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A Set-Membership Approach to Parametric Synthesis of Reliable Stabilising Controllers

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Outline

1. Control Synthesis
2. Reachability Computation
3. Example
4. Conclusion

Outline

Nonlinear dynamical system

$$\Sigma : \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad \mathbf{x}(t_0) \in \mathbb{X}_0 \in \mathbb{R}^n$$

Template for the control law

$$\mathbf{u} = \mathbf{K}(\mathbf{x}, \mathbf{k})$$

Stabilisation around point vector $\mathbf{x}_{op}$

Synthesis using simplified model $\dot{\mathbf{x}} \approx \widehat{\mathbf{f}}(\mathbf{x}, \mathbf{u}) \longrightarrow \widehat{\mathbf{k}}$.

Certificates

- Validate nominal control parameter vector $\widehat{\mathbf{k}}$ on actual model $\mathbf{f}(.,.)$
- Robustness margin for the control parameter vector

Closed-loop system

$$\Sigma_{cl}(\mathbf{k}): \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{K}(\mathbf{x}, \mathbf{k})) \equiv \mathbf{f}_{cl}(\mathbf{x}, \mathbf{k}), \quad \mathbf{x}(t_0) \in \mathbb{X}_0 \in \mathbb{R}^n$$

Specifications on $\mathbf{k}$

$\mathbf{x}_{op}$: a stable operating point. $\mathcal{V}_\tau(\mathbf{x}_{op})$: a given **target set**.

$\Rightarrow$ $\mathcal{V}_\tau(\mathbf{x}_{op})$ is positive invariant set.

$\Rightarrow$ $\mathcal{V}_\tau(\mathbf{x}_{op})$ is finite time reachable, settling time $t_0 \leq t_s \leq \tau$

$$\forall \mathbf{x}_0 \in \mathbb{X}_0, \forall t \geq \tau, \mathbf{x}(t, \mathbf{x}_0, t_0, \mathbf{k}) \in \mathcal{V}_\tau(\mathbf{x}_{op})$$

Set of stabilising controller parameter vectors

$$\mathbb{K}_\tau = \{\mathbf{k} \mid \forall \mathbf{x}_0 \in \mathbb{X}_0, \forall t \geq \tau, \mathbf{x}(t, \mathbf{x}_0, t_0, \mathbf{k}) \in V_\tau(\mathbf{x}_{op})\}$$

Local asymptotic stability

Linearisation of $\Sigma_{cl}(\mathbf{k})$ at $\mathbf{x}_{op}$ is asymptotically stable

$\Rightarrow \Sigma_{cl}(\mathbf{k})$ is asymptotically stable.

Theorem (Berman & Plemmons, 1994)

Metzler matrix $\mathbf{M}$ is stable if $\exists \mathbf{d} > 0, \exists \mathbf{b} > 0, \mathbf{Md} \leq -\mathbf{b}$

LAS([k]). Local asymptotic stability for $[\mathbf{k}] = [\underline{\mathbf{k}}, \overline{\mathbf{k}}]$

$$\exists \mathbf{M}_k = \mathbf{M}(\underline{\mathbf{k}}, \overline{\mathbf{k}}) \text{ Metzler, s.t. } \forall \mathbf{k} \in [\mathbf{k}], \nabla_x \mathbf{f}_{cl}(\mathbf{x}_{op}, \mathbf{k}) \leq \mathbf{M}_k$$

If Metzler matrix $\mathbf{M}_k$ is stable, then $\Sigma_{cl}(\mathbf{k})$ is stable for all $\mathbf{k} \in [\mathbf{k}]$.

Solution set at time τ

$$\mathcal{X}_\tau(\mathbf{k}) = \{\mathbf{x}(\tau, \mathbf{x}_0, t_0, \mathbf{k}),\ \mathbf{x}_0 \in \mathbb{X}_0\}$$

Finite-time reachability for parameter vector $\mathbf{k}$

$\mathcal{X}_\tau(\mathbf{k}) \subseteq V_\tau(\mathbf{x}_{op}) \Rightarrow \mathcal{V}_\tau(\mathbf{x}_{op})$ is finite time reachable by $\Sigma_{cl}(\mathbf{k})$

FTReach([k]). Finite-time reachability for [k]

$\mathcal{X}_\tau([\mathbf{k}]) \subseteq V_\tau(\mathbf{x}_{op}) \Rightarrow \mathcal{V}_\tau(\mathbf{x}_{op})$ is FTReach by $\Sigma_{cl}(\mathbf{k})\ \forall \mathbf{k} \in [\mathbf{k}]$

FTUnReach([k]). Finite-time unreachability for [k]

$\mathcal{X}_\tau([\mathbf{k}]) \cap V_\tau(\mathbf{x}_{op}) = \emptyset \Rightarrow \mathcal{V}_\tau(\mathbf{x}_{op})$ is FTUnReach by $\Sigma_{cl}(\mathbf{k})\,\forall \mathbf{k} \in [\mathbf{k}]$

Feasible parameter box [k]

If **LAS**([**k**]) and **FTReach**([**k**]) $\Rightarrow [\mathbf{k}] \subseteq \mathbb{K}_\tau$

UnFeasible parameter box [k]

If **FTUnReach**([**k**]) $\Rightarrow [\mathbf{k}] \cap \mathbb{K}_\tau = \emptyset$

SM BnB algorithm

Algorithm 1: Control Synthesis

input : $\mathbb{X}_0$, f_{cl}, ϵ, $\mathbb{K}_0$
output: List = Inner approximation $\mathbb{K}$

Running list of boxes $\mathcal{L} \leftarrow [\mathbb{K}_0]$;
while $\mathcal{L} \neq \emptyset$ **do**
    pick first box $[\mathbf{k}]$ from the list;
    if FTUnReach($[\mathbf{k}]$) **then**
        discard box;
    else if FTReach($[\mathbf{k}]$) $\wedge$ LAS($[\mathbf{k}]$) **then**
        store box $[\mathbf{k}]$ in list $\mathbb{K}$;
    else if width($[\mathbf{k}]$) $\leq \epsilon$;
     then
        discard box ;
    else
        bisect $[\mathbf{k}]$ and store new boxes in $\mathcal{L}$;
    end if
end while

Outline

Set integration

$$\Sigma_{cl}(\mathbf{k}): \quad \dot{\mathbf{x}} = \mathbf{f}_{cl}(\mathbf{x}, \mathbf{k}), \quad \mathbf{x}(t_0) \in \mathbb{X}_0 \in \mathbb{R}^n, \mathbf{k} \in [\mathbf{k}]$$

Interval Taylor methods (N.S.Nedialkov, R.Rihm, R.J.Lohner, ...)

$t_0 < t_1 < \ldots < \tau < \ldots, \; h_j = t_{j+1} - t_j$.

$$[\mathbf{x}_{j+1}] = [\mathbf{x}_j] + \sum_{i=1}^{m-1} h_j^i \mathbf{f}_{cl}^{[i]}([x_j]) + h_j^k \mathbf{f}_{cl}^{[m]}([\tilde{x}_j])$$

Comparison theorems for differential inequalities

Use Müller's theorem (Ramdani 2009) and build bracketing systems

$$\begin{cases} \dot{\overline{\mathbf{x}}} = \overline{\mathbf{f}}_{cl}(\underline{\mathbf{x}}, \overline{\mathbf{x}}, \underline{\mathbf{k}}, \overline{\mathbf{k}}) \\ \dot{\underline{\mathbf{x}}} = \underline{\mathbf{f}}_{cl}(\underline{\mathbf{x}}, \overline{\mathbf{x}}, \underline{\mathbf{k}}, \overline{\mathbf{k}}), \quad \mathbf{dim} = 2n, \quad \mathbf{x}(t) \in [\underline{\mathbf{x}}(t), \overline{\mathbf{x}}(t)] \end{cases}$$

Dual set integration

$$\Sigma_{cl}(\mathbf{k}): \quad \dot{\mathbf{x}} = \mathbf{f}_{cl}(\mathbf{x}, \mathbf{k}), \quad \mathbf{x}(t_0) \in \mathbb{X}_0 \in \mathbb{R}^n, \mathbf{k} \in [\mathbf{k}]$$

$(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}, \quad (\mathbf{x}_{10}, \mathbf{x}_{20}) = \mathbf{x}_0 \in \mathbb{X}_0$

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2), \quad \mathbf{x}_{10} \in [\mathbf{x}_{10}] \tag{1}$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2), \quad \mathbf{x}_{20} \in [\mathbf{x}_{20}] \tag{2}$$

Assumption A1

It exists a bounding method that, when applied to the vector field $\mathbf{f}_1$, generates an enclosure of bounded width for the reachable set of (1).

Assumption A2

The diagonal entries of Jacobian matrix of $\mathbf{f}_2$ are strictly negative

$$\forall j \in \{1, \ldots, n_2\}, \ \frac{\partial f_{2j}}{\partial x_{2j}} < 0 \tag{3}$$

Dual set integration

$$\Sigma_{cl}(\mathbf{k}): \quad \dot{\mathbf{x}} = \mathbf{f}_{cl}(\mathbf{x}, \mathbf{k}), \quad \mathbf{x}(t_0) \in \mathbb{X}_0 \in \mathbb{R}^n, \, \mathbf{k} \in [\mathbf{k}]$$

$$\dot{\overline{\mathbf{x}}}_1 = \mathbf{f}_1(\overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2), \quad \overline{\mathbf{x}}_{10} = \mathrm{Sup}([\mathbf{x}_{10}]) \tag{4}$$

$$\dot{\underline{\mathbf{x}}}_1 = \mathbf{f}_1(\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2), \quad \underline{\mathbf{x}}_{10} = \mathrm{Inf}([\mathbf{x}_{10}]) \tag{5}$$

$$\dot{\mathbf{x}}_2 \in \mathbf{f}_2([\mathbf{x}_1], \mathbf{x}_2), \quad \mathbf{x}_{20} \in [\mathbf{x}_{20}] \tag{6}$$

Theorem (Meslem & Ramdani, IMA MCI 34(1) 2017)

Consider (1)-(2) that fulfills A1 and A2. Assume for a given integration step-size h_j, it exists solutions $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 > 0$

$$\begin{aligned} \left\| \mathbf{I_2} + h_j \mathbf{J_{22}}(\mathbf{f}_2; [\mathbf{x}_j]) \right\| &\leq \beta_1 & (7) \\ h_j \| \mathbf{J}_{21}(\mathbf{f}_2; [\mathbf{x}_j]) \| &\leq \beta_2 & (8) \\ h_j^2 \mathrm{w}\left(\mathbf{f}_2^{[2]}([\tilde{\mathbf{x}}_j])\right) &\leq \beta_3 \mathrm{w}([\mathbf{x}_j]) & (9) \\ \beta_1 + \beta_2 + \beta_3 &\leq 1 & (10) \end{aligned}$$

where $\mathbf{J}_2 = (\mathbf{J}_{21}, \mathbf{J}_{22})$. *Then, for integration step-size h_j, one obtains*

(i) $\mathrm{w}([\mathbf{x}_{2,j+1}]) \leq \mathrm{w}([\mathbf{x}_j])$, *(the latter obtained using interval Taylor method),*

(ii) $[\mathbf{x}_{1,j+1}] = [\underline{\mathbf{x}}_{1,j+1}, \overline{\mathbf{x}}_{1,j+1}]$
is a tight solution set of $\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_{1,j} \in [\mathbf{x}_{1,j}]$

Proof sketch

- Second-order Taylor expansion + Mean value form
- Linear mapping and width of intervals
- Assumptions A1 and A2.

Dual set integration - Illustrative example

$$\dot{\mathbf{x}} = \begin{bmatrix} -a_1 & -a_2 & 0 & 0 & 0 \\ a_3 & -a_4 & 1 & 0 & 0 \\ 0 & 0 & -a_5 & a_6 & 0 \\ 0 & 0 & -a_7 & -a_8 & 0 \\ 0 & 0 & 0 & 0 & -a_9 \end{bmatrix} \mathbf{x} + \begin{bmatrix} b_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

Decomposition:

$$\dot{\mathbf{x}}_1 = \begin{bmatrix} -a_5 & a_6 & 0 \\ -a_7 & -a_8 & 0 \\ 0 & 0 & -a_9 \end{bmatrix} \mathbf{x}_1$$

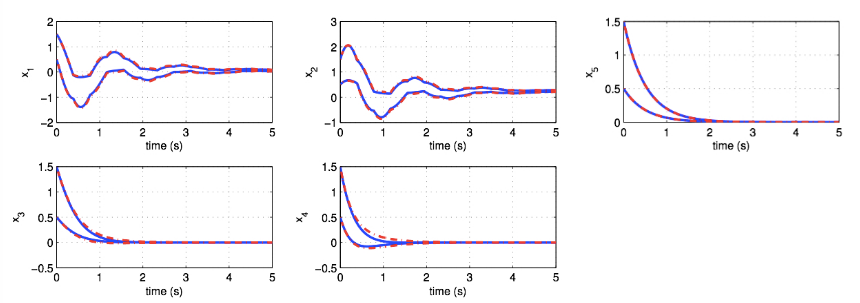
$$\dot{\mathbf{x}}_2 = \begin{bmatrix} -a_1 & -a_2 \\ a_3 & -a_4 \end{bmatrix} \mathbf{x}_2 + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} u$$

Dual set integration - Illustrative example

- Assumption A1 is satisfied.
  System ($\mathbf{f}_1$) is not monotone but associated Metzler matrix is stable.

$$\mathbf{e} = \overline{\mathbf{x}} - \underline{\mathbf{x}} \quad \Rightarrow \quad \dot{\mathbf{e}}_1 = \begin{bmatrix} -a_5 & a_6 & 0 \\ a_7 & -a_8 & 0 \\ 0 & 0 & -a_9 \end{bmatrix} \mathbf{e}_1$$

- Assumption A2 is satisfied.



Outline

- Drosophila circadian rhythms described by Goldbeter model (Angeli & Sontag, 2008)

Nonlinear dynamical system

$$
\begin{aligned}
\dot{x}_1 &= u - \frac{v_m x_1}{\gamma_m + x_1} \\
\dot{x}_2 &= \gamma_s x_1 - \frac{v_1 x_2}{\gamma_1 + x_2} + \frac{v_2 x_3}{\gamma_2 + x_3} \\
\dot{x}_3 &= \frac{v_1 x_2}{\gamma_1 + x_2} - \frac{v_2 x_3}{\gamma_2 + x_3} - \frac{v_3 x_3}{\gamma_3 + x_3} + \frac{v_4 x_4}{\gamma_4 + x_4} \\
\dot{x}_4 &= \frac{v_3 x_3}{\gamma_3 + x_3} - \frac{v_4 x_4}{\gamma_4 + x_4} - \Gamma_1 x_4 + \Gamma_2 x_5 - \frac{v_d x_4}{\gamma_d + x_4} \\
\dot{x}_5 &= \Gamma_1 x_4 - \Gamma_2 x_5
\end{aligned}
$$

The template for the proposed stabilizing controller

$$
u = \frac{k_1}{k_2^n + x_5^n}, \quad n = 4
$$

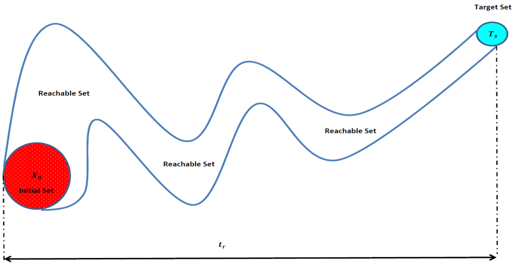
Specifications

$$\mathbb{K}_\tau = \{\mathbf{k} \mid \forall \mathbf{x}_0 \in \mathbb{X}_0, \forall t \geq \tau, \mathbf{x}(t, \mathbf{x}_0, t_0, \mathbf{k}) \in V_\tau(\mathbf{x}_{op})\}$$

$$\mathbb{X}_0 = [0.1, 0.4] \times [0.6, 2.4] \times [0.85, 3.4] \times [0.25, 1] \times [0.5, 2]$$

$\tau = 50s$

$V_\tau(\mathbf{x}_{op}) = [0.78, 0.82] \times [0.29, 0.32] \times [0.15, 0.22] \times [0.08, 0.11] \times [0.10, 0.15]$

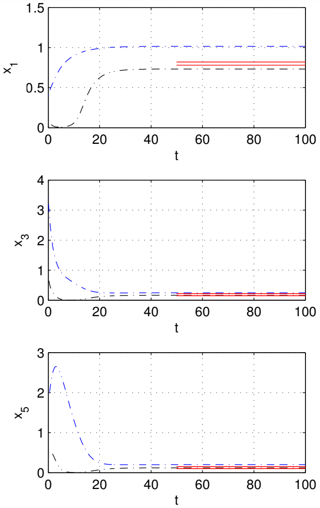


$$\left\{\begin{array}{rcl} \dot{\overline{x}}_1 & = & \frac{\overline{k}_1}{\underline{k}_2^n + \underline{x}_5^n} - \frac{v_m \overline{x}_1}{\gamma_m + \overline{x}_1} \\ \dot{\overline{x}}_2 & = & \gamma_s \overline{x}_1 - \frac{v_1 \overline{x}_2}{\gamma_1 + \overline{x}_2} + \frac{v_2 \overline{x}_3}{\gamma_2 + \overline{x}_3} \\ \dot{\overline{x}}_3 & = & \frac{v_1 \overline{x}_2}{\gamma_1 + \overline{x}_2} - \frac{v_2 \overline{x}_3}{\gamma_2 + \overline{x}_3} - \frac{v_3 \overline{x}_3}{\gamma_3 + \overline{x}_3} + \frac{v_4 \overline{x}_4}{\gamma_4 + \overline{x}_4} \\ \dot{\overline{x}}_4 & = & \frac{v_3 \overline{x}_3}{\gamma_3 + \overline{x}_3} - \frac{v_4 \overline{x}_4}{\gamma_4 + \overline{x}_4} - \Gamma_1 \overline{x}_4 + \Gamma_2 \overline{x}_5 - \frac{v_d \overline{x}_4}{\gamma_d + \overline{x}_4} \\ \dot{\overline{x}}_5 & = & \Gamma_1 \overline{x}_4 - \Gamma_2 \overline{x}_5 \end{array}\right. \quad (11)$$

$$\left\{\begin{array}{rcl} \dot{\underline{x}}_1 & = & \frac{\underline{k}_1}{\overline{k}_2^n + \overline{x}_5^n} - \frac{v_m \underline{x}_1}{\gamma_m + \underline{x}_1} \\ \dot{\underline{x}}_2 & = & \gamma_s \underline{x}_1 - \frac{v_1 \underline{x}_2}{\gamma_1 + \underline{x}_2} + \frac{v_2 \underline{x}_3}{\gamma_2 + \underline{x}_3} \\ \dot{\underline{x}}_3 & = & \frac{v_1 \underline{x}_2}{\gamma_1 + \underline{x}_2} - \frac{v_2 \underline{x}_3}{\gamma_2 + \underline{x}_3} - \frac{v_3 \underline{x}_3}{\gamma_3 + \underline{x}_3} + \frac{v_4 \underline{x}_4}{\gamma_4 + \underline{x}_4} \\ \dot{\underline{x}}_4 & = & \frac{v_3 \underline{x}_3}{\gamma_3 + \underline{x}_3} - \frac{v_4 \underline{x}_4}{\gamma_4 + \underline{x}_4} - \Gamma_1 \underline{x}_4 + \Gamma_2 \underline{x}_5 - \frac{v_d \underline{x}_4}{\gamma_d + \underline{x}_4} \\ \dot{\underline{x}}_5 & = & \Gamma_1 \underline{x}_4 - \Gamma_2 \underline{x}_5 \end{array}\right. \quad (12)$$

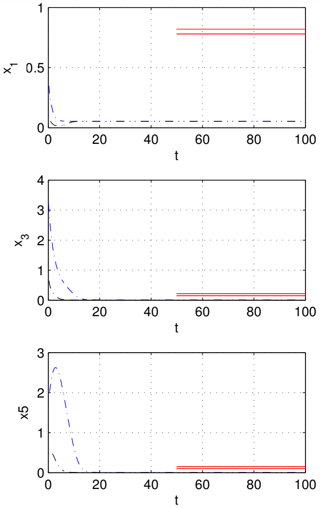
Specifications may be satisfied, can bisect

$[\mathbf{k}] = [0.39,\ 0.402] \times [0.98,\ 1.002]$



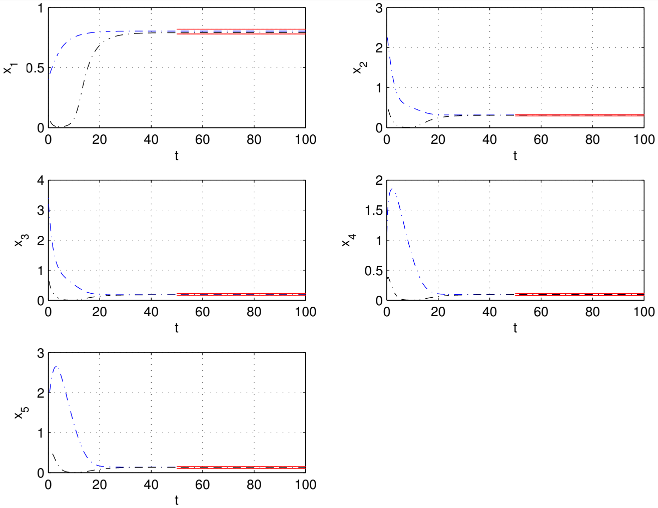
Specifications are not satisfied

$[\mathbf{k}] = [1,\ 1.002] \times [2,\ 2.002]$



Specifications are satisfied

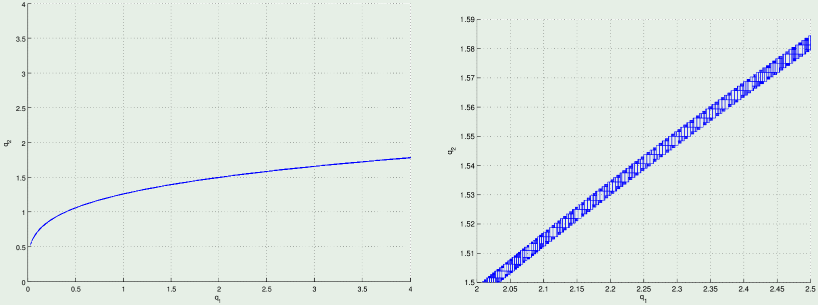
$[\mathbf{k}] = [0.4,\ 0.401] \times [1,\ 1.001]$



$\mathbb{X}_0 = [0.1, 0.4] \times [0.6, 2.4] \times [0.85, 3.4] \times [0.25, 1] \times [0.5, 2]$, $\tau = 50s$
$V_\tau(\mathbf{x}_{op}) = [0.78, 0.82] \times [0.29, 0.32] \times [0.15, 0.22] \times [0.08, 0.11] \times [0.10, 0.15]$

$$\mathbb{K}_0 = [0, 4] \times [0, 4], \quad \epsilon = 0.001, \quad u = \frac{k_1}{k_2^4 + x_5^4}$$

Inner solution set $\mathbb{K}$



Outline

Concluding Remarks

- A SM BnB + Reachability approach

Future work

Further the method to address

- safety and reach-avoid specifications
- presence of disturbances (system, measurement ...)
- hybrid systems
- periodic x-triggered control

Main references

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- D. Angeli, E.D. Sontag, Oscillations in I/O monotone systems, IEEE Transactions on Circuits and Systems, Special Issue on Systems Biology 55, 66–176, 2008.
- N. Ramdani, N. Meslem, Y. Candau, A hybrid bounding method for computing an over-approximation for the reachable space of uncertain nonlinear systems, IEEE Transaction on Automatic Control 54(10), 2352–2364, 2009.

Table: Parameter values

Parameter	Value	Parameter	Value
Γ_2	1.3	Γ_1	1.9
v_1	3.2	v_2	1.58
v_3	5	v_4	2.5
q_1	- -	γ_m	0.5
γ_s	0.38	v_d	0.95
γ_d	0.2	n	4
γ_1	2	γ_2	2
γ_3	2	γ_4	2
q_2	- -	v_m	0.65