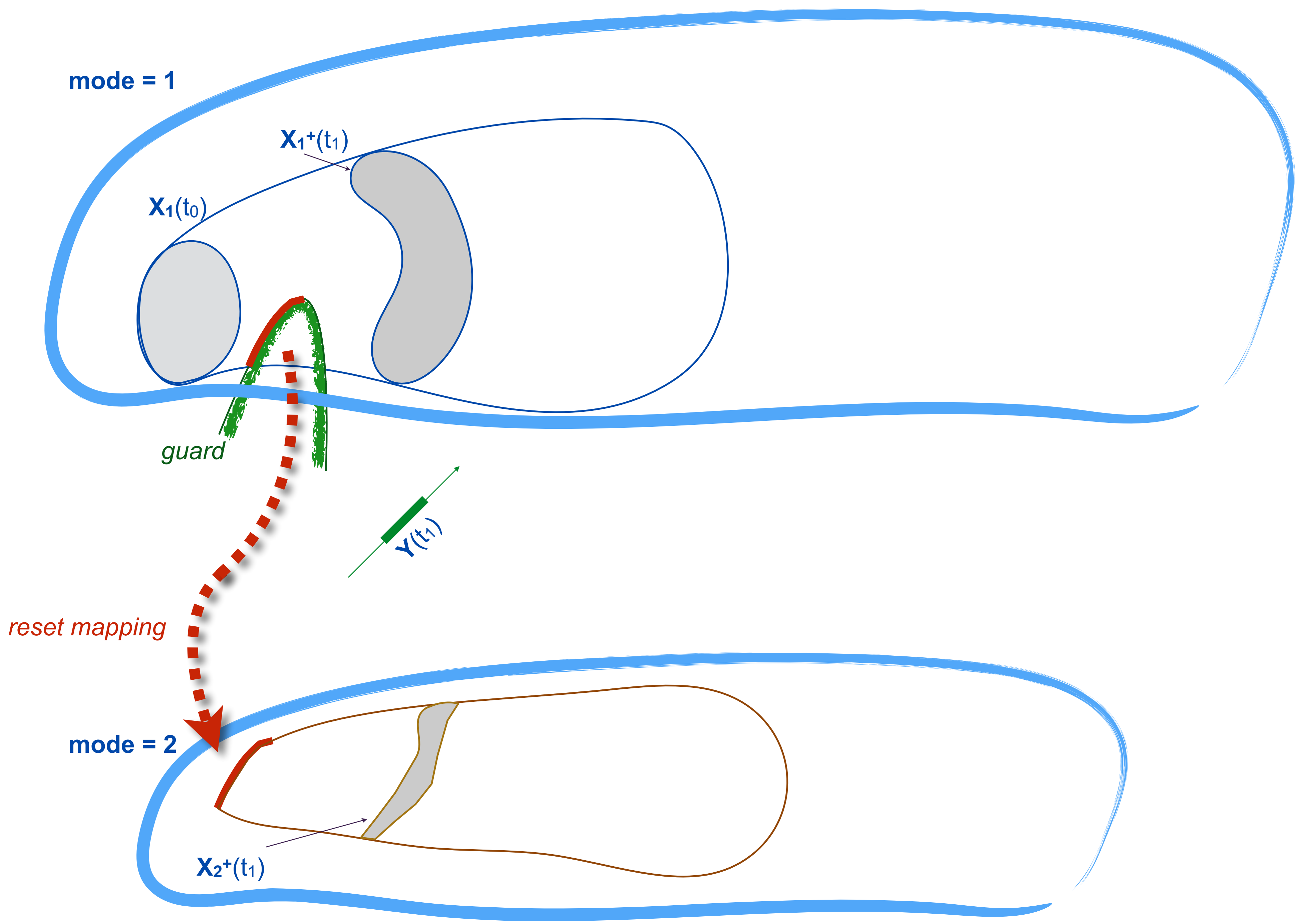
Reachability-based approach



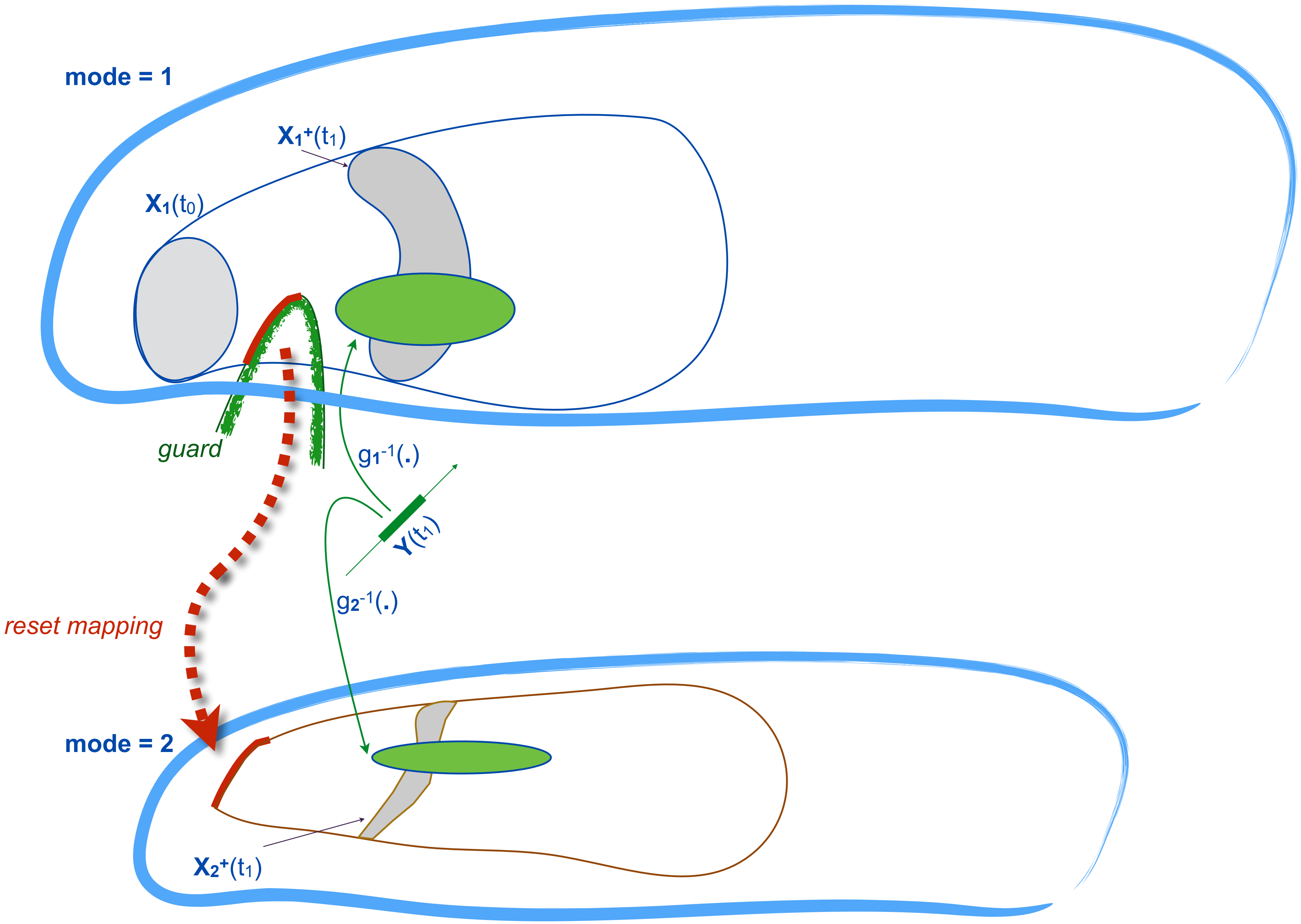
Reachability-based approach



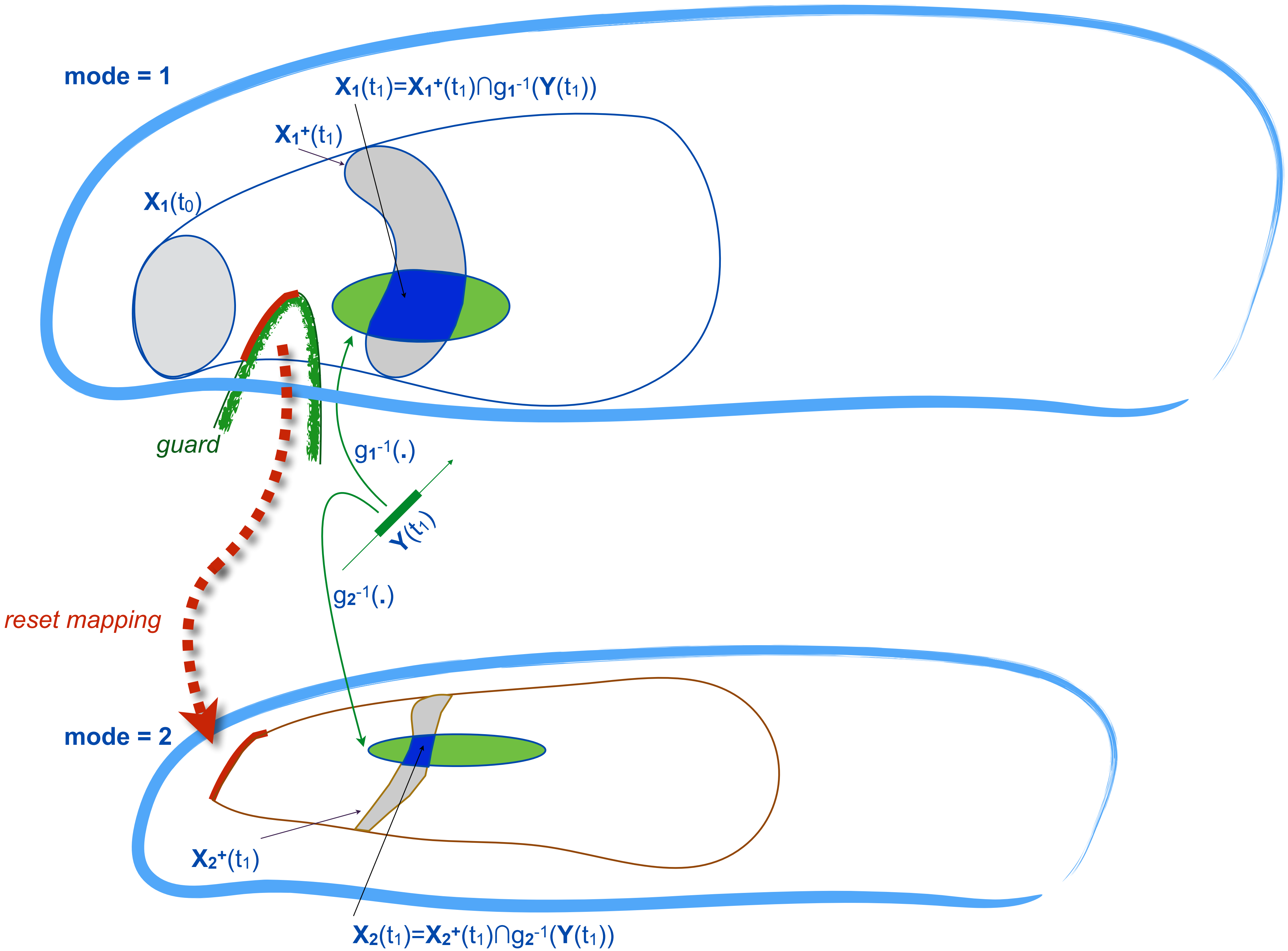
Reachability-based approach



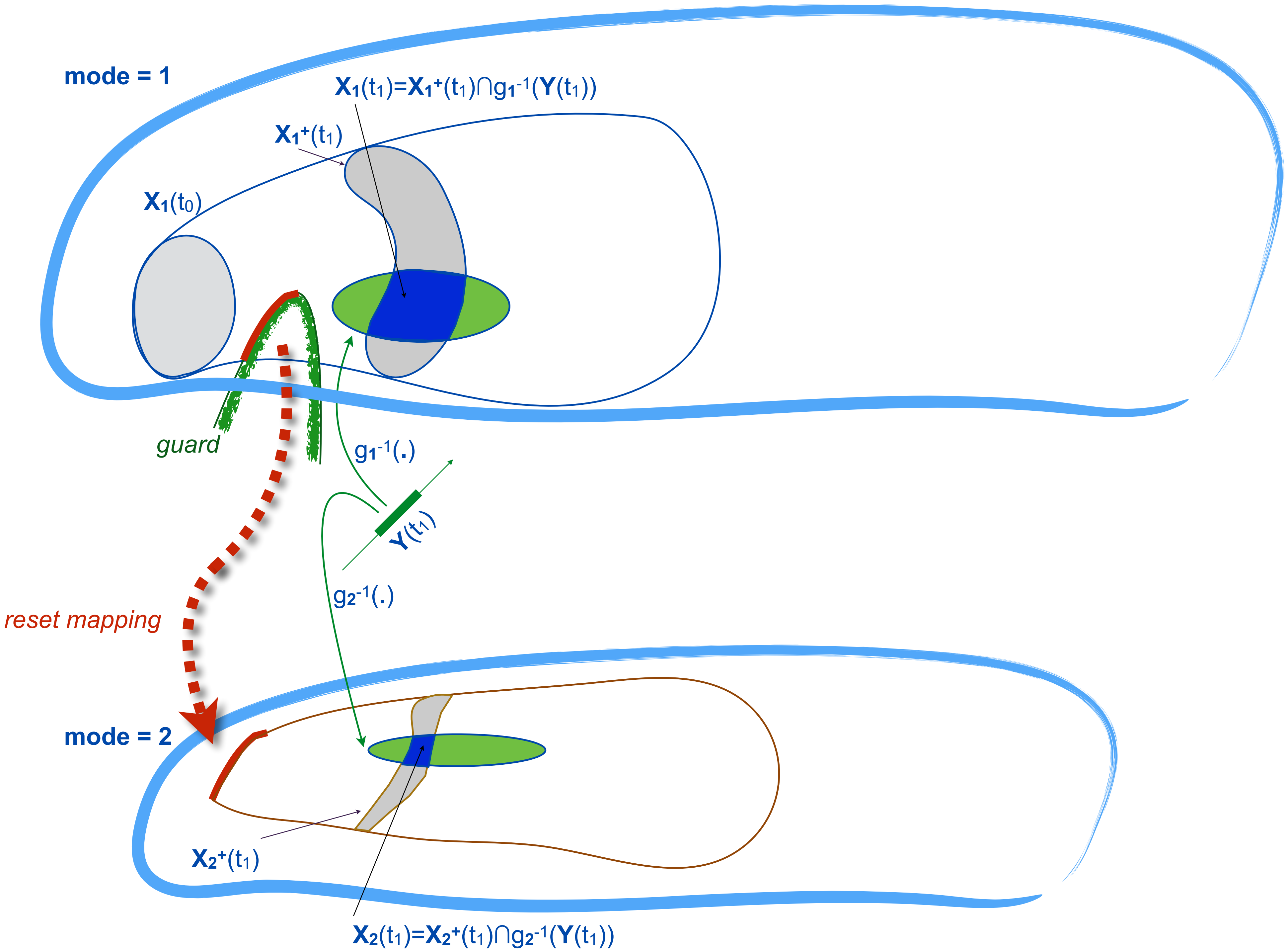
Reachability-based approach



Reachability-based approach



Reachability-based approach

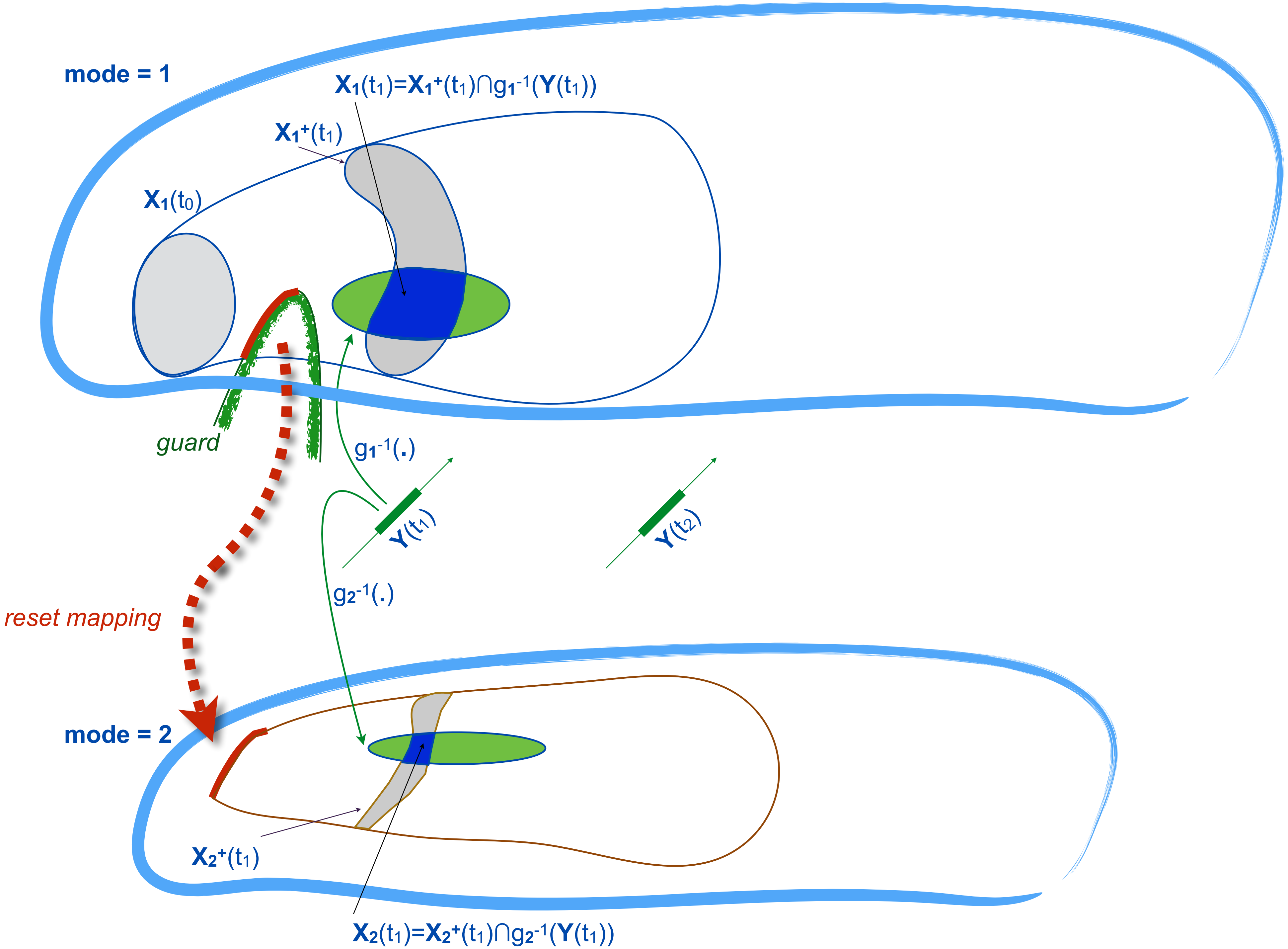


Reconstructed Hybrid State:

t_1

$\{q=1, \mathbf{X_1}(t_1)\} \cup \{q=2, \mathbf{X_2}(t_1)\}$

Reachability-based approach



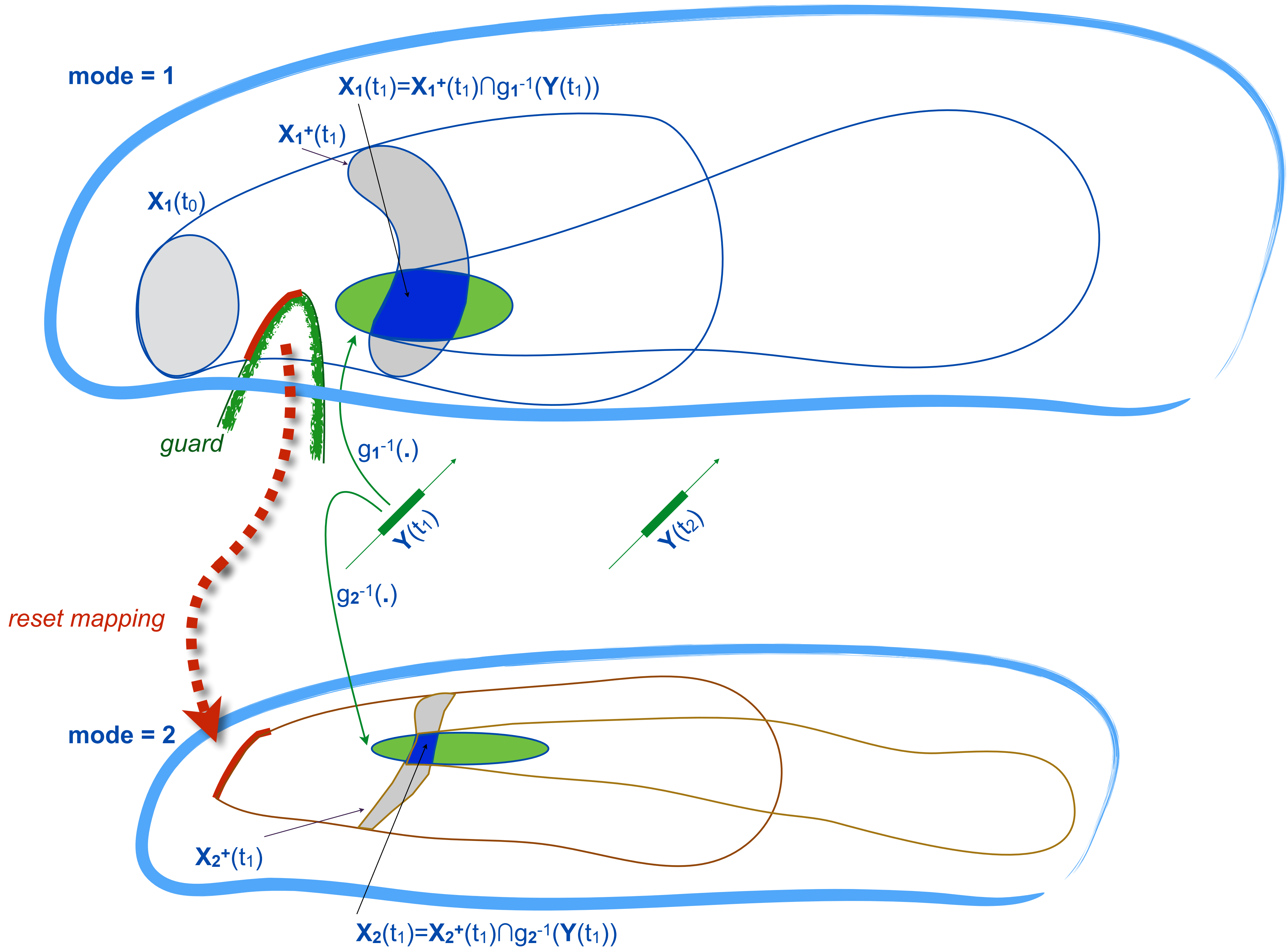
Reconstructed Hybrid State:

t_1

$\{q=1, \mathbf{X_1}(t_1)\} \cup \{q=2, \mathbf{X_2}(t_1)\}$

LAAS CNRS
PRISME
Laboratoire Pluridisciplinaire de Recherche
Ingénierie des Systèmes, Mécanique, Énergétique

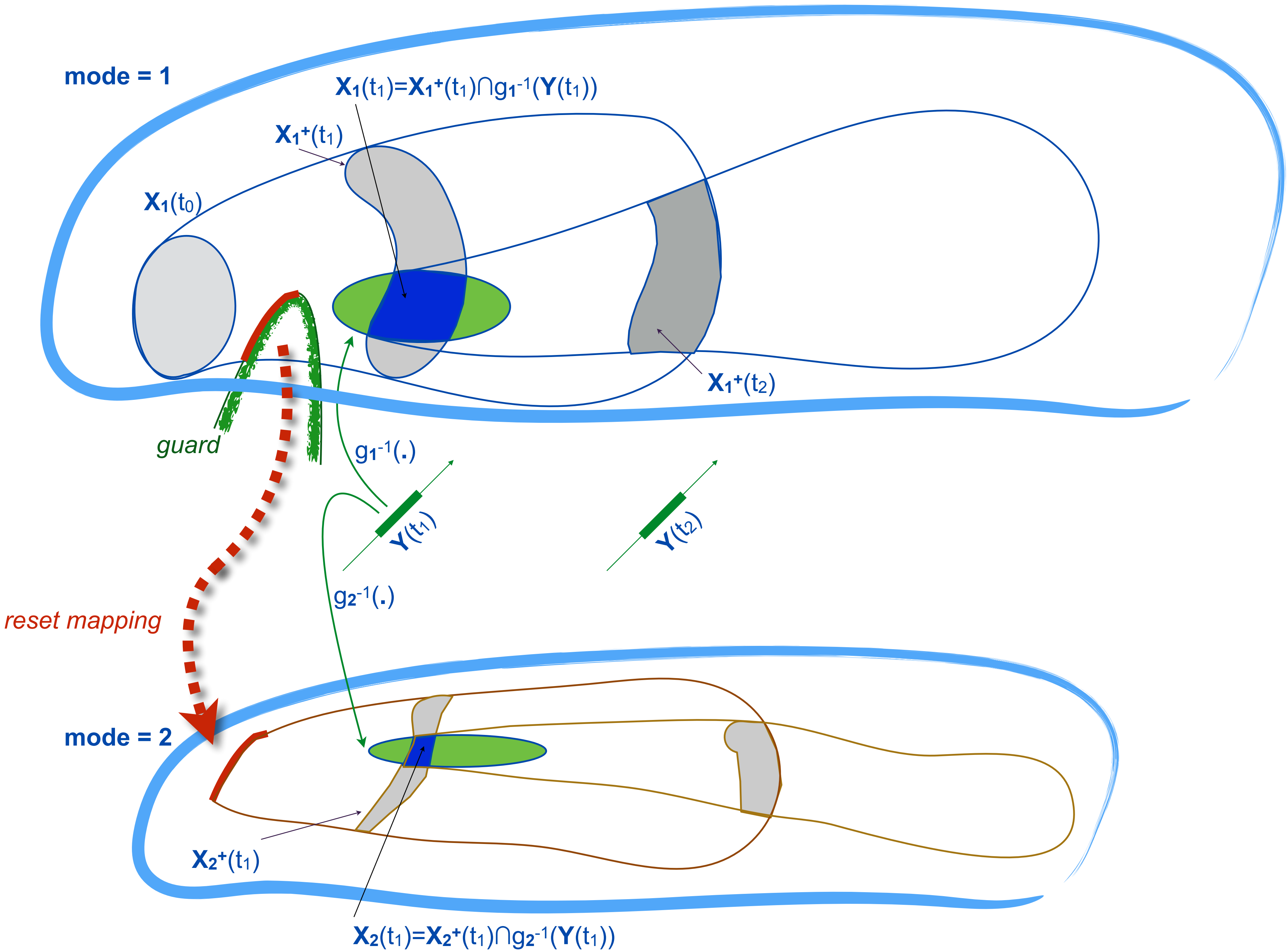
Reachability-based approach



Reconstructed Hybrid State:

t_1
$\{q=1, \mathbf{X_1}(t_1)\} \cup \{q=2, \mathbf{X_2}(t_1)\}$

Reachability-based approach

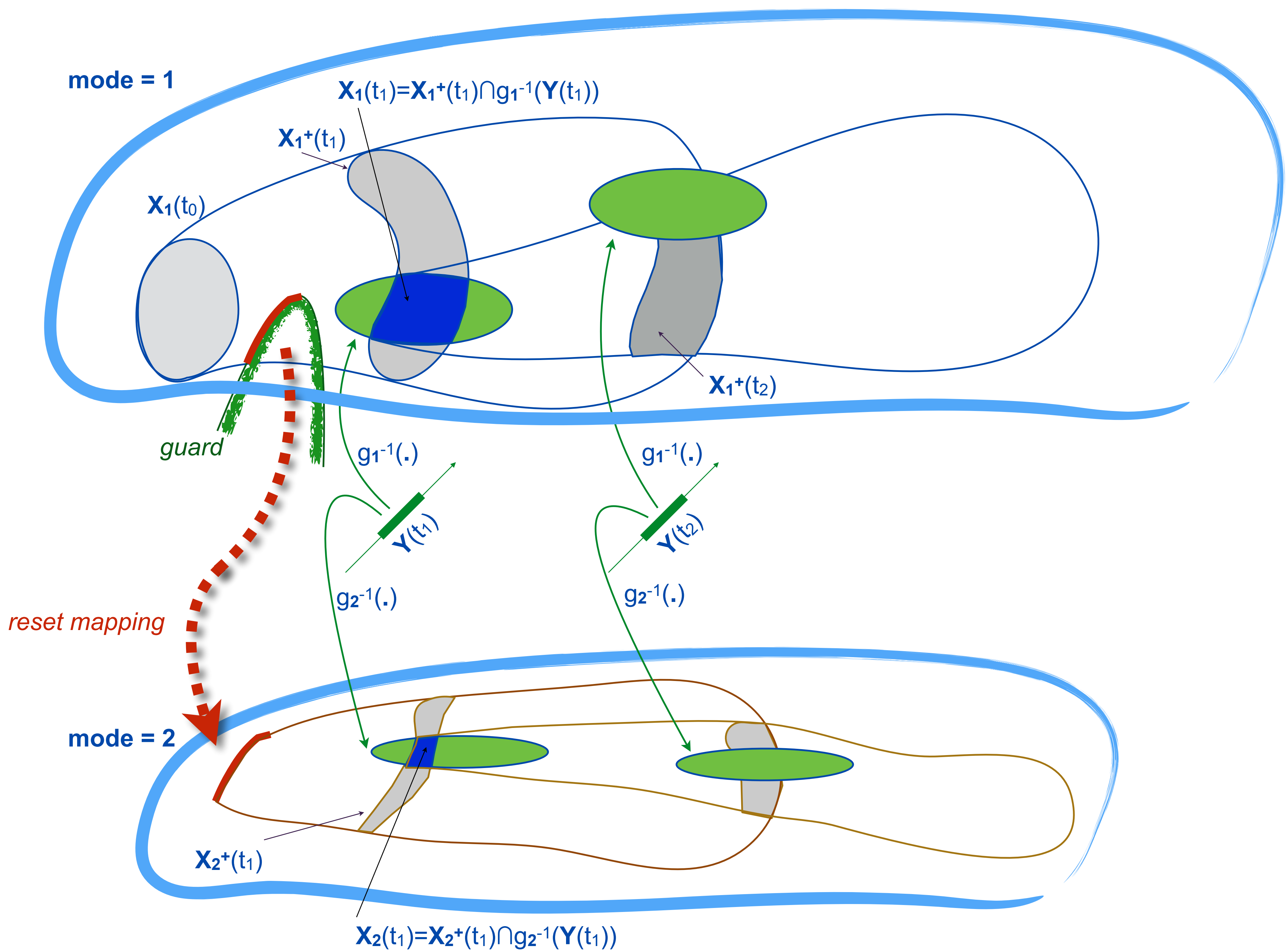


Reconstructed Hybrid State:

t_1

$\{q=1, \mathbf{X_1}(t_1)\} \cup \{q=2, \mathbf{X_2}(t_1)\}$

Reachability-based approach

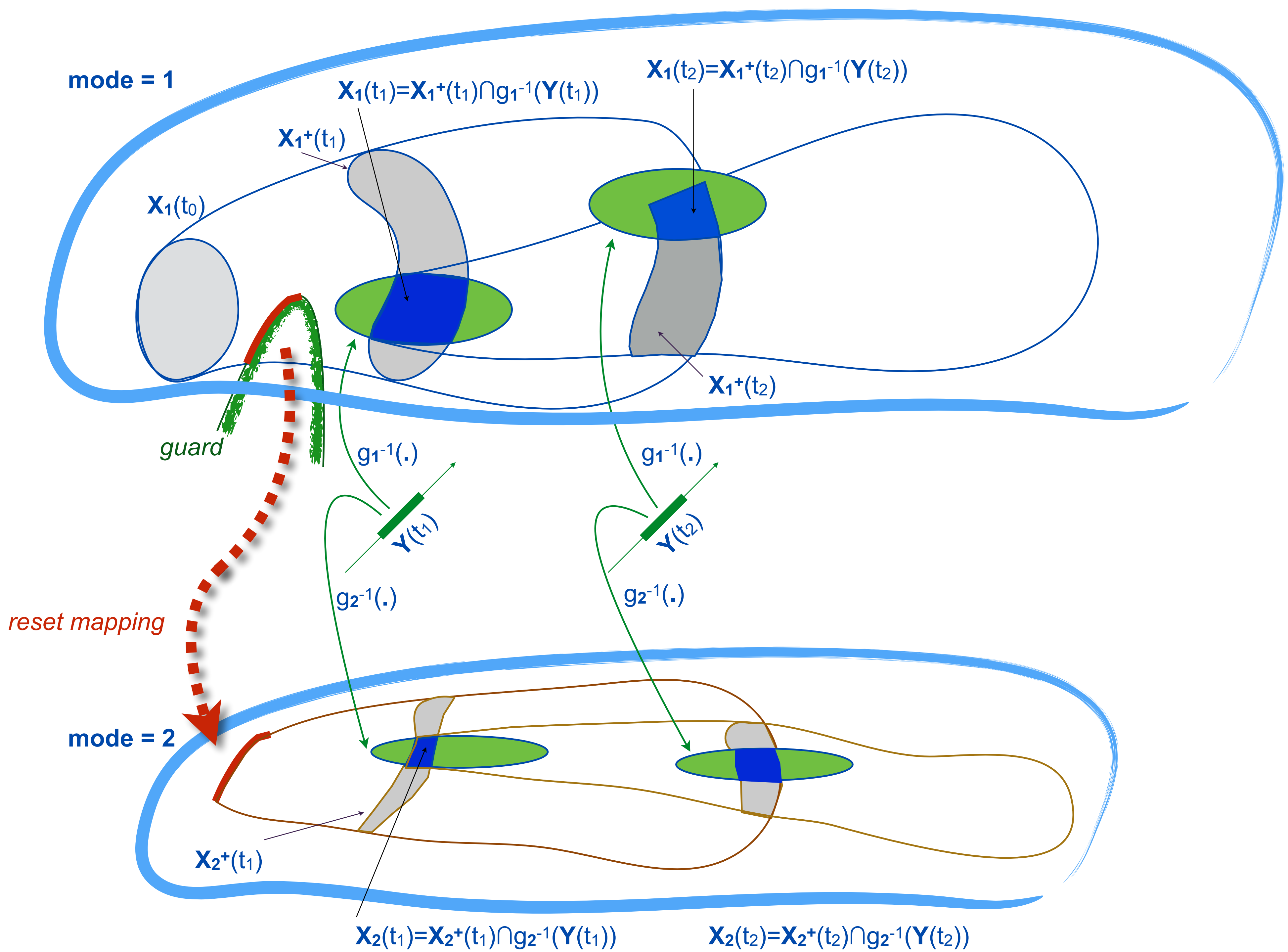


Reconstructed Hybrid State:

t_1

$\{q=1, \mathbf{X_1}(t_1)\} \cup \{q=2, \mathbf{X_2}(t_1)\}$

Reachability-based approach

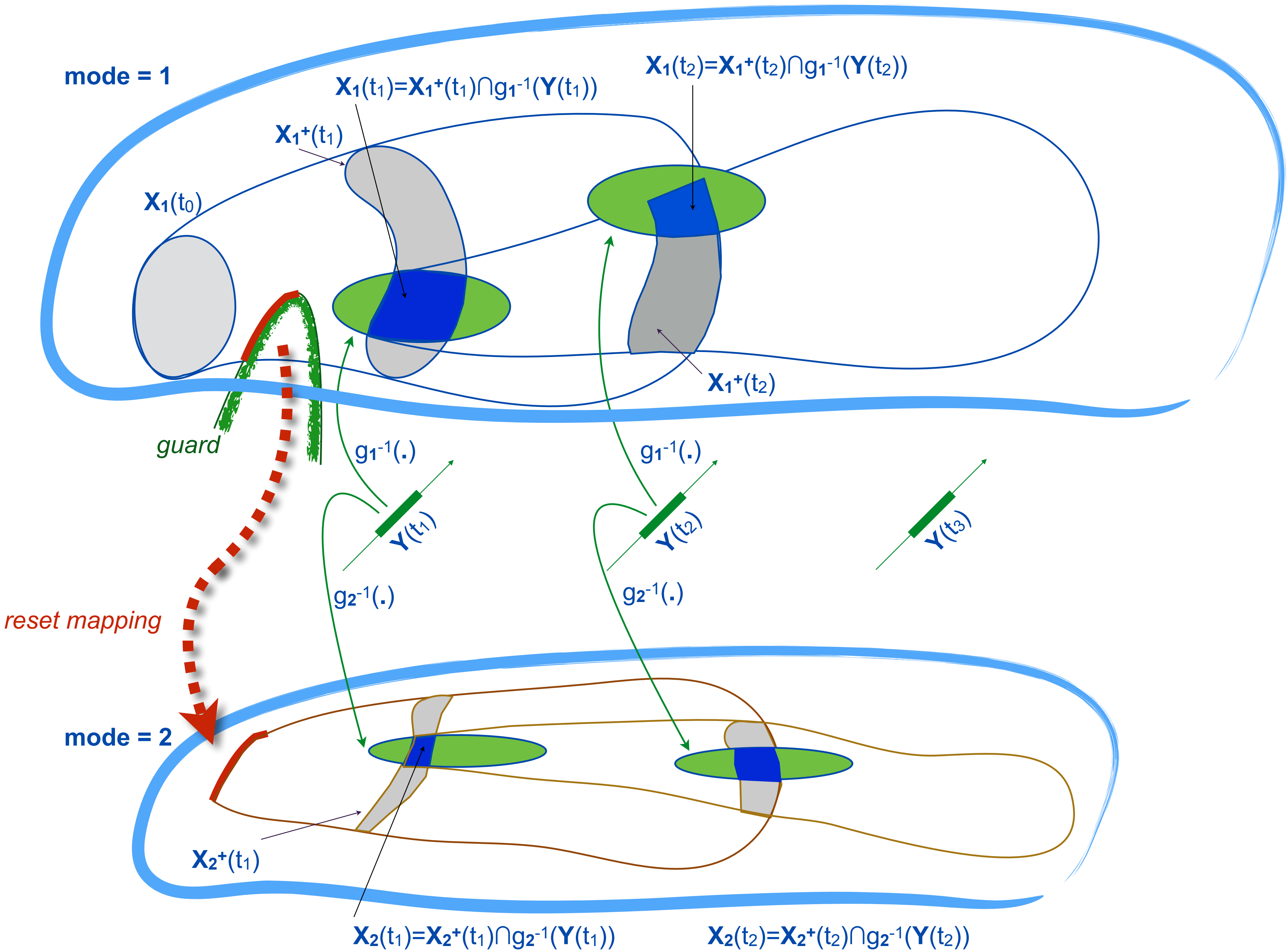


Reconstructed Hybrid State:

t_1
$\{q=1, \mathbf{X_1}(t_1)\} \cup \{q=2, \mathbf{X_2}(t_1)\}$

t_2
$\{q=1, \mathbf{X_1}(t_2)\} \cup \{q=2, \mathbf{X_2}(t_2)\}$

Reachability-based approach



Reconstructed Hybrid State:

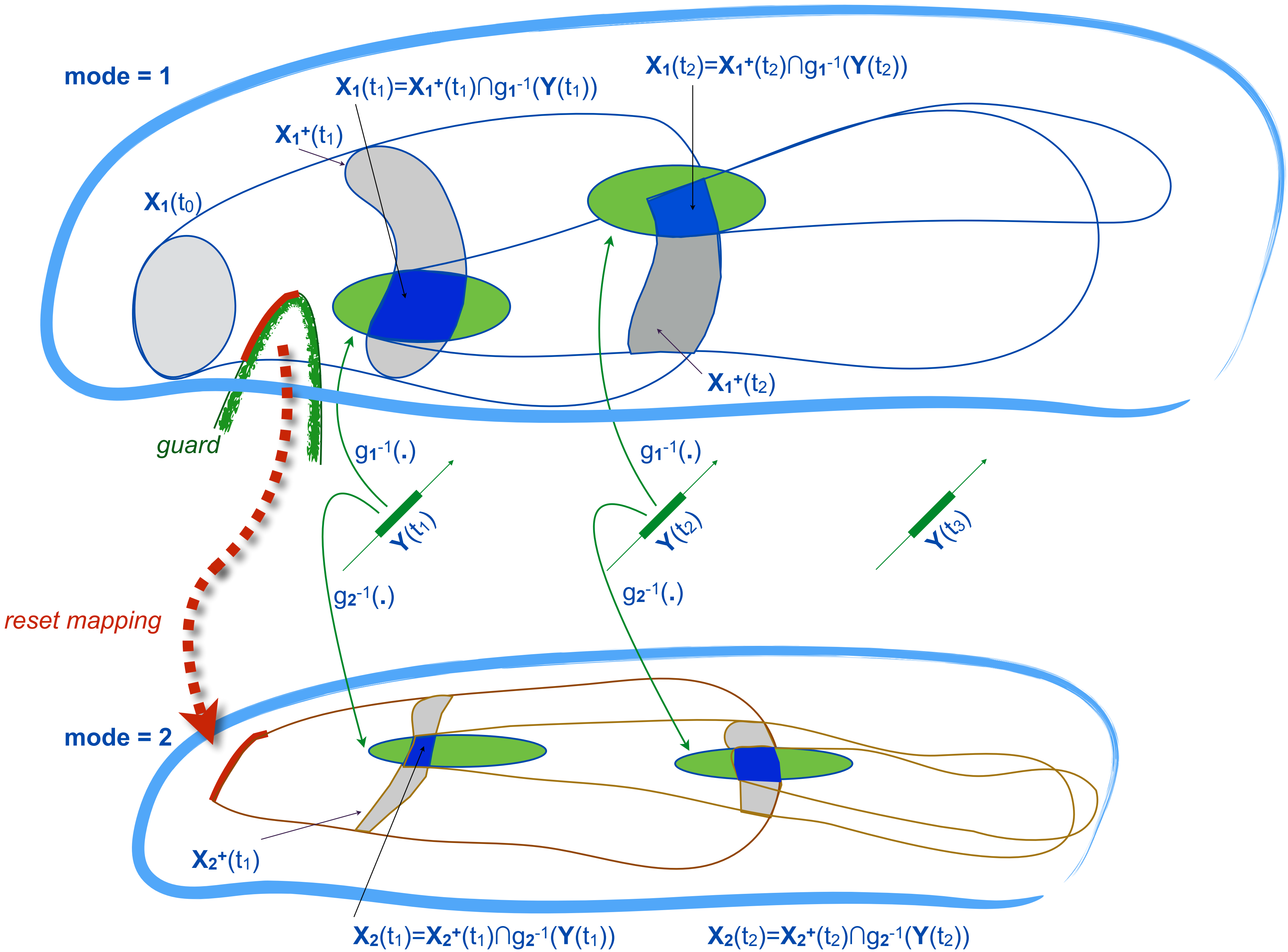
t_1

$\{q=1, \mathbf{X_1}(t_1)\} \cup \{q=2, \mathbf{X_2}(t_1)\}$

t_2

$\{q=1, \mathbf{X_1}(t_2)\} \cup \{q=2, \mathbf{X_2}(t_2)\}$

Reachability-based approach

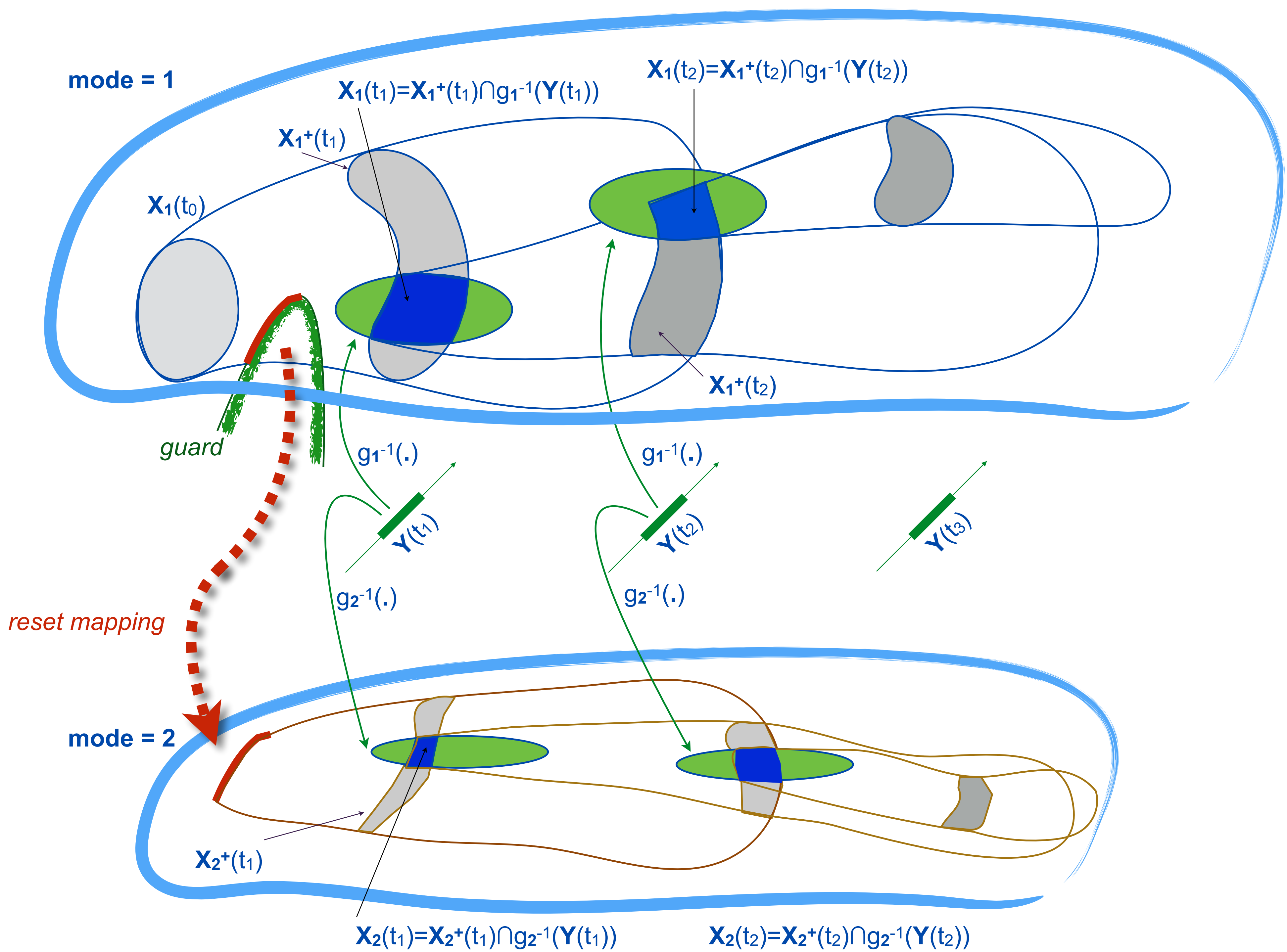


Reconstructed Hybrid State:

t_1
$\{q=1, X_1(t_1)\} \cup \{q=2, X_2(t_1)\}$

t_2
$\{q=1, X_1(t_2)\} \cup \{q=2, X_2(t_2)\}$

Reachability-based approach

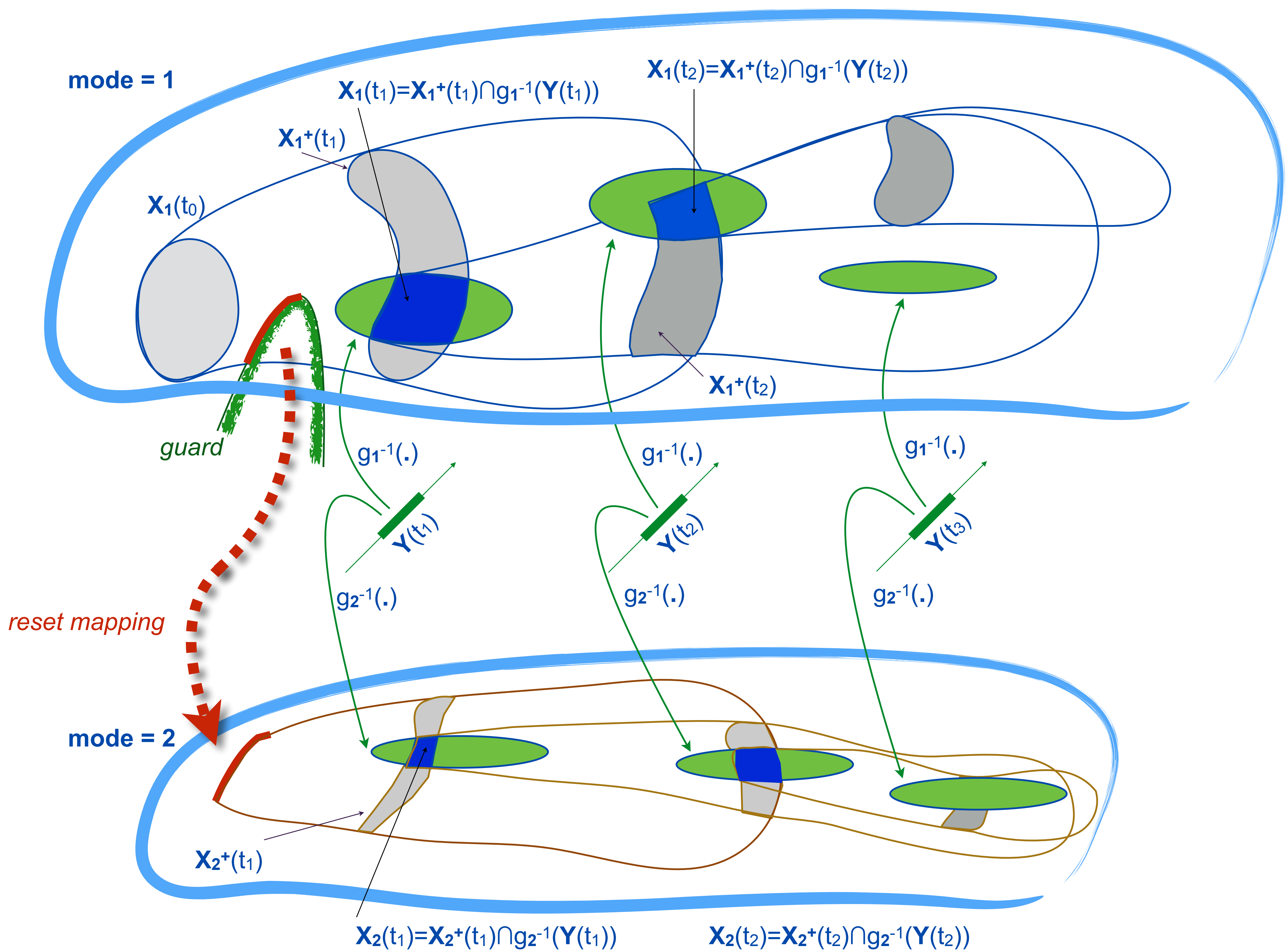


Reconstructed Hybrid State:

t_1
$\{q=1, X_1(t_1)\} \cup \{q=2, X_2(t_1)\}$

t_2
$\{q=1, X_1(t_2)\} \cup \{q=2, X_2(t_2)\}$

Reachability-based approach

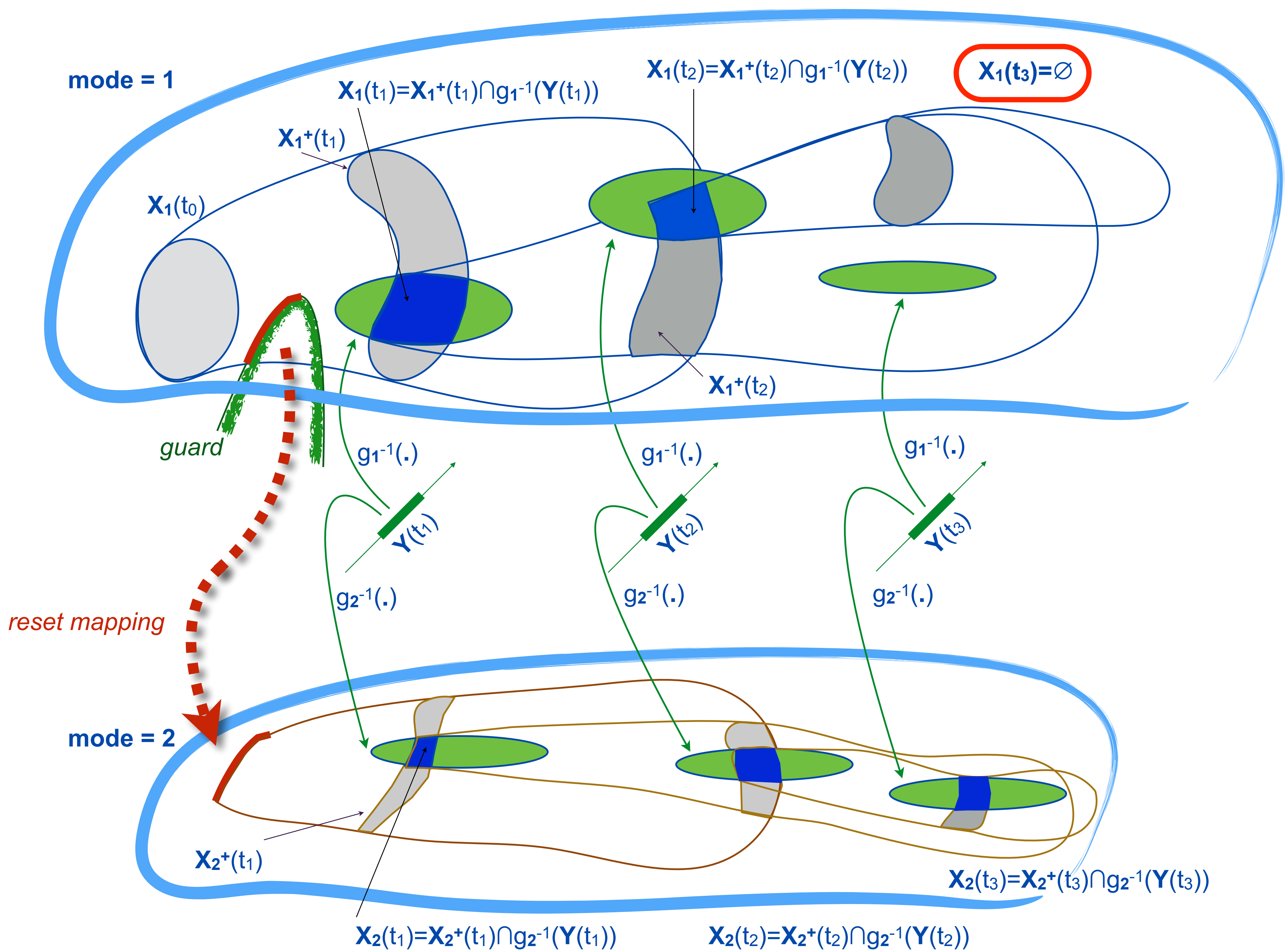


Reconstructed Hybrid State:

t_1
$\{q=1, \mathbf{X_1}(t_1)\} \cup \{q=2, \mathbf{X_2}(t_1)\}$

t_2
$\{q=1, \mathbf{X_1}(t_2)\} \cup \{q=2, \mathbf{X_2}(t_2)\}$

Reachability-based approach



Reconstructed Hybrid State:

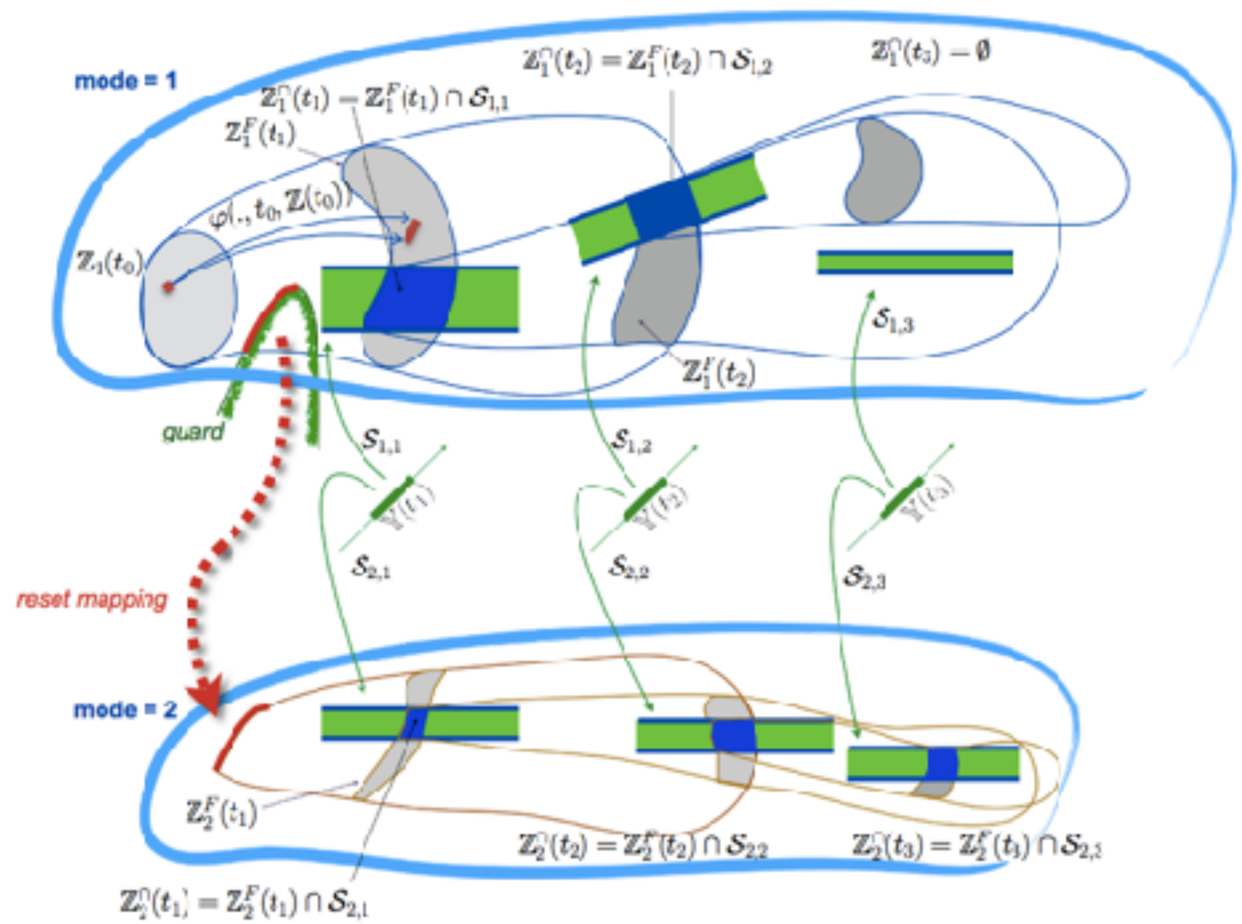
t_1
$\{q=1, X_1(t_1)\} \cup \{q=2, X_2(t_1)\}$

t_2
$\{q=1, X_1(t_2)\} \cup \{q=2, X_2(t_2)\}$

t_3
$\{q=2, X_2(t_3)\}$

t_{34}
…

Hybrid State Estimation



M. Maïga, N. Ramdani, L. Travé-Massuyès, C. Combastel,
***A comprehensive method for reachability analysis of uncertain nonlinear hybrid systems**, IEEE Transactions on Automatic Control, vol. 61, n.9, 2016. Pages 2341-2356*

N. Ramdani, L. Travé-Massuyès, C. Jauberthie.
Mode discernibility and bounded-error state estimation for nonlinear hybrid systems
Automatica, vol. 91, 2018. Pages 118–125

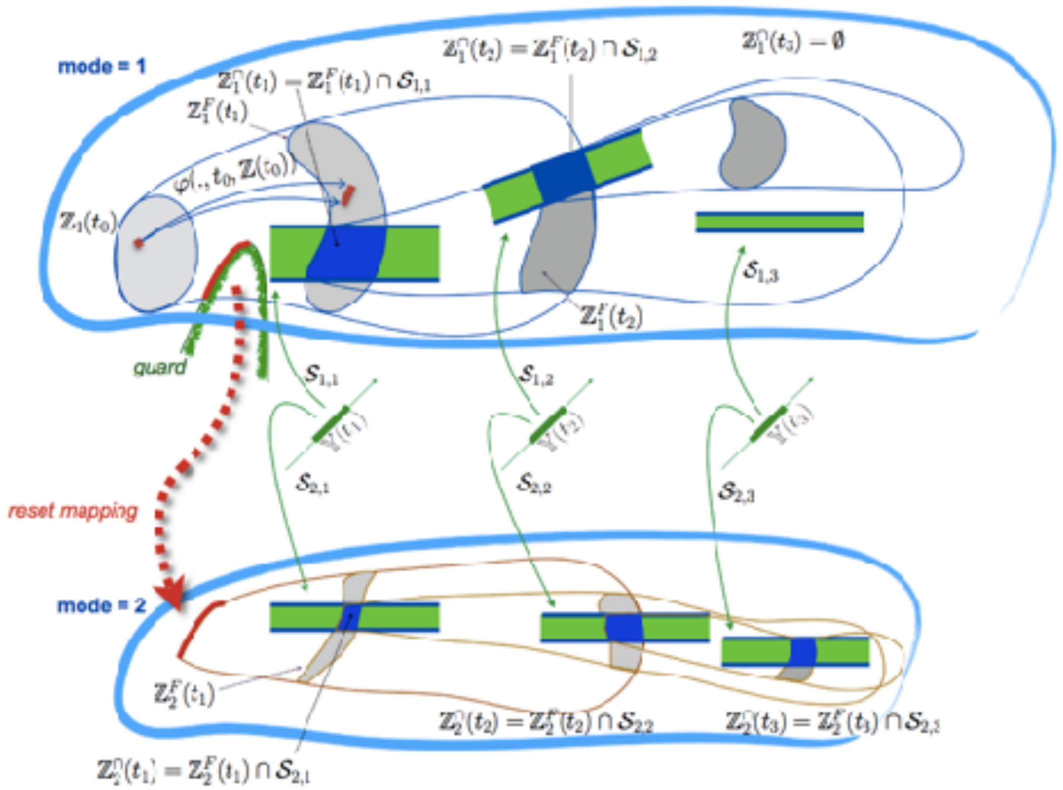
- **Mode discernibility**

Mode discernibility

Definition (Mode discernibility (Babaali & Pappas (2005))

Two different modes q_1 and q_2 are discernible over $T > 0$ if whenever $q([0,\ T]) = q_1$ and $q'([0,\ T]) = q_2$,

$$q_1 \neq q_2 \Rightarrow \exists u, \forall \chi_0, \forall \chi_0', y_q([0,\ T];\ 0, \chi_0, u) \neq y_{q'}([0,\ T];\ 0, \chi_0', u).$$



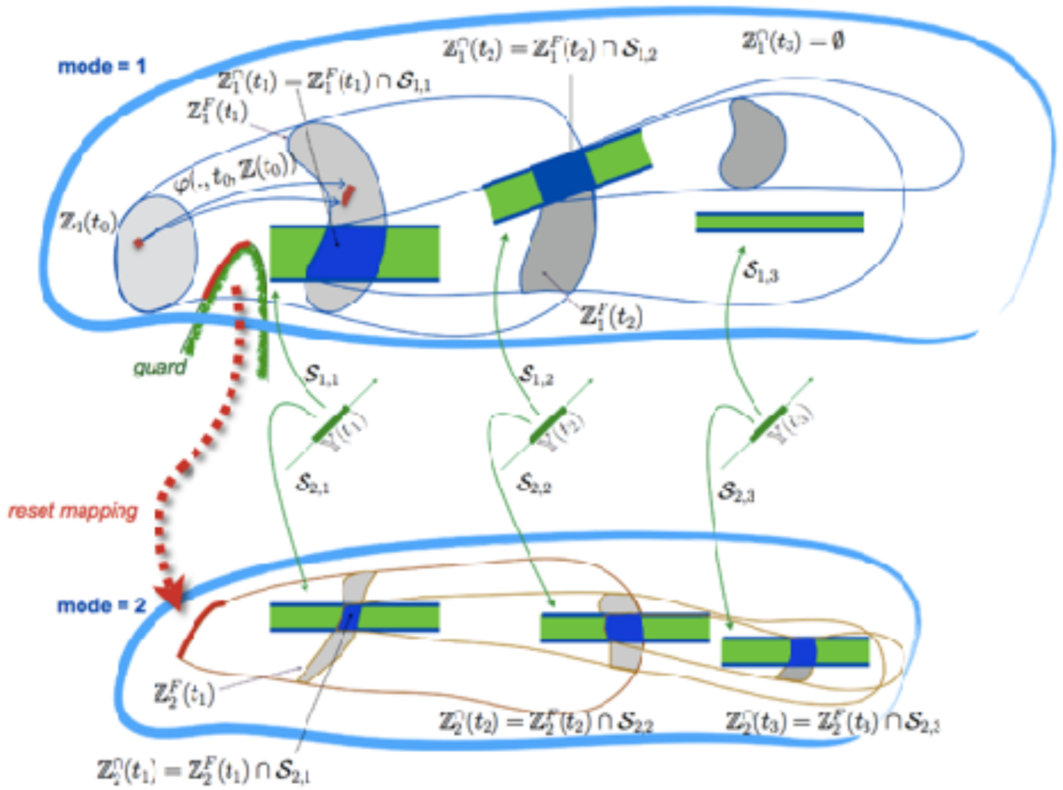
Mode discernibility

Collection of continuous models

mode $q \in \mathbb{Q}$

$$\begin{cases} \text{flow}(q): \dot{z}(t) = f_q(z(t), u(t)), \\ \text{output}(q): y(t) = \mu_q^\top z(t), \end{cases}$$

where $\mu_q \in \mathbb{R}^{n \times n_y}$.



LAAS CNRS
PRISME Laboratoire Pluridisciplinaire de Recherche Ingénierie des Systèmes, Mécanique, Énergétique

Mode discernibility

The **composite continuous model**, $s \in \mathbb{Q}$: $(n_q = |\mathbb{Q}|)$

$$\dot{z}(t) = \mathcal{F}(z(t), s, u(t)) = \sum_{i=1}^{n_q} \frac{\prod_{j=1, j \neq i}^{n_q} (s - q_j)}{\prod_{j=1, j \neq i}^{n_q} (q_i - q_j)} f_{q_i}(z, u), \qquad (1)$$

$z_s(t, t_0, z_0, u) =$ solution of IVP ODE (1)...

The **composite output model** :

$$y(t) = \mathcal{Y}_s(t, t_0, z_0, u) = \sum_{i=1}^{n_q} \frac{\prod_{j=1, j \neq i}^{n_q} (s - q_j)}{\prod_{j=1, j \neq i}^{n_q} (q_i - q_j)} \eta_{q_i}^{\top} z_s(t, t_0, z_0, u). \qquad (2)$$

Mode discernibility

Theorem (Mode discernibility (Ramdani, Travé-Massuyès, Jauberthie, 2018))

If the scalar parameter s *in system* (1)-(2) *is identifiable, then the hybrid modes* q_i $i \in \mathbb{Q}$ *are discernible.*

Parameter identifiability

Let us consider a controlled dynamical system described by:

$$\dot{z} = \mathfrak{f}(z, p, u), \tag{3}$$

$$y = \mathfrak{g}(z, p), \tag{4}$$

where :

- $z(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$, $p \in \mathbb{P} \subseteq \mathbb{R}^{n_p}$,

The mappings $\mathfrak{f}$ and $\mathfrak{g}$ are real, analytic and infinitely differentiable on $\mathbb{M}$, where $\mathbb{M}$ is an open set of $\mathbb{R}^n$.

Parameter identifiability

Definition (Ljung and Glad 1994)

The parameter p_i *of model* (3)-(4) *is globally identifiable if there exists* $u(t) \in \mathbb{R}^{n_u}$ *such that for all* $(\hat{p}_i, p_i^*) \in \mathbb{P}^2$, $\hat{p}_i \neq p_i^*$*:*

$$(\forall t \in [0, T],\ y(t, \hat{p}_i, u) = y(t, p_i^*, u)) \Rightarrow (\hat{p}_i = p_i^*),$$

and the parameter vector p *is globally identifiable in* $\mathbb{P}$ *if all its components* p_i *are globally identifiable in* $\mathbb{P}^{n_p}$*.*

Checking identifiability

Method based on differential algebra (Kolchin and al., 1973)

- elimination order $\{p\} < \{y, u\} < \{x\}$ ($\Rightarrow$ eliminate unmeasured state variables),
- Rosenfeld-Groebner algorithm = elimination algorithm (Boulier et al., 1997),

Checking identifiability

Method based on differential algebra (Kolchin and al., 1973)

- elimination order $\{p\} < \{y, u\} < \{x\}$
  ($\Rightarrow$ eliminate unmeasured state variables),
- Rosenfeld-Groebner algorithm $=$ elimination algorithm
  (Boulier et al., 1997),

Regular differential chain

$\Rightarrow$ relations between inputs, outputs and parameters:

$$\mathscr{R}_i(y, u, p) = m_0^i(y, u) + \sum_{k=1}^{n_i} \theta_k^i(p) m_k^i(y, u), \quad i = 1, \ldots, n_y$$

$\rightarrow$ Exhaustive summary of $\mathscr{R}_i$:
$(\theta_k^i)_{1 \leq k \leq n_i}$ are rational in p, $\theta_\alpha^i \neq \theta_\beta^i$ $(\alpha \neq \beta)$,

$\rightarrow$ $(m_k^i)_{1 \leq k \leq n_i}$ are differential polynomials with respect to y and u and $m_0^i \neq 0$.

Wronskian

Consider $\Delta R(y,u)$ that denotes the **functional determinant** formed from the $\{m_k(y,u)\}_{1\leq k\leq \bar{n}}$ and given by the the **Wronskian**

$$\Delta R(y,u) = \left| \begin{pmatrix} m_1(y,u) & \dots & m_n(y,u) \\ m_1(y,u)^{(1)} & \dots & m_n(y,u)^{(1)} \\ & \ddots & \\ m_1(y,u)^{(\bar{n}-1)} & \dots & m_n(y,u)^{(\bar{n}-1)} \end{pmatrix} \right|$$

Proposition :

If $\Delta R(y,u) \neq 0$ then $\{m_k(y,u)\}_k$ are linearly independent.

Parameter identifiability

Theorem (Denis-Vidal et al. 2001)

Assume that the functional determinant $\Delta R(y, u)$ is not identically equal to zero. If the mapping

$$\phi : p \mapsto (\theta_1(p), \ldots, \theta_n(p))$$

is injective then the parameter p is globally identifiable.

Note: If $n_y \geq 1$, the corresponding $(\theta_k^i)_{1 \leq k \leq n_i}$ must be added to the image of the function ϕ.

Mode discernibility

Theorem (Mode discernibility (Ramdani, Travé-Massuyès, Jauberthie, 2018))

If the scalar parameter s in system (1)-(2) *is identifiable, then the hybrid modes q_i $i \in \mathbb{Q}$ are discernible.*

Note:

- The identifiability condition of this theorem applies to the parameter s, which is scalar.
- This theorem does not consider mode invariants that may be used to discriminate two different modes. It is thus not a necessary condition for mode discernibility.

LAAS CNRS

PRISME
Laboratoire Pluridisciplinaire de Recherche
Ingénierie des Systèmes, Mécanique, Énergétique

- **Example**

State Estimation

Hybrid Mass-Spring

- Unknown initial mode.

- Composite model :

$$\begin{cases} \dot{x}_1(t) &= x_2(t) - x_4(t), \\ \dot{x}_2(t) &= \frac{s-q_1}{q_0-q_1}(-\kappa_1 x_1(t)) \\ &\quad + \frac{s-q_0}{q_1-q_0}(-\kappa_1 x_1(t) - \kappa_2 x_2(t) + \kappa_2 x_4(t)), \\ \dot{x}_3(t) &= x_4(t), \\ \dot{x}_4(t) &= \frac{s-q_1}{q_0-q_1}(\kappa_3 x_1(t) - \kappa_5 x_3(t) - \kappa_6 x_4(t)) \\ &\quad + \frac{s-q_0}{q_1-q_0}(\kappa_3 x_1(t) + \kappa_4 x_2(t) - \kappa_5 x_3(t) \\ &\quad - (\kappa_4 + \kappa_6) x_4(t)). \end{cases}$$

$$\begin{cases} y_1(t) &= x_1(t), \\ y_2(t) &= x_3(t). \end{cases}$$

Mode discernibility

Hybrid Mass-Spring

- **Regular Differential Chain**

Mode discernibility

Hybrid Mass-Spring

Regular Differential Chain

$$\mathscr{R}_i(y,u,p) = m_0^i(y,u) + \sum_{k=1}^{n_i} \theta_k^i(p) m_k^i(y,u), \quad i = 1,\ldots,n_y$$

$$\begin{cases} R_1^{\Sigma_{q_0/q_1}}(y,s) = & (q_0 - q_1)\ddot{y}_1(t) + q_0(\kappa_2 + \kappa_4)\dot{y}_1(t) \\ & +(q_0 - q_1)(\kappa_1 + \kappa_3)y_1(t) \\ & +(q_1 - q_0)\kappa_6\dot{y}_2(t) + (q_1 - q_0)\kappa_5 y_2(t) \\ & -s\,(\kappa_4 - \kappa_2)\dot{y}_1(t), \\ R_2^{\Sigma_{q_0/q_1}}(y,s) = & (q_0 - q_1)\ddot{y}_2(t) + (q_0 - q_1)\kappa_6\dot{y}_2(t) \\ & +(q_0 - q_1)\kappa_5 y_2(t) + (q_1 - q_0)\kappa_3 y_1(t) \\ & -\kappa_4 q_0 \dot{y}_1(t)\kappa_6 + s\,\kappa_4\dot{y}_1(t)\kappa_6. \end{cases}$$

LAAS CNRS
PRISME
Laboratoire Pluridisciplinaire de Recherche
Ingénierie des Systèmes, Mécanique, Énergétique

Mode discernibility

- **Hybrid Mass-Spring**
  - **Wronskien non-vanishing if** $\dot{y}_1(t) \neq 0$
  - **Exhaustive summary**

$$\theta_1^1(s) = s$$

$$\theta_1^2(s) = s$$

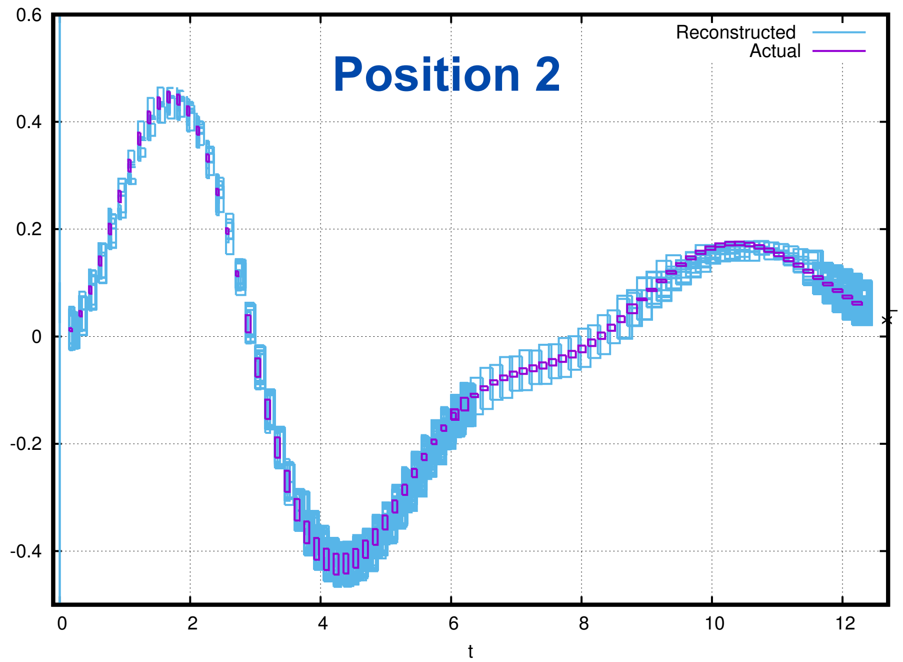
$\phi : p \mapsto (\theta_1(p), \ldots, \theta_n(p))$ **is bijective.**

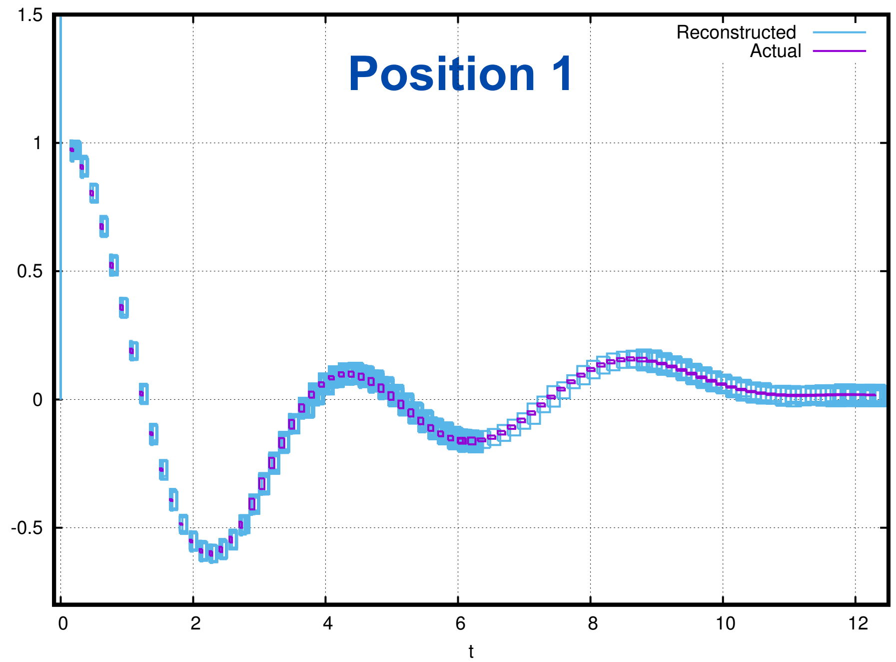
then the modes are discernible.

State Estimation

Hybrid Mass-Spring

- **Unknown initial mode.** ***CPU time 242s***

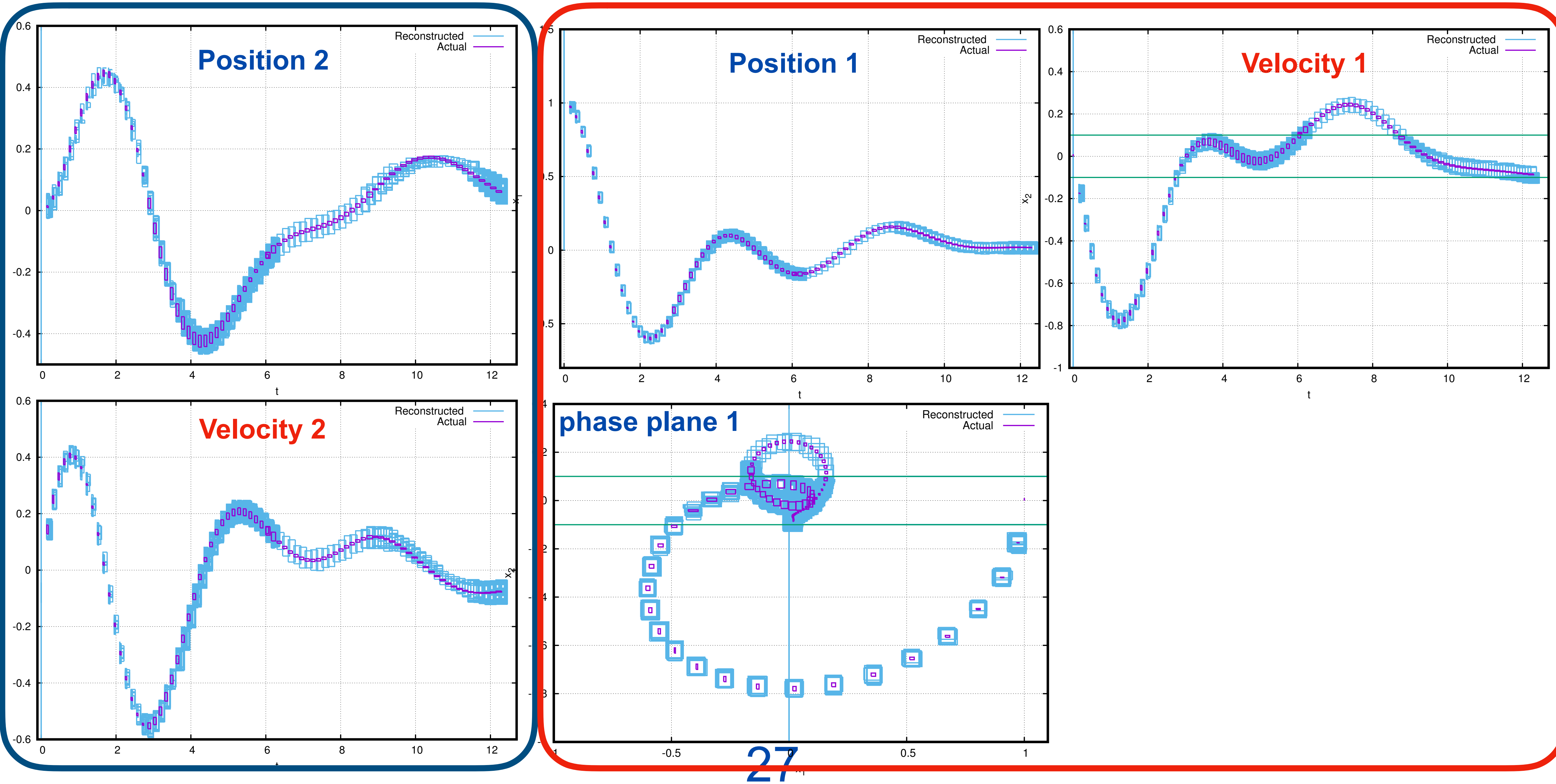




State Estimation

Hybrid Mass-Spring

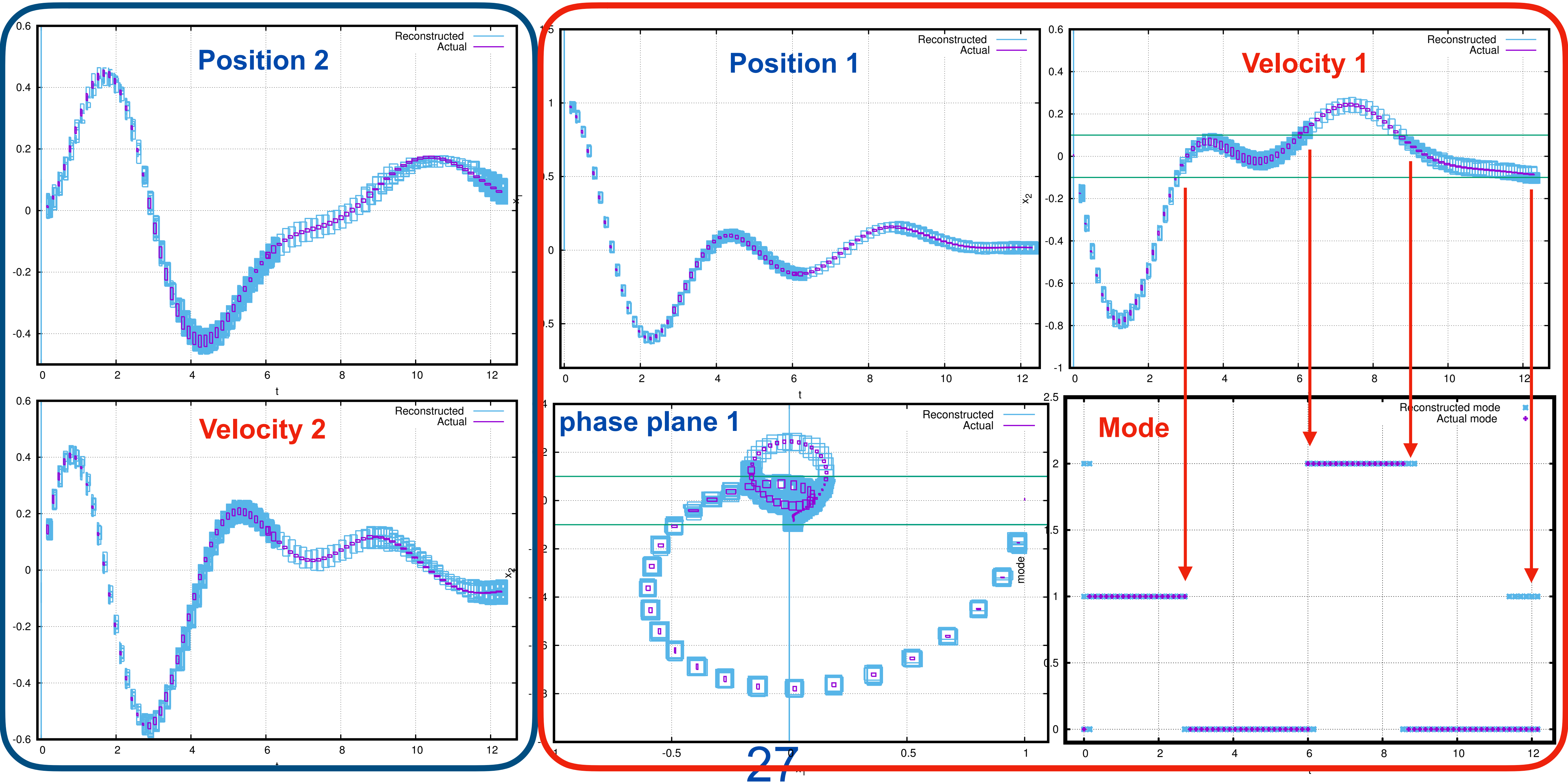
- **Unknown initial mode.** ***CPU time 242s***



State Estimation

Hybrid Mass-Spring

- Unknown initial mode. *CPU time 242s*

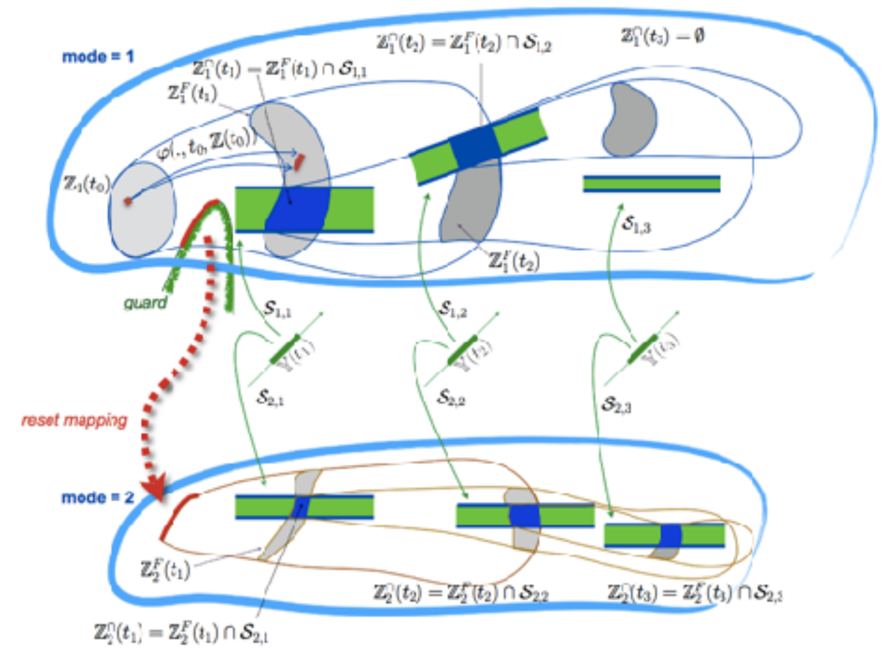


- **Future work**

Research directions

- Explore extension to robust estimation with sporadic or self-triggered sampling.
- Further the methods for embedded FDI.

Selected References



- *N. Ramdani, L. Travé-Massuyès, & C. Jauberthie.* ***Mode discernibility and bounded-error state estimation for nonlinear hybrid systems****, Automatica 91, 2018. 118-125*
- *M. Maïga, N. Ramdani, L. Travé-Massuyès, & C. Combastel,* ***A comprehensive method for reachability analysis of uncertain nonlinear hybrid systems****, IEEE Transactions on Automatic Control, vol. 61, n.9, 2016. pp. 2341-2356*
- *L. Denis-Vidal, G. Joly-Blanchard, G., & C. Noiret, C.* ***Some effective approaches to check identifiability of uncontrolled nonlinear systems,*** *Mathematics and Computers in Simulation, 57, 2001, pp. 35–44.*

- **Thank you !**