Reliable propagation of time uncertainties in dynamical systems

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SWIM
27$^{\mathrm{th}}$ July 2018, Rostock

Section 1

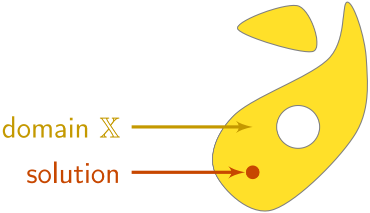
Constraint programming for dynamical systems

Constraint programming for dynamical systems

Constraint programming in a nutshell

Example in $\mathbb{R}^2$:

- system solving described by a *constraint network*
- **variables** (vectors $\mathbf{x} \in \mathbb{R}^n$) belonging to **domains** $\mathbb{X}$



Constraint network:

Variables: $\mathbf{x}$
Constraints:

Domains: $\mathbb{X}$

Constraint programming for dynamical systems

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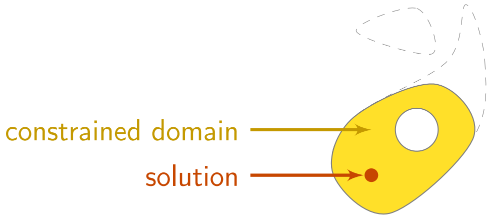

Constraint network:

Variables: $\mathbf{x}$

Constraints:

1. $\mathcal{L}_1(\mathbf{x})$

Domains: $\mathbb{X}$

Constraint programming for dynamical systems

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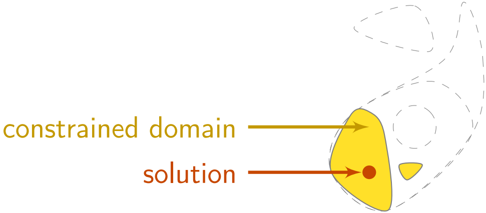

Constraint network:

Variables: $\mathbf{x}$

Constraints:

1. $\mathcal{L}_1(\mathbf{x})$
2. $\mathcal{L}_2(\mathbf{x})$

Domains: $\mathbb{X}$

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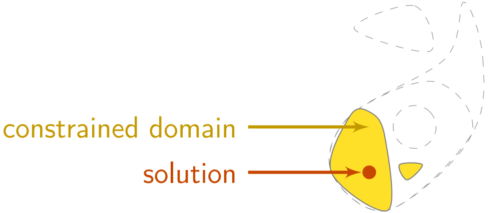

Constraint network:

$$\begin{cases} \textbf{Variables: } \mathbf{x} \\ \textbf{Constraints:} \\ \quad 1.\ \mathcal{L}_1(\mathbf{x}) \\ \quad 2.\ \mathcal{L}_2(\mathbf{x}) \\ \quad 3.\ \dots \\ \textbf{Domains: } \mathbb{X} \end{cases}$$

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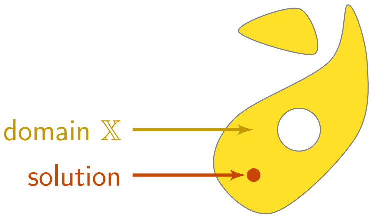

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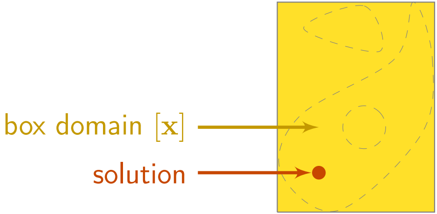
- representable domains: interval-vectors $[\mathbf{x}] \in \mathbb{IR}^n$



Constraint network:

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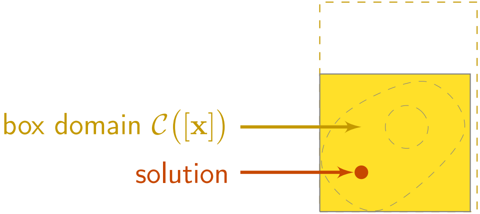
- resolution by **contractors**, $\mathcal{C}_{\mathcal{L}}([\mathbf{x}])$



Constraint network:

Variables: $\mathbf{x}$

Constraints:

1. $\mathcal{L}_1(\mathbf{x}) \rightarrow \mathcal{C}_1([\mathbf{x}])$
2. $\mathcal{L}_2(\mathbf{x})$
3. ...

Domains: $[\mathbf{x}]$

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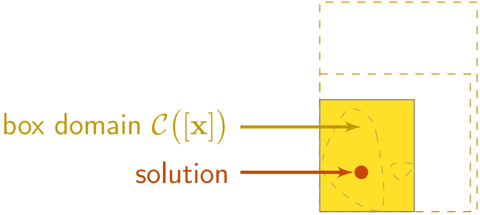

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Domains: $[\mathbf{x}]$

Constraint programming for dynamical systems

Extension to dynamical systems

Only few work on **constraints for dynamical systems**:

- Hickey 2000
- Janssen, Van Hentenryck, and Deville 2002
- Cruz and Barahona 2003

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New approach:

- variables: **trajectories**, $\mathbf{x}(\cdot) : \mathbb{R} \to \mathbb{R}^n$
- domains: **tubes**, $[\mathbf{x}](\cdot) : \mathbb{R} \to \mathbb{IR}^n$

Set-membership state estimation with fleeting data
F. Le Bars, J. Sliwka, L. Jaulin, O. Reynet *Automatica*, 2012

Solving Non-Linear Constraint Satisfaction Problems Involving Time-Dependant Functions
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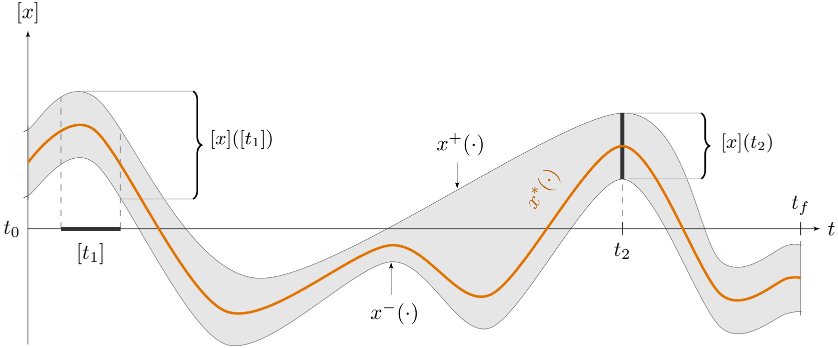
Our contribution:

- develop **primitive dynamical contractors**

Constraint programming for dynamical systems

Tubes

Tube $[x](\cdot)$: interval of trajectories $[x^-(\cdot), x^+(\cdot)]$
such that $\forall t \in \mathbb{R},\ x^-(t) \leqslant x^+(t)$



Tube $[x](\cdot)$ enclosing an uncertain trajectory $x^*(\cdot)$

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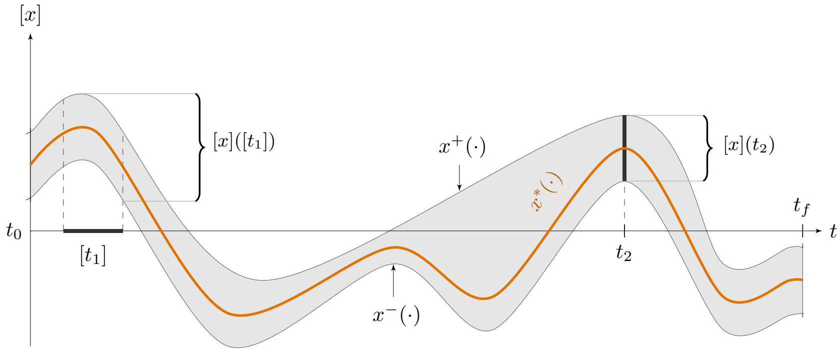
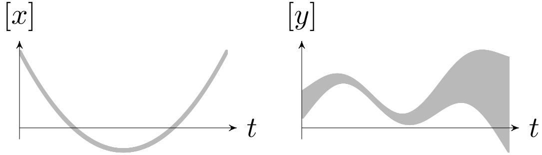


Tube $[x](\cdot)$ enclosing an uncertain trajectory $x^*(\cdot)$

- dot notation $(\cdot)$

Constraint programming for dynamical systems

Tubes arithmetic



Constraint programming for dynamical systems

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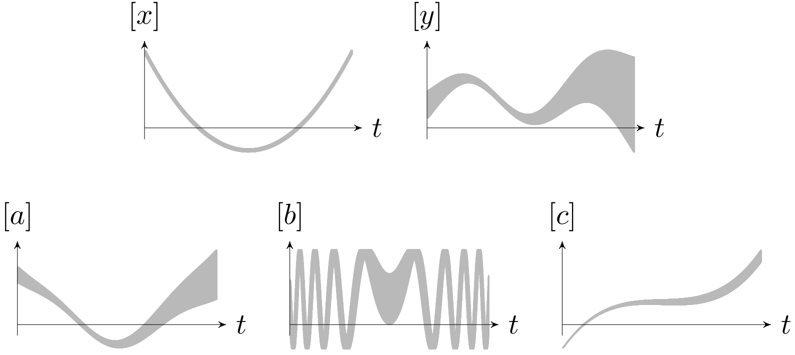

$$[a](\cdot) = [x](\cdot) + [y](\cdot)$$

$$[b](\cdot) = \sin\big([x](\cdot)\big)$$

$$[c](\cdot) = \int_0^{\cdot} [x](\tau)d\tau$$

Constraint programming for dynamical systems

Tube contractor

Contractor on boxes can be extended to sets of trajectories (tubes).

Definition

A contractor $\mathcal{C}_{\mathcal{L}}$ applied on a tube $[x](\cdot)$ aims at removing infeasible trajectories according to a given constraint $\mathcal{L}$ so that:

Constraint programming for dynamical systems

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$$(i) \quad \forall t \in [t_0, t_f],\ \mathcal{C}_{\mathcal{L}}\big([x](t)\big) \subseteq [x](t) \qquad \text{(contraction)}$$

Constraint programming for dynamical systems

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$$(ii) \quad \begin{pmatrix} \mathcal{L}\big(x(\cdot)\big) \\ x(\cdot) \in [x](\cdot) \end{pmatrix} \implies x(\cdot) \in \mathcal{C}_{\mathcal{L}}\big([x](\cdot)\big) \qquad \text{(consistency)}$$

Constraint programming for dynamical systems

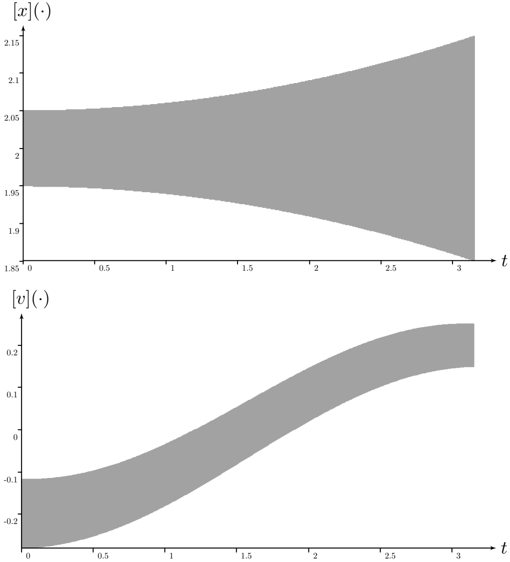
Constraint $\dot{x}(\cdot) = v(\cdot)$

Differential constraint:

$$\mathcal{L}_{\frac{d}{dt}}\big(x(\cdot), v(\cdot)\big) : \dot{x}(\cdot) = v(\cdot)$$

Constraint programming for dynamical systems

Constraint $\dot{x}(\cdot) = v(\cdot)$



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Related contractor $\mathcal{C}_{\frac{d}{dt}}$:

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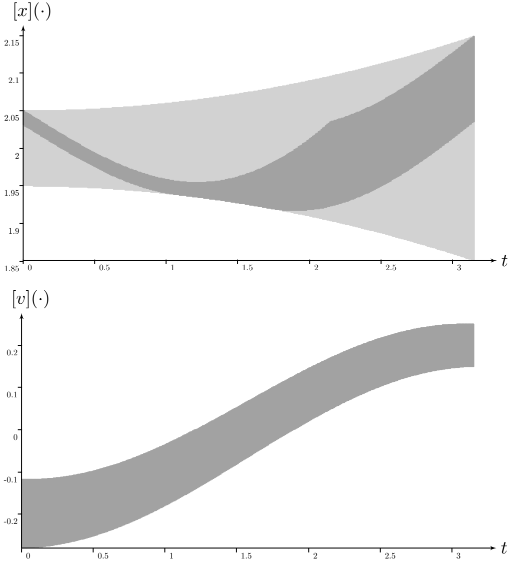

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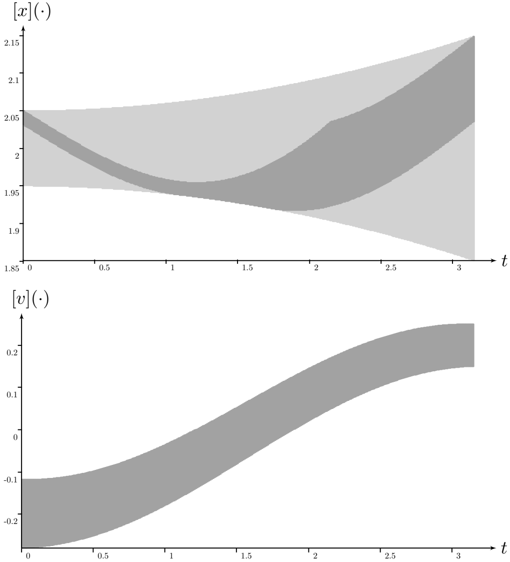

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Guaranteed computation of robot trajectories

Rohou, Jaulin, Mihaylova, Le Bars, Veres

Robotics and Autonomous Systems, 2017

Constraint programming for dynamical systems

State estimation

Classical formalization:

$$\begin{cases} \dot{\mathbf{x}}(\cdot) = \mathbf{f}\big(\mathbf{x}(\cdot), \mathbf{u}(\cdot)\big) & \text{(evolution)} \\ z = g\big(\mathbf{x}(t)\big) & \text{(observations)} \end{cases}$$

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Decomposition:

1. $\mathbf{v}(\cdot) = \mathbf{f}\big(\mathbf{x}(\cdot), \mathbf{u}(\cdot)\big)$
2. $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$
3. $y(\cdot) = g(\mathbf{x}(\cdot))$
4. $z = y(t)$

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Constraints:

1. $\mathcal{L}_{\mathbf{f}}\big(\mathbf{v}(\cdot), \mathbf{x}(\cdot), \mathbf{u}(\cdot)\big)$ (arithmetic composition)
2. $\mathcal{L}_{\frac{d}{dt}}\big(\mathbf{x}(\cdot), \mathbf{v}(\cdot)\big)$
3. $\mathcal{L}_{g}\big(y(\cdot), \mathbf{x}(\cdot)\big)$ (arithmetic composition)

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3. $\mathcal{L}_{g}\big(y(\cdot), \mathbf{x}(\cdot)\big)$ (arithmetic composition)
4. $\mathcal{L}_{\text{eval}}\big(t, z, y(\cdot)\big)$

Section 2

Constraint $\mathcal{L}_{\text{eval}}$: $z = y(t)$

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Definition

$$\mathcal{L}_{\text{eval}} : \begin{cases} \textbf{Variables: } t,\ z,\ y(\cdot) \\ \textbf{Constraints:} \\ \quad 1.\ z = y(t) \\ \\ \textbf{Domains: } [t],\ [z],\ [y](\cdot) \end{cases}$$

$\mathcal{L}_{\text{eval}}$ equivalent to:

$\exists t \in [t],\ \exists z \in [z],\ \exists y(\cdot) \in [y](\cdot) \ \mid\ z = y(t)$

Reliable non-linear state estimation involving time uncertainties
S. Rohou, L. Jaulin, L. Mihaylova, F. Le Bars, S. M. Veres. *Automatica*, 2018

Constraint $\mathcal{L}_{\text{eval}}$: $z = y(t)$

Definition

$$\mathcal{L}_{\text{eval}} : \begin{cases} \textbf{Variables: } t,\ z,\ y(\cdot),\ w(\cdot) \\ \textbf{Constraints:} \\ \quad 1.\ z = y(t) \\ \quad 2.\ \dot{y}(\cdot) = w(\cdot) \\ \textbf{Domains: } [t],\ [z],\ [y](\cdot),\ [w](\cdot) \end{cases}$$

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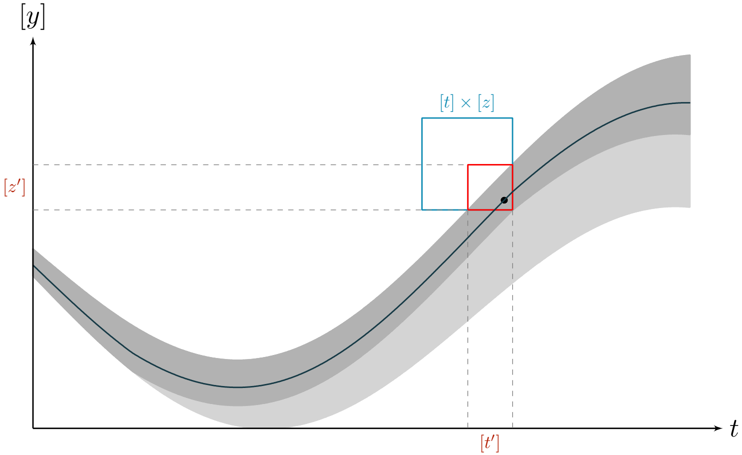
Constraint $\mathcal{L}_{\text{eval}}$: $z = y(t)$

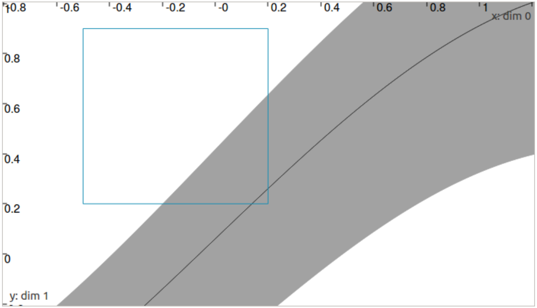
$\mathcal{C}_{\text{eval}}$: illustration



Bounded evaluation with contractions of $[y](\cdot)$ and both $[t]$ and $[z]$ by means of $\mathcal{C}_{\text{eval}}$.
The tube's contracted part is depicted in light gray.

Constraint $\mathcal{L}_{\text{eval}}$: $z = y(t)$

$$\mathcal{C}_{\text{eval}}\big([t],[z],[y](\cdot),[w](\cdot)\big)$$



Definition:

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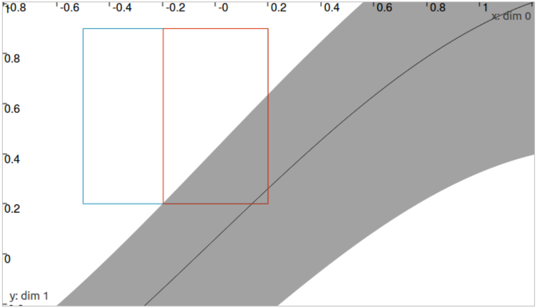

Definition:

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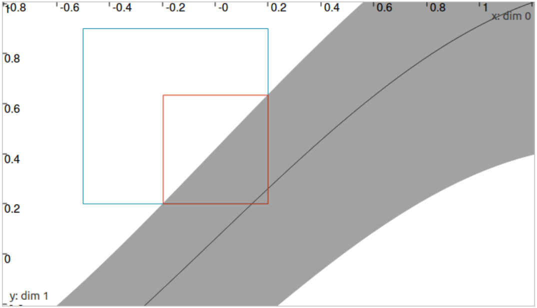

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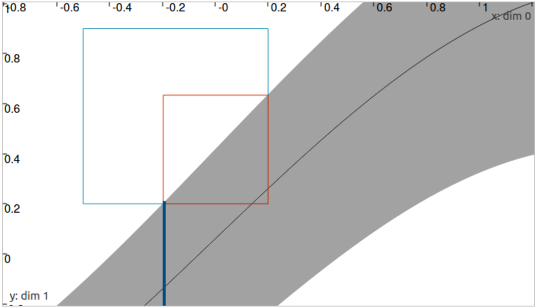

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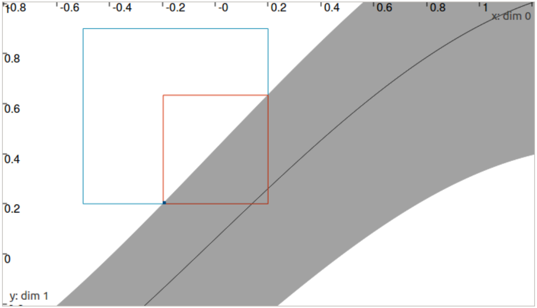

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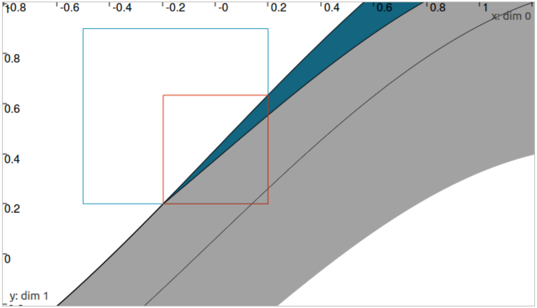

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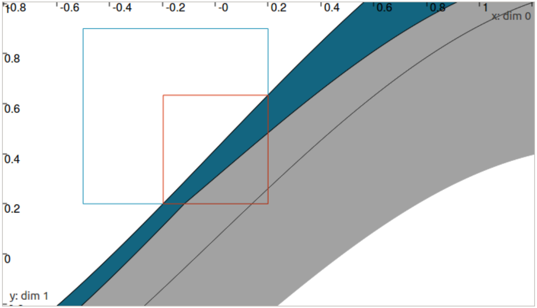

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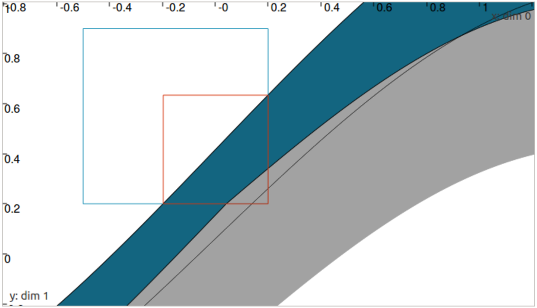
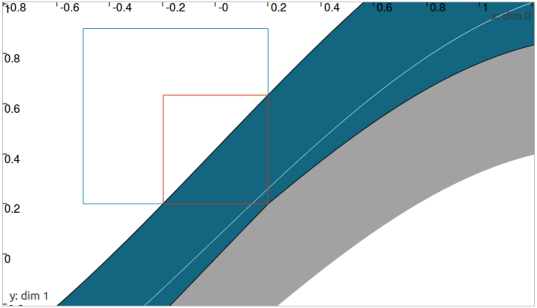


Definition:

$$\begin{pmatrix} [t] \\ [z] \\ [y](\cdot) \\ [w](\cdot) \end{pmatrix} \xmapsto{\mathcal{C}_{\text{eval}}} \begin{pmatrix} [t] \cap [y]^{-1}([z]) \\ [z] \cap [y]([t]) \\ \bigsqcup_{t_1 \in [t]} \left(([y](t_1) \cap [z]) + \int_{t_1}^{\cdot} [w](\tau) d\tau \right) \end{pmatrix}$$

Constraint $\mathcal{L}_{\text{eval}}$: $z = y(t)$

$$\mathcal{C}_{\text{eval}}\big([t],[z],[y](\cdot),[w](\cdot)\big)$$

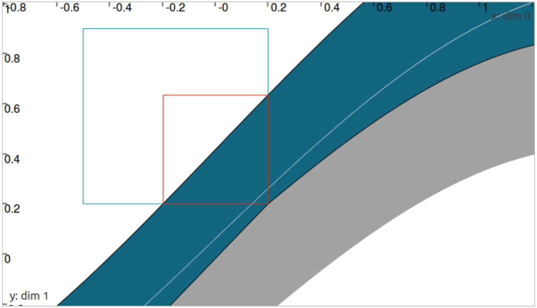


Definition:

$$\begin{pmatrix} [t] \\ [z] \\ [y](\cdot) \\ [w](\cdot) \end{pmatrix} \xmapsto{\mathcal{C}_{\text{eval}}} \begin{pmatrix} [t] \cap [y]^{-1}([z]) \\ [z] \cap [y]([t]) \\ \displaystyle\bigsqcup_{t_1 \in [t]} \left(([y](t_1) \cap [z]) + \int_{t_1}^{\cdot} [w](\tau) d\tau \right) \\ \end{pmatrix}$$

Constraint $\mathcal{L}_{\text{eval}}$: $z = y(t)$

$$\mathcal{C}_{\text{eval}}\big([t], [z], [y](\cdot), [w](\cdot)\big)$$

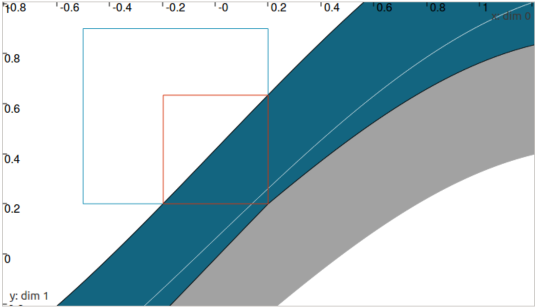


Definition:

$$\begin{pmatrix} [t] \\ [z] \\ [y](\cdot) \\ [w](\cdot) \end{pmatrix} \xmapsto{\mathcal{C}_{\text{eval}}} \begin{pmatrix} [t] \cap [y]^{-1}([z]) \\ [z] \cap [y]([t]) \\ [y](\cdot) \cap \bigsqcup_{t_1 \in [t]} \left(([y](t_1) \cap [z]) + \int_{t_1}^{\cdot} [w](\tau) d\tau \right) \\ \end{pmatrix}$$

Constraint $\mathcal{L}_{\text{eval}}$: $z = y(t)$

$$\mathcal{C}_{\text{eval}}\big([t], [z], [y](\cdot), [w](\cdot)\big)$$



Definition:

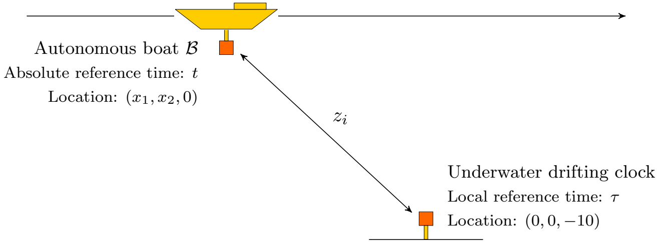
$$\begin{pmatrix} [t] \\ [z] \\ [y](\cdot) \\ [w](\cdot) \end{pmatrix} \xmapsto{\mathcal{C}_{\text{eval}}} \begin{pmatrix} [t] \cap [y]^{-1}([z]) \\ [z] \cap [y]([t]) \\ [y](\cdot) \cap \bigsqcup_{t_1 \in [t]} \left(([y](t_1) \cap [z]) + \int_{t_1}^{\cdot} [w](\tau) d\tau \right) \\ [w](\cdot) \end{pmatrix}$$

Section 3

Application: drifting clock

Application: drifting clock

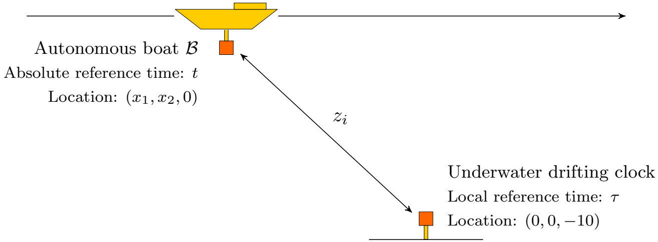
Underwater system equipped with a low-cost drifting clock



- absolute time reference represented by t
- underwater clock providing a drifting value τ:
  - $\tau = h(t)$
  - unknown: $h(t) = 0.045t^2 + 0.98t$

Application: drifting clock

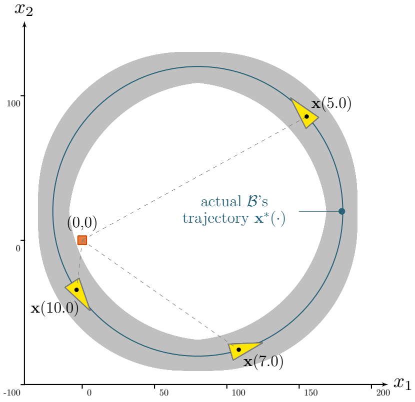
Underwater system equipped with a low-cost drifting clock



- absolute time reference represented by t
- underwater clock providing a drifting value τ:
  - $\tau = h(t)$
  - unknown: $h(t) = 0.045t^2 + 0.98t$
  - clock's datasheet: $\dot{h}(t) \in [0.08, 0.12] \cdot t + [0.97, 1.08]$

Application: drifting clock

Boat $\mathcal{B}$ following a preprogrammed trajectory



Top view of $\mathcal{B}$'s traj. $\mathbf{x}^*(\cdot)$ and tube $[\mathbf{x}](\cdot)$ around the underwater beacon in $(0,0)$.

Preprogrammed trajectory $\mathbf{x}(\cdot)$ (ephemeris) assumed as:

$$\mathbf{x}(\cdot) \in \begin{pmatrix} [70,90] \\ [10,30] \end{pmatrix} + 100 \begin{pmatrix} \cos(\cdot) \\ \sin(\cdot) \end{pmatrix}$$

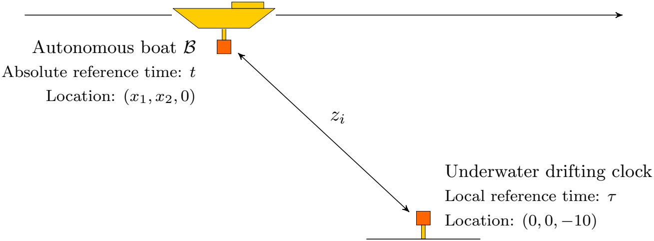
Bounded $\mathcal{B}$'s velocities:

$$\mathbf{v}(\cdot) \in \frac{1}{10} \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix} + 100 \begin{pmatrix} -\sin(\cdot) \\ \cos(\cdot) \end{pmatrix}$$

Application: drifting clock

List of measurements $(\tau_i, [z_i])$

i	τ_i	$[z_i]$	i	τ_i	$[z_i]$
1	1.57	$[152.47, 156.47]$	5	9.88	$[167.09, 171.09]$
2	3.34	$[34.67, 38.67]$	6	12.46	$[60.03, 64.03]$
3	5.32	$[102.38, 106.38]$	7	15.25	$[78.76, 82.76]$
4	7.50	$[184.45, 188.45]$	8	18.24	$[175.88, 179.88]$



Application: drifting clock

Variables:

Domains:

Constraints:

Application: drifting clock

Variables: $\mathbf{x}(\cdot)$, $\mathbf{v}(\cdot)$

Domains: $[\mathbf{x}](\cdot)$, $[\mathbf{v}](\cdot)$

Constraints:

1. Boat's positions:
   - $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$

Application: drifting clock

Variables: $\mathbf{x}(\cdot)$, $\mathbf{v}(\cdot)$, $y(\cdot)$

Domains: $[\mathbf{x}](\cdot)$, $[\mathbf{v}](\cdot)$, $[y](\cdot)$

Constraints:

1. Boat's positions:
   - $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$
2. Beacon-boat distance function:
   - $y(\cdot) = \sqrt{x_1(\cdot)^2 + x_2(\cdot)^2 + (-10)^2}$

Application: drifting clock

Variables: $\mathbf{x}(\cdot)$, $\mathbf{v}(\cdot)$, $y(\cdot)$, $h(\cdot)$, $\phi(\cdot)$

Domains: $[\mathbf{x}](\cdot)$, $[\mathbf{v}](\cdot)$, $[y](\cdot)$, $[h](\cdot)$, $[\phi](\cdot)$

Constraints:

1. Boat's positions:
   - $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$
2. Beacon-boat distance function:
   - $y(\cdot) = \sqrt{x_1(\cdot)^2 + x_2(\cdot)^2 + (-10)^2}$
3. Drifting time function:
   - $\dot{h}(\cdot) \in [\phi](\cdot)$ (clock's datasheet)
   - $h(0) = 0$ (no drift at first)

Application: drifting clock

Variables: $\mathbf{x}(\cdot)$, $\mathbf{v}(\cdot)$, $y(\cdot)$, $h(\cdot)$, $\phi(\cdot)$, $\{(t_i, z_i)\}$

Domains: $[\mathbf{x}](\cdot)$, $[\mathbf{v}](\cdot)$, $[y](\cdot)$, $[h](\cdot)$, $[\phi](\cdot)$, $\{([t_i], [z_i])\}$

Constraints:

1. Boat's positions:
   - $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$
2. Beacon-boat distance function:
   - $y(\cdot) = \sqrt{x_1(\cdot)^2 + x_2(\cdot)^2 + (-10)^2}$
3. Drifting time function:
   - $\dot{h}(\cdot) \in [\phi](\cdot)$ (clock's datasheet)
   - $h(0) = 0$ (no drift at first)
4. Evaluations:
   - $\tau_i = h(t_i)$
   - $z_i = y(t_i)$

Application: drifting clock

Variables: $\mathbf{x}(\cdot)$, $\mathbf{v}(\cdot)$, $y(\cdot)$, $h(\cdot)$, $\phi(\cdot)$, $\{(t_i, z_i)\}$

Domains: $[\mathbf{x}](\cdot)$, $[\mathbf{v}](\cdot)$, $[y](\cdot)$, $[h](\cdot)$, $[\phi](\cdot)$, $\{([t_i], [z_i])\}$

Constraints:

1. Boat's positions:
   - $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$ ⟶ $\mathcal{C}_{\frac{d}{dt}}\left([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\right)$
2. Beacon-boat distance function:
   - $y(\cdot) = \sqrt{x_1(\cdot)^2 + x_2(\cdot)^2 + (-10)^2}$
3. Drifting time function:
   - $\dot{h}(\cdot) \in [\phi](\cdot)$ (clock's datasheet)
   - $h(0) = 0$ (no drift at first)
4. Evaluations:
   - $\tau_i = h(t_i)$
   - $z_i = y(t_i)$

Contractor programming algorithm:

$\mathcal{C}_{\frac{d}{dt}}\left([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\right)$

Application: drifting clock

Variables: $\mathbf{x}(\cdot)$, $\mathbf{v}(\cdot)$, $y(\cdot)$, $h(\cdot)$, $\phi(\cdot)$, $\{(t_i, z_i)\}$

Domains: $[\mathbf{x}](\cdot)$, $[\mathbf{v}](\cdot)$, $[y](\cdot)$, $[h](\cdot)$, $[\phi](\cdot)$, $\{([t_i],[z_i])\}$

Constraints:

1. Boat's positions:
   - $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$ → $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\big)$
2. Beacon-boat distance function:
   - $y(\cdot) = \sqrt{x_1(\cdot)^2 + x_2(\cdot)^2 + (-10)^2}$ → $\mathcal{C}_{\text{dist}}\big([y](\cdot), [\mathbf{x}](\cdot)\big)$
3. Drifting time function:
   - $\dot{h}(\cdot) \in [\phi](\cdot)$ (clock's datasheet)
   - $h(0) = 0$ (no drift at first)
4. Evaluations:
   - $\tau_i = h(t_i)$
   - $z_i = y(t_i)$

Contractor programming algorithm:

Application: drifting clock

Variables: $\mathbf{x}(\cdot)$, $\mathbf{v}(\cdot)$, $y(\cdot)$, $h(\cdot)$, $\phi(\cdot)$, $\{(t_i, z_i)\}$

Domains: $[\mathbf{x}](\cdot)$, $[\mathbf{v}](\cdot)$, $[y](\cdot)$, $[h](\cdot)$, $[\phi](\cdot)$, $\{([t_i], [z_i])\}$

Constraints:

1. Boat's positions:
   - $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$ ⟶ $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\big)$
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   - $y(\cdot) = \sqrt{x_1(\cdot)^2 + x_2(\cdot)^2 + (-10)^2}$ ⟶ $\mathcal{C}_{\text{dist}}\big([y](\cdot), [\mathbf{x}](\cdot)\big)$
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   - $\dot{h}(\cdot) \in [\phi](\cdot)$ (clock's datasheet)
   - $h(0) = 0$ (no drift at first)

   ⟶ $\mathcal{C}_{\frac{d}{dt}}\big([h](\cdot), [\phi](\cdot)\big)$
4. Evaluations:
   - $\tau_i = h(t_i)$
   - $z_i = y(t_i)$

Contractor programming algorithm:

Application: drifting clock

Variables: $\mathbf{x}(\cdot)$, $\mathbf{v}(\cdot)$, $y(\cdot)$, $h(\cdot)$, $\phi(\cdot)$, $\{(t_i, z_i)\}$

Domains: $[\mathbf{x}](\cdot)$, $[\mathbf{v}](\cdot)$, $[y](\cdot)$, $[h](\cdot)$, $[\phi](\cdot)$, $\{([t_i], [z_i])\}$

Constraints:

Contractor programming algorithm:

1. Boat's positions:
   - $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$ ⟶ $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\big)$
2. Beacon-boat distance function:
   - $y(\cdot) = \sqrt{x_1(\cdot)^2 + x_2(\cdot)^2 + (-10)^2}$ ⟶ $\mathcal{C}_{\text{dist}}\big([y](\cdot), [\mathbf{x}](\cdot)\big)$
3. Drifting time function:
   - $\dot{h}(\cdot) \in [\phi](\cdot)$ (clock's datasheet) ⟶ $\mathcal{C}_{\frac{d}{dt}}\big([h](\cdot), [\phi](\cdot)\big)$
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   - $\tau_i = h(t_i)$ ⟶ $\mathcal{C}_{\text{eval}}\big([t_i], \tau_i, [h](\cdot), [\phi](\cdot)\big)$
   - $z_i = y(t_i)$

Application: drifting clock

Variables: $\mathbf{x}(\cdot)$, $\mathbf{v}(\cdot)$, $y(\cdot)$, $h(\cdot)$, $\phi(\cdot)$, $\{(t_i, z_i)\}$

Domains: $[\mathbf{x}](\cdot)$, $[\mathbf{v}](\cdot)$, $[y](\cdot)$, $[h](\cdot)$, $[\phi](\cdot)$, $\{([t_i], [z_i])\}$

Constraints:

1. Boat's positions:
   - $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$ → $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\big)$
2. Beacon-boat distance function:
   - $y(\cdot) = \sqrt{x_1(\cdot)^2 + x_2(\cdot)^2 + (-10)^2}$ → $\mathcal{C}_{\text{dist}}\big([y](\cdot), [\mathbf{x}](\cdot)\big)$
3. Drifting time function:
   - $\dot{h}(\cdot) \in [\phi](\cdot)$ (clock's datasheet) → $\mathcal{C}_{\frac{d}{dt}}\big([h](\cdot), [\phi](\cdot)\big)$
   - $h(0) = 0$ (no drift at first)
4. Evaluations:
   - $\tau_i = h(t_i)$ → $\mathcal{C}_{\text{eval}}\big([t_i], \tau_i, [h](\cdot), [\phi](\cdot)\big)$
   - $z_i = y(t_i)$ → $\mathcal{C}_{\text{eval}}\big([t_i], [z_i], [y](\cdot), [w](\cdot)\big)$

Contractor programming algorithm:

Application: drifting clock

Variables: $\mathbf{x}(\cdot)$, $\mathbf{v}(\cdot)$, $y(\cdot)$, $h(\cdot)$, $\phi(\cdot)$, $\{(t_i, z_i)\}$, $w(\cdot)$

Domains: $[\mathbf{x}](\cdot)$, $[\mathbf{v}](\cdot)$, $[y](\cdot)$, $[h](\cdot)$, $[\phi](\cdot)$, $\{([t_i],[z_i])\}$, $[w](\cdot)$

Constraints:

1. Boat's positions:
   - $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$ → $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\big)$
2. Beacon-boat distance function:
   - $y(\cdot) = \sqrt{x_1(\cdot)^2 + x_2(\cdot)^2 + (-10)^2}$ → $\mathcal{C}_{\text{dist}}\big([y](\cdot), [\mathbf{x}](\cdot)\big)$
   - $\dot{y}(\cdot) = w(\cdot)$
3. Drifting time function:
   - $\dot{h}(\cdot) \in [\phi](\cdot)$ (clock's datasheet) → $\mathcal{C}_{\frac{d}{dt}}\big([h](\cdot), [\phi](\cdot)\big)$
   - $h(0) = 0$ (no drift at first)
4. Evaluations:
   - $\tau_i = h(t_i)$ → $\mathcal{C}_{\text{eval}}\big([t_i], \tau_i, [h](\cdot), [\phi](\cdot)\big)$
   - $z_i = y(t_i)$ → $\mathcal{C}_{\text{eval}}\big([t_i], [z_i], [y](\cdot), [w](\cdot)\big)$

Contractor programming algorithm:

Application: drifting clock

Variables: $\mathbf{x}(\cdot)$, $\mathbf{v}(\cdot)$, $y(\cdot)$, $h(\cdot)$, $\phi(\cdot)$, $\{(t_i, z_i)\}$, $w(\cdot)$

Domains: $[\mathbf{x}](\cdot)$, $[\mathbf{v}](\cdot)$, $[y](\cdot)$, $[h](\cdot)$, $[\phi](\cdot)$, $\{([t_i],[z_i])\}$, $[w](\cdot)$

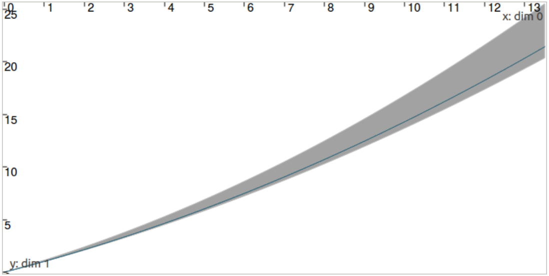
Constraints:

Contractor programming algorithm:

1. Boat's positions:
   - $\dot{\mathbf{x}}(\cdot) = \mathbf{v}(\cdot)$ ⟶ $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot),[\mathbf{v}](\cdot)\big)$
2. Beacon-boat distance function:
   - $y(\cdot) = \sqrt{x_1(\cdot)^2 + x_2(\cdot)^2 + (-10)^2}$ ⟶ $\mathcal{C}_{\text{dist}}\big([y](\cdot),[\mathbf{x}](\cdot)\big)$
   - $\dot{y}(\cdot) = w(\cdot)$
   - $w(\cdot) = (x_1(\cdot)\cdot v_1(\cdot) + x_2(\cdot)\cdot v_2(\cdot))/y(\cdot)$ ⟶ $\mathcal{C}_{\text{ddist}}\big([w](\cdot),[\mathbf{x}](\cdot)\big)$
3. Drifting time function:
   - $\dot{h}(\cdot) \in [\phi](\cdot)$ (clock's datasheet) ⟶ $\mathcal{C}_{\frac{d}{dt}}\big([h](\cdot),[\phi](\cdot)\big)$
   - $h(0) = 0$ (no drift at first)
4. Evaluations:
   - $\tau_i = h(t_i)$ ⟶ $\mathcal{C}_{\text{eval}}\big([t_i],\tau_i,[h](\cdot),[\phi](\cdot)\big)$
   - $z_i = y(t_i)$ ⟶ $\mathcal{C}_{\text{eval}}\big([t_i],[z_i],[y](\cdot),[w](\cdot)\big)$

Application: drifting clock

Resolution: enclosing absolute time references $[t_i]$



Tube $[h](\cdot)$: clock's drift.

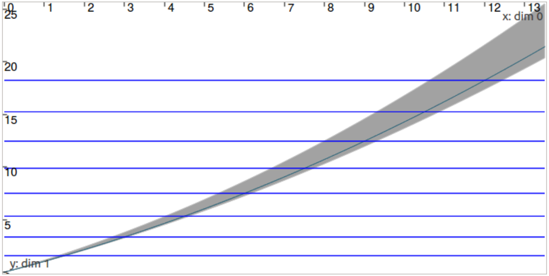
Contractor programming algorithm:

- $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\big)$
- $\mathcal{C}_{\text{dist}}\big([y](\cdot), [\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\text{ddist}}\big([w](\cdot), [\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\frac{d}{dt}}\big([h](\cdot), [\phi](\cdot)\big)$
- $\mathcal{C}_{\text{eval}}\big([t_i], \tau_i, [h](\cdot), [\phi](\cdot)\big)$
- $\mathcal{C}_{\text{eval}}\big([t_i], [z_i], [y](\cdot), [w](\cdot)\big)$

Application: drifting clock

Resolution: enclosing absolute time references $[t_i]$

- $[t_i]$ initialized to $[-\infty, \infty]$



Tube $[h](\cdot)$: clock's drift.
Blue lines: temporal references $[t_i] \times \tau_i$.

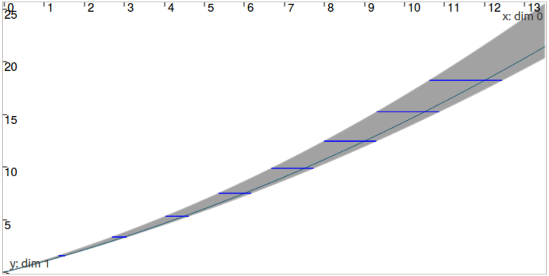
Contractor programming algorithm:

- $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\big)$
- $\mathcal{C}_{\text{dist}}\big([y](\cdot), [\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\text{ddist}}\big([w](\cdot), [\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\frac{d}{dt}}\big([h](\cdot), [\phi](\cdot)\big)$
- $\mathcal{C}_{\text{eval}}\big([t_i], \tau_i, [h](\cdot), [\phi](\cdot)\big)$
- $\mathcal{C}_{\text{eval}}\big([t_i], [z_i], [y](\cdot), [w](\cdot)\big)$

Application: drifting clock

Resolution: enclosing absolute time references $[t_i]$

- $[t_i]$ initialized to $[-\infty, \infty]$
- $\mathcal{C}_{\text{eval}}\big([t_i], \tau_i, [h](\cdot), [\phi](\cdot)\big)$



Tube $[h](\cdot)$: clock's drift.
Blue lines: temporal references $[t_i] \times \tau_i$.

Contractor programming algorithm:

- $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\big)$
- $\mathcal{C}_{\text{dist}}\big([y](\cdot), [\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\text{ddist}}\big([w](\cdot), [\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\frac{d}{dt}}\big([h](\cdot), [\phi](\cdot)\big)$
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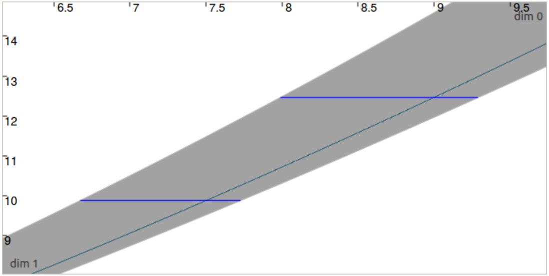
Application: drifting clock

Resolution: enclosing absolute time references $[t_i]$

- $[t_i]$ initialized to $[-\infty, \infty]$
- $\mathcal{C}_{\text{eval}}\big([t_i], \tau_i, [h](\cdot), [\phi](\cdot)\big)$



Tube $[h](\cdot)$: clock's drift.
Blue lines: temporal references $[t_i] \times \tau_i$.

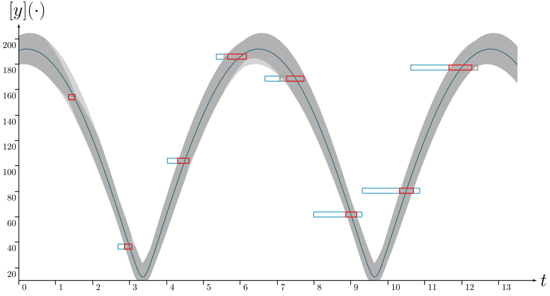
Contractor programming algorithm:

- $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\big)$
- $\mathcal{C}_{\text{dist}}\big([y](\cdot), [\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\text{ddist}}\big([w](\cdot), [\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\frac{d}{dt}}\big([h](\cdot), [\phi](\cdot)\big)$
- $\mathcal{C}_{\text{eval}}\big([t_i], \tau_i, [h](\cdot), [\phi](\cdot)\big)$
- $\mathcal{C}_{\text{eval}}\big([t_i], [z_i], [y](\cdot), [w](\cdot)\big)$

Application: drifting clock

Resolution: contracting the times $[t_i]$ from $[y](\cdot)$

- $\mathcal{C}_{\text{eval}}\big([t_i],[z_i],[y](\cdot),[w](\cdot)\big)$



Tube $[y](\cdot)$: reliable prevision of the distances between the boat and the beacon.

Boxes: measurements $[t_i]\times[z_i]$.

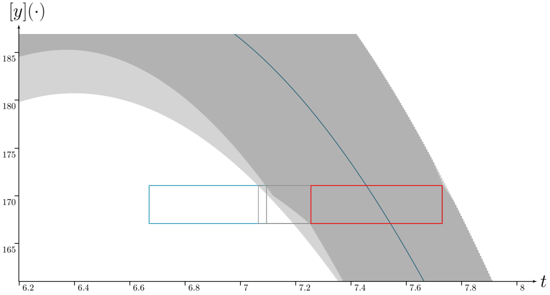
Contractor programming algorithm:

- $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot),[\mathbf{v}](\cdot)\big)$
- $\mathcal{C}_{\text{dist}}\big([y](\cdot),[\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\text{ddist}}\big([w](\cdot),[\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\frac{d}{dt}}\big([h](\cdot),[\phi](\cdot)\big)$
- $\mathcal{C}_{\text{eval}}\big([t_i],\tau_i,[h](\cdot),[\phi](\cdot)\big)$
- $\mathcal{C}_{\text{eval}}\big([t_i],[z_i],[y](\cdot),[w](\cdot)\big)$

Application: drifting clock

Resolution: contracting the times $[t_i]$ from $[y](\cdot)$

- $\mathcal{C}_{\text{eval}}\big([t_i],[z_i],[y](\cdot),[w](\cdot)\big)$



Tube $[y](\cdot)$: reliable prevision of the distances between the boat and the beacon.

Boxes: measurements $[t_i] \times [z_i]$.

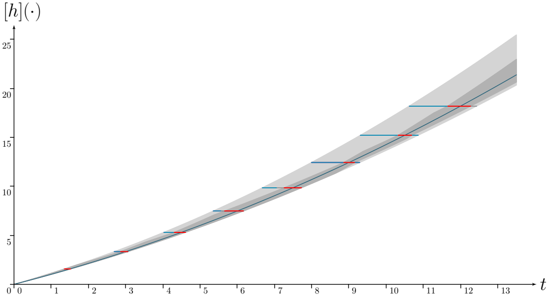
Contractor programming algorithm:

- $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot),[\mathbf{v}](\cdot)\big)$
- $\mathcal{C}_{\text{dist}}\big([y](\cdot),[\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\text{ddist}}\big([w](\cdot),[\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\frac{d}{dt}}\big([h](\cdot),[\phi](\cdot)\big)$
- $\mathcal{C}_{\text{eval}}\big([t_i],\tau_i,[h](\cdot),[\phi](\cdot)\big)$
- $\mathcal{C}_{\text{eval}}\big([t_i],[z_i],[y](\cdot),[w](\cdot)\big)$

Application: drifting clock

Resolution: propagating the $[t_i]$ to contract $[h](\cdot)$

- $\mathcal{C}_{\text{eval}}\big([t_i], \tau_i, [h](\cdot), [\phi](\cdot)\big)$



Tube $[h](\cdot)$: clock's drift.
Horizontal lines: temporal references $[t_i] \times \tau_i$.

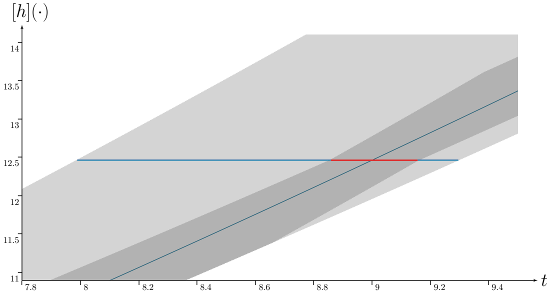
Contractor programming algorithm:

- $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\big)$
- $\mathcal{C}_{\text{dist}}\big([y](\cdot), [\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\text{ddist}}\big([w](\cdot), [\mathbf{x}](\cdot)\big)$
- $\mathcal{C}_{\frac{d}{dt}}\big([h](\cdot), [\phi](\cdot)\big)$
- $\mathcal{C}_{\text{eval}}\big([t_i], \tau_i, [h](\cdot), [\phi](\cdot)\big)$
- $\mathcal{C}_{\text{eval}}\big([t_i], [z_i], [y](\cdot), [w](\cdot)\big)$

Application: drifting clock

Resolution: propagating the $[t_i]$ to contract $[h](\cdot)$

- $\mathcal{C}_{\text{eval}}\big([t_i], \tau_i, [h](\cdot), [\phi](\cdot)\big)$



Tube $[h](\cdot)$: clock's drift.
Horizontal lines: temporal references $[t_i] \times \tau_i$.

Contractor programming algorithm:

- $\mathcal{C}_{\frac{d}{dt}}\big([\mathbf{x}](\cdot), [\mathbf{v}](\cdot)\big)$
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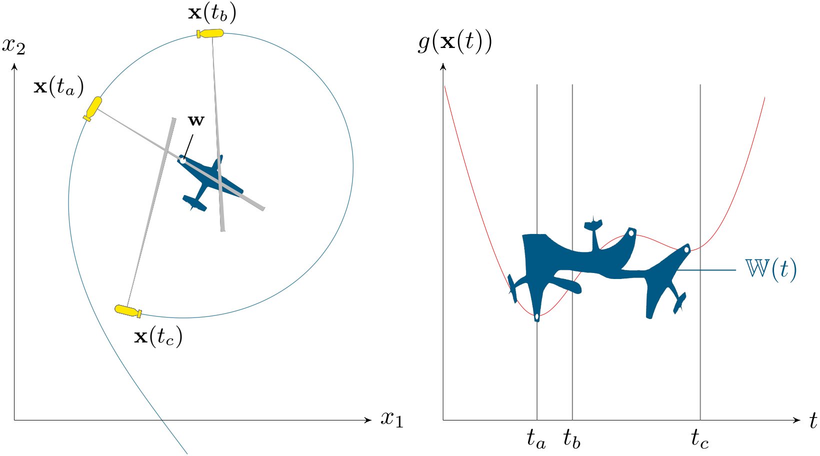
Application: drifting clock

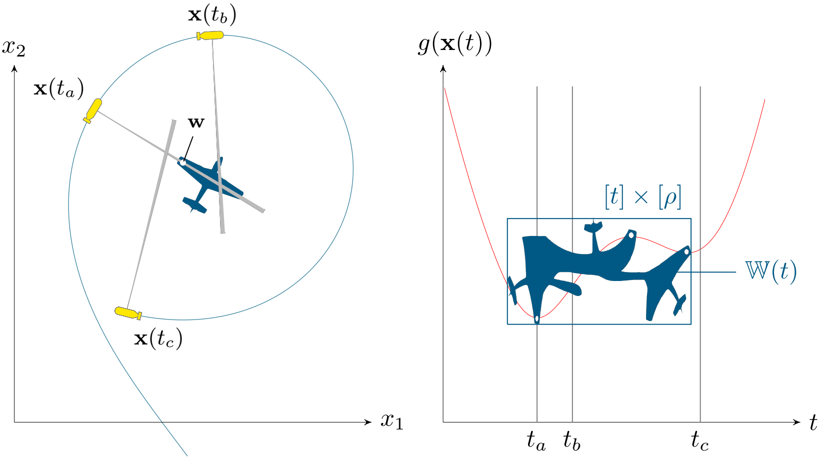
Time uncertainties in state estimation



A robot $\mathcal{R}$ perceiving a plane wreck with a side scan sonar.

Application: drifting clock

Time uncertainties in state estimation



A robot $\mathcal{R}$ perceiving a plane wreck with a side scan sonar.

Section 4

Conclusions

Conclusion

To conclude:

- original method to deal with (strong) **time uncertainties**
- **non-linear** and **differential** systems
- elementary tool in the **contractor programming** framework
- $\mathcal{C}_{\textrm{eval}}$ now allows one to consider state estimation problems from a **temporal point of view** where the time t becomes an unknown variable to be estimated

Prospects:

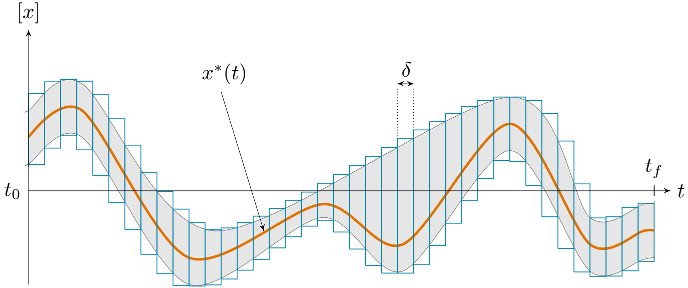
- wreck-based localization problem

Conclusions

Tubex library

An open-source C++ library based on IBEX and providing tools for constraint programming over dynamical systems.

- `Tube`, `TubeVector`, ...
- contractors $\mathcal{C}_{\frac{d}{dt}}$, $\mathcal{C}_{\text{eval}}$, $\mathcal{C}_{\text{delay}}$, ...
- robotic tools and applications



http://www.simon-rohou.fr/research/tubex-lib/

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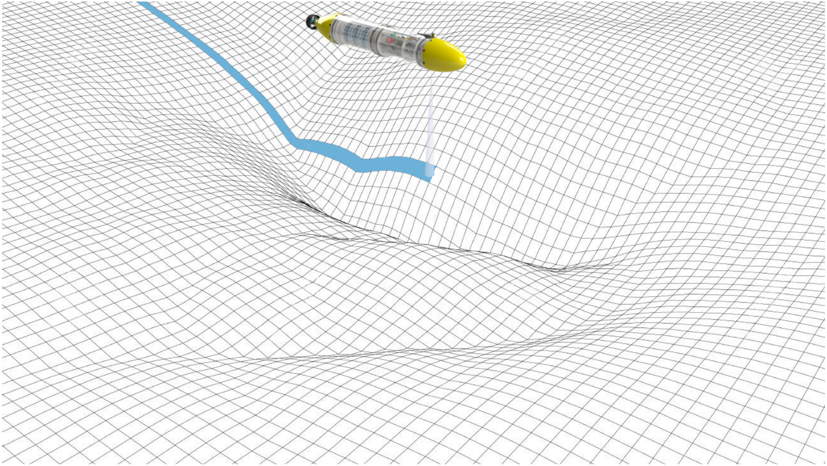
Section 5

Appendices

Appendices

Robot localization → temporal resolution

Trajectory $\mathbf{p}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^2$ crossed at times t_1, t_2: $\mathbf{p}(t_1) = \mathbf{p}(t_2)$.



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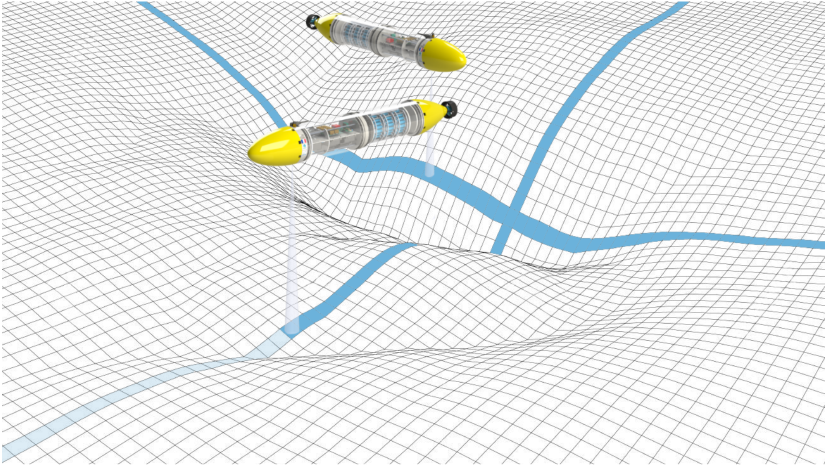


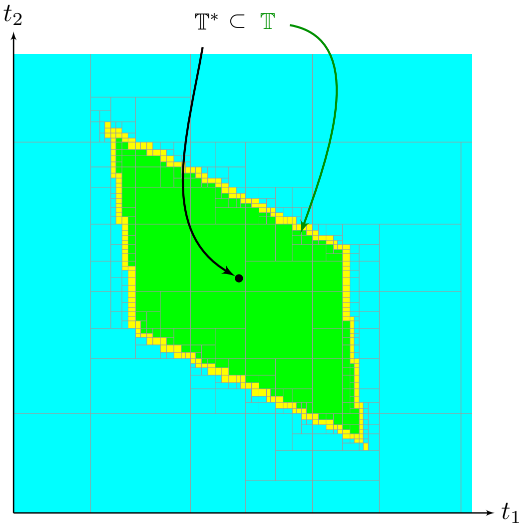
Appendices

Robot localization → temporal resolution

Constraint:

- $\mathbf{p}(t_1) = \mathbf{p}(t_2)$
- $t_1 \in [t_1]$, $t_2 \in [t_2]$

1. approximation of a temporal set $\mathbb{T}$ with evolution constraints
2. contraction of $\mathbb{T}$ thanks to exteroceptive measurements (ex: bathymetry)



Appendices

Robot localization $\rightarrow$ temporal resolution

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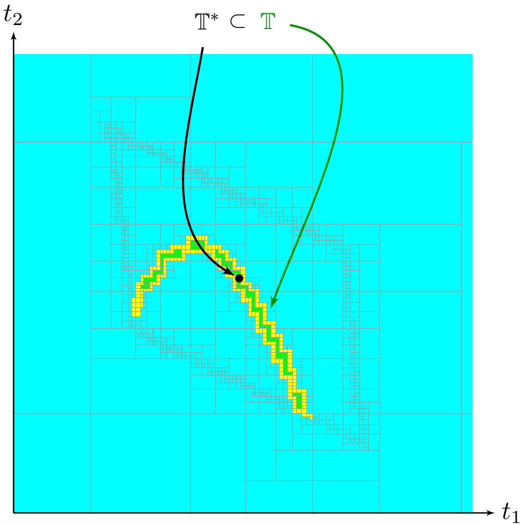

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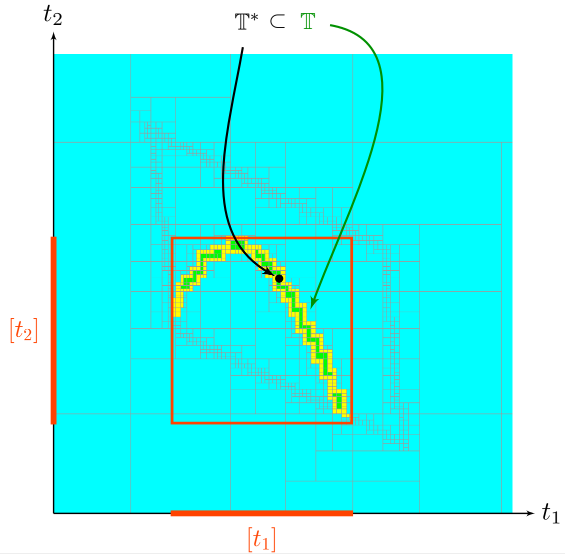

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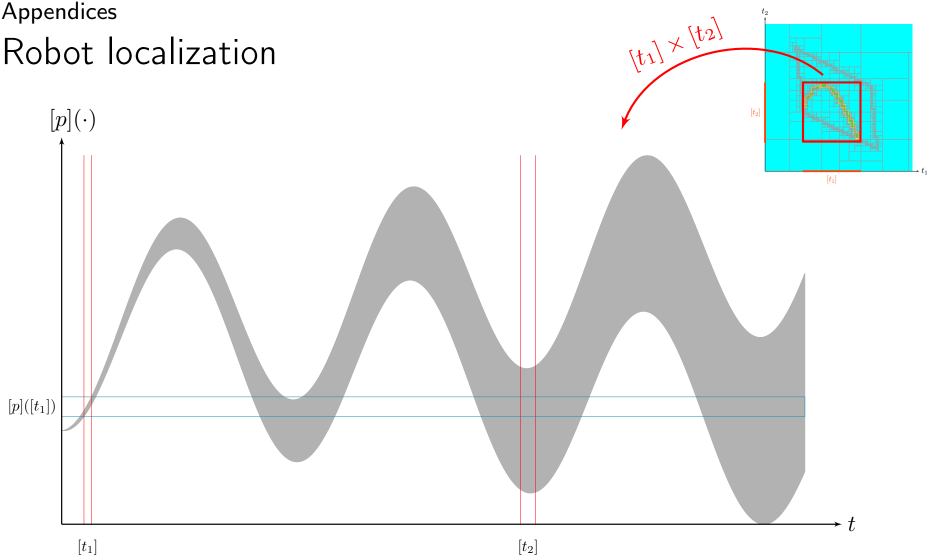
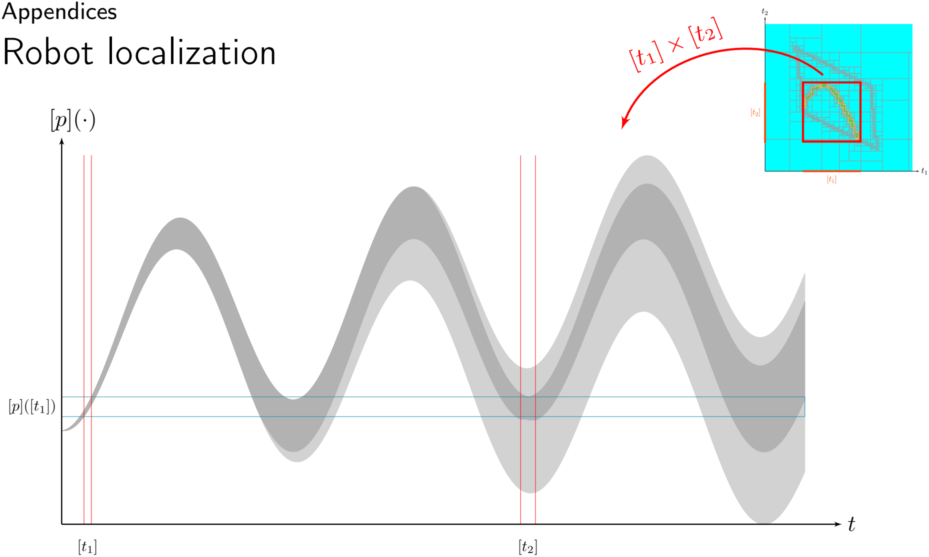
Robot localization



Constraint $\mathcal{L}_{t_1,t_2}\big(t_1, t_2, \mathbf{p}(\cdot), \mathbf{w}(\cdot)\big)$: $\begin{cases} \mathbf{p}(t_1) = \mathbf{p}(t_2) \\ \dot{\mathbf{p}}(\cdot) = \mathbf{w}(\cdot) \end{cases}$

Appendices

Robot localization



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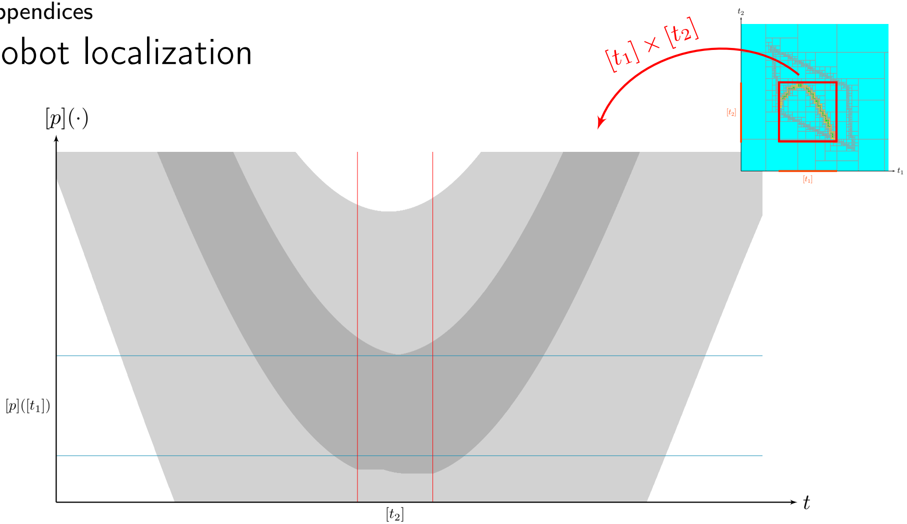
Robot localization



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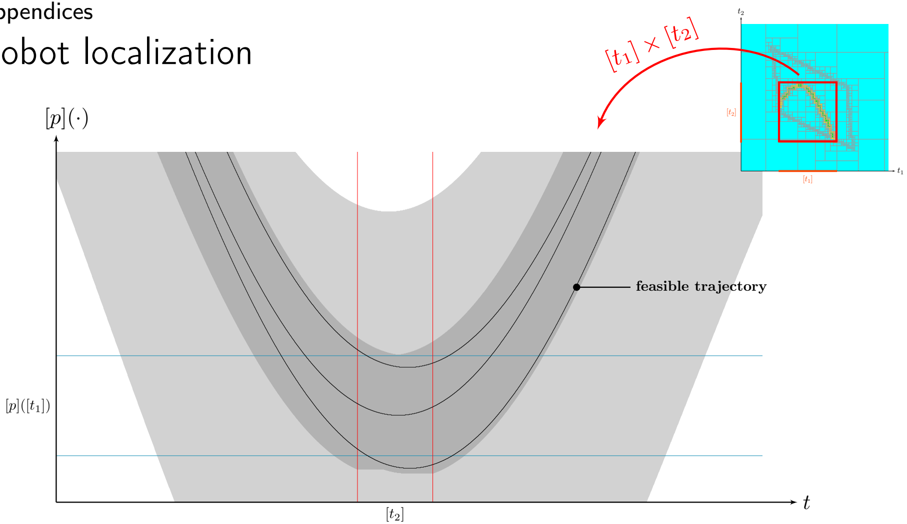
Robot localization



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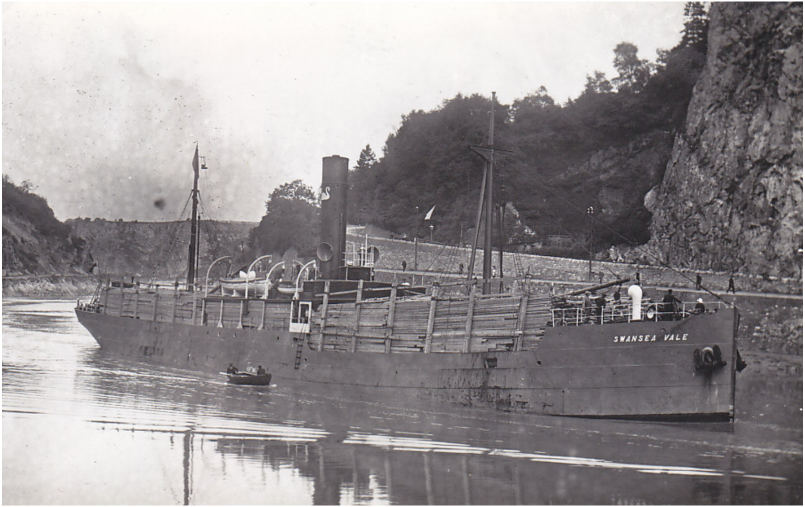
Time uncertainties in state estimation

Application example: wreck based localization



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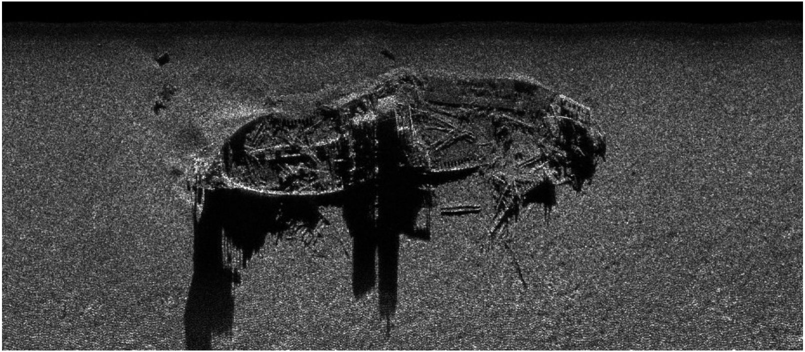
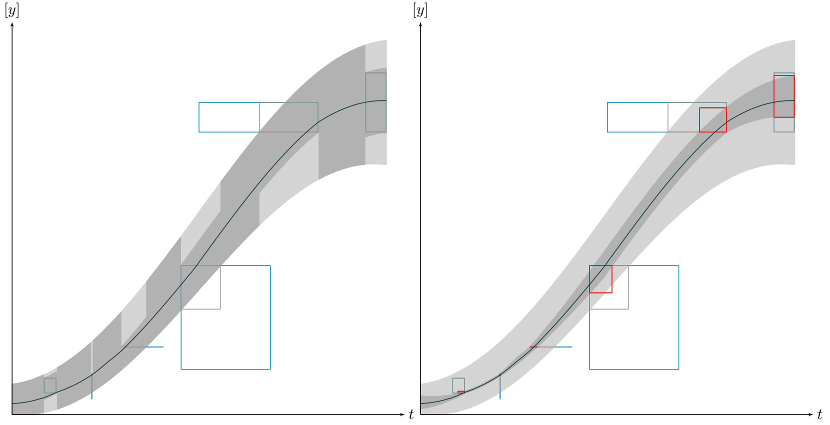


The *Swansea* wreck perceived with a side scan sonar (Rade de Brest).
The ship's funnel and superstructures cause wide shadowed areas: the darkest parts of the sonar image.
Copyrights: SHOM, DGA-TN Brest, Michel Legris.

Appendices

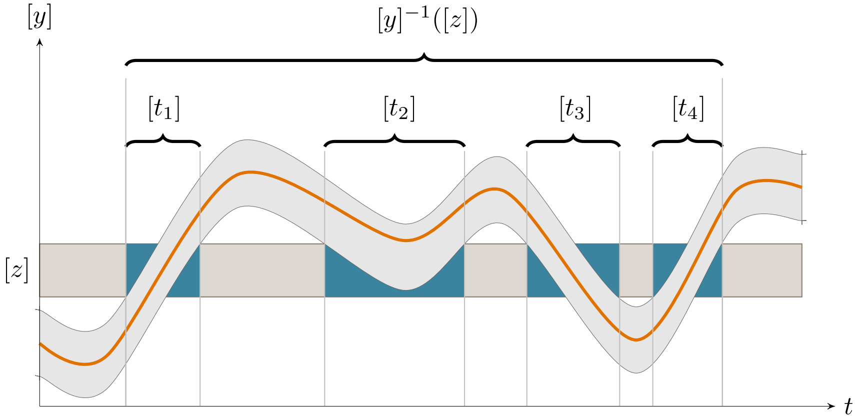
Several evaluations: fixed point iteration



Left: one iteration. Right: fixed point result.

Appendices

Tube inversion



Tube set-inversion $[y]^{-1}([z]) = \bigsqcup_{z\in[z]} \{t \mid y \in [y](t)\}$.