ROPO Extreme.
A method in Set-membership State Estimation

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Motivation

Given the following dynamic state,

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{E}\boldsymbol{\omega}_k$$

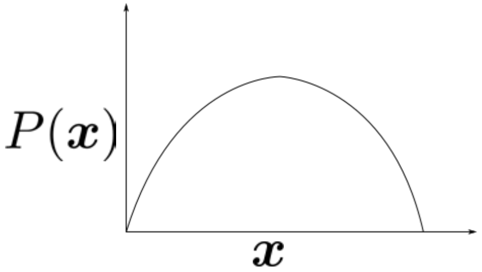
Inaccuracy in Sensors

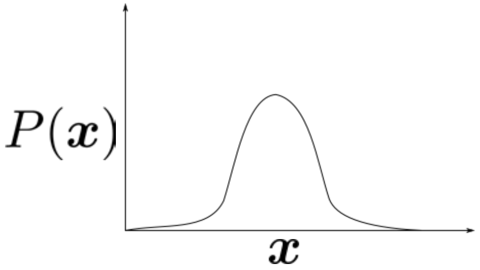
$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{F}\boldsymbol{\nu}_k$$

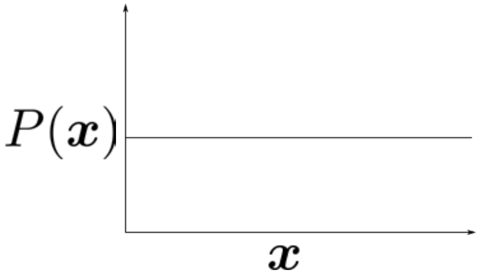
States
$\boldsymbol{x}_k \in \mathcal{X}_k$

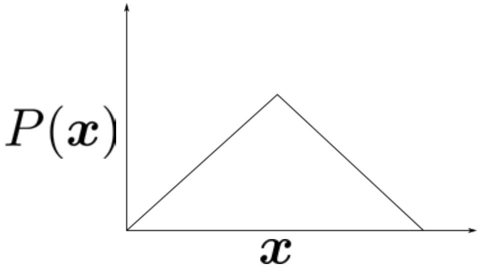
Uncertainties and nonlinnearities
$\boldsymbol{\omega}_k \in \Omega$

Noise
$\boldsymbol{\nu}_k \in \Psi$









There is no way to get the value of x_k.

How to design an observer that guarantees the state of the system?

Motivation

The Robust Estimation Problem.

- EKF, Luenberger Observer, MHE, SSE, etc.

Set-membership State Estimation.

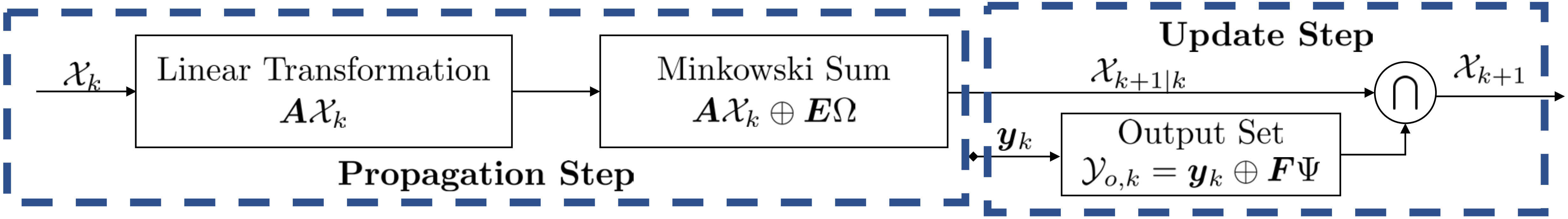
- Needs only measurement and uncertainties bounds.

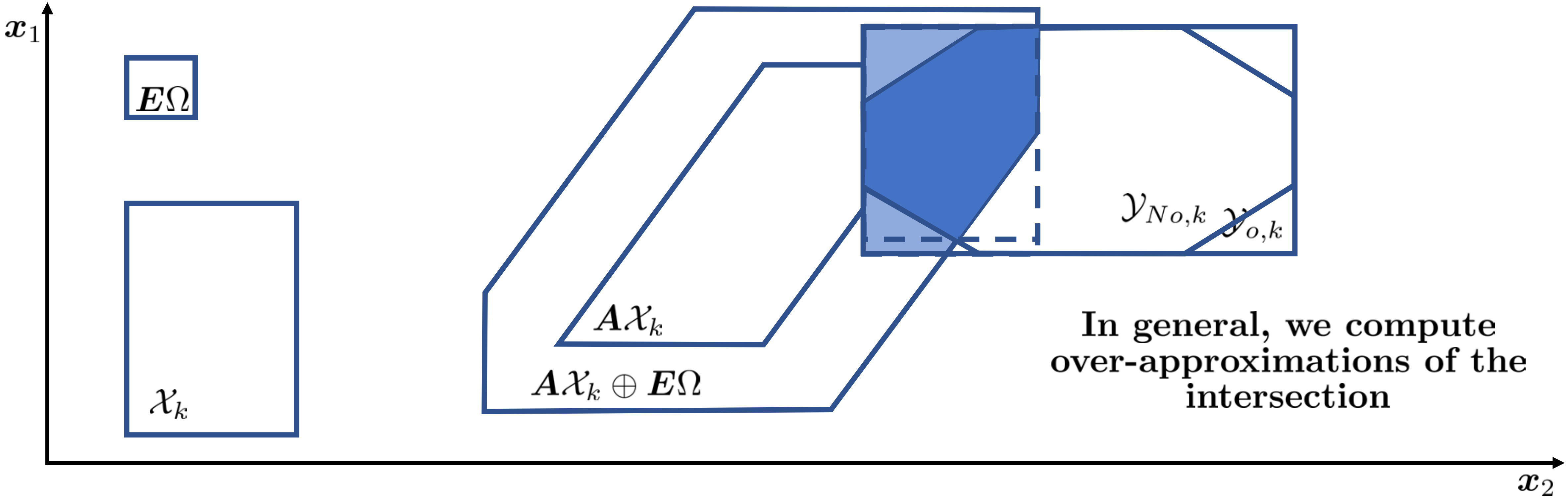
- Parallelotopes.
- Zonotopes.
- Ellipsoids.
- Polytopes.

All set-based estimation techniques require a trade-off between

- Accuracy.
- Low Complexity and Computational Cost.

Set-membership State Estimation (SSE)





In general, we compute over-approximations of the intersection

Parallelotopes

Generator Form

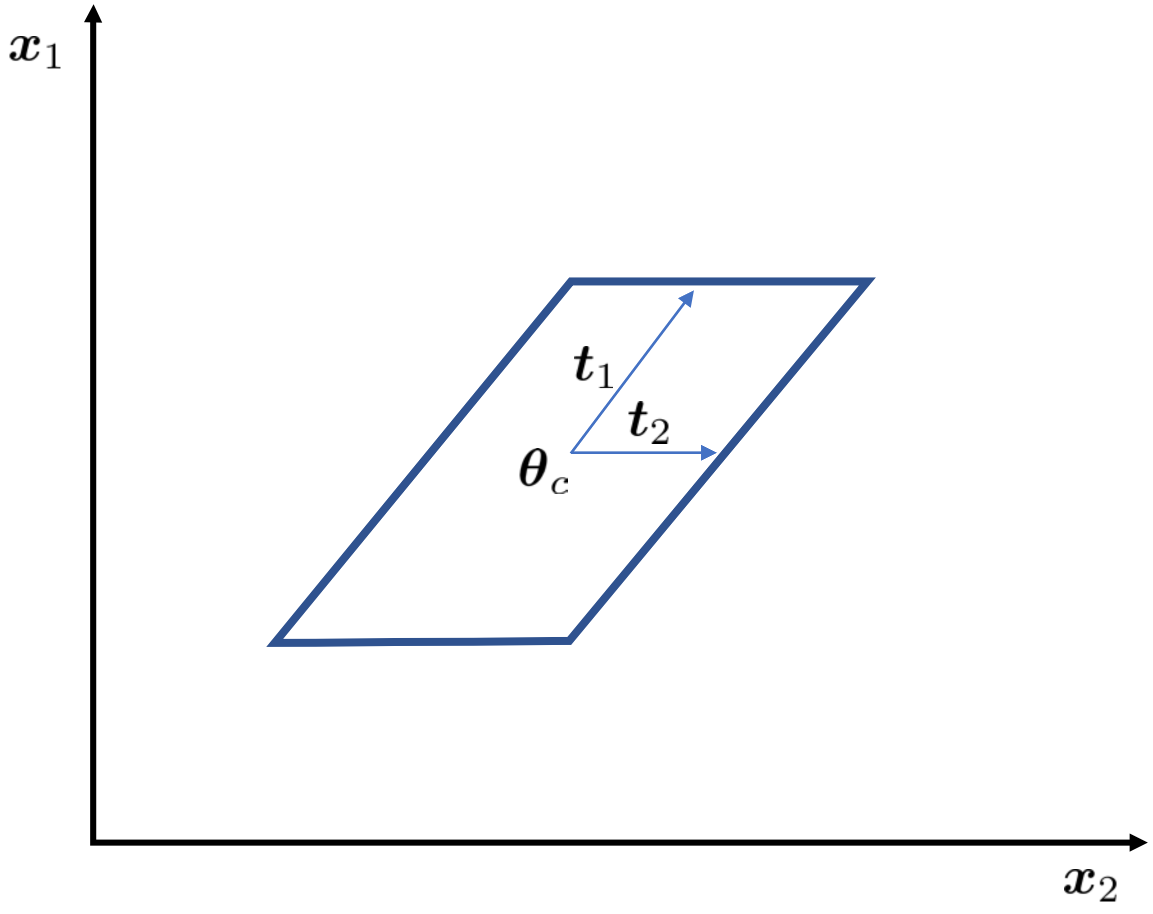
$$\mathcal{P}(\boldsymbol{T},\boldsymbol{\theta}_c) = \{\boldsymbol{x} = \boldsymbol{T}\boldsymbol{v} + \boldsymbol{\theta}_c, ||\boldsymbol{v}||_\infty \leq \boldsymbol{1}\}$$

$\boldsymbol{T}$ – Generator matrix $[\boldsymbol{t}_1 \ \ \boldsymbol{t}_2 \ \ \ldots \ \ \boldsymbol{t}_n]$

$\boldsymbol{\theta}_c$ – Center of a parallelotope.

Half-space Representation

$$\mathcal{P}(\boldsymbol{G}_p, \boldsymbol{h}_p) = \left\{\boldsymbol{x} \in \mathbb{R}^n | \begin{bmatrix} \boldsymbol{P} \\ -\boldsymbol{P} \end{bmatrix} \boldsymbol{x} \leq \begin{bmatrix} \boldsymbol{1}+\boldsymbol{c} \\ \boldsymbol{1}-\boldsymbol{c} \end{bmatrix} \right\}$$

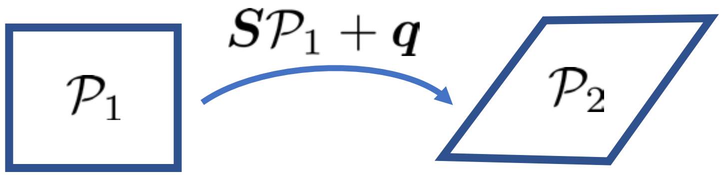


Parallelotopes

Linear Transformation

$$\mathcal{P}_2(\boldsymbol{T}_2, \boldsymbol{\theta}_{c,2}) = \boldsymbol{S}\mathcal{P}_1(\boldsymbol{T}_1, \boldsymbol{\theta}_{c,1}) + \boldsymbol{q}$$

$$\boldsymbol{T}_2 = \boldsymbol{S}\boldsymbol{T}_1 \qquad \boldsymbol{\theta}_{c,2} = \boldsymbol{S}\boldsymbol{\theta}_{c,1} + \boldsymbol{q}$$

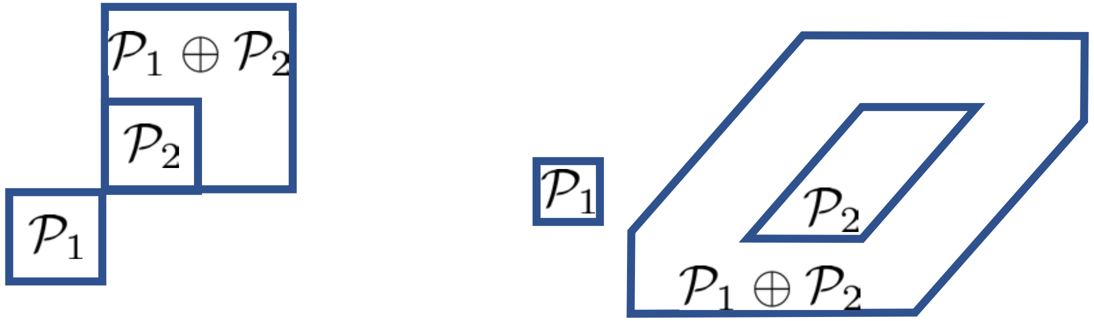


Minkowski sum

$$\mathcal{P}_1(\boldsymbol{T}_1, \boldsymbol{\theta}_{c,1}) \oplus \mathcal{P}_2(\boldsymbol{T}_2, \boldsymbol{\theta}_{c,2})$$

$$\boldsymbol{T}_s = \begin{bmatrix} \boldsymbol{T}_1 & \boldsymbol{T}_2 \end{bmatrix} \qquad \boldsymbol{\theta}_{c,s} = \boldsymbol{\theta}_{c,1} + \boldsymbol{\theta}_{c,2}$$

This is not a parallelotope



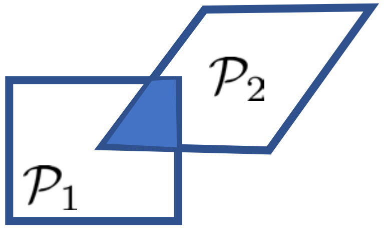
Intersection

$$\mathcal{P}_1(\boldsymbol{T}_1, \boldsymbol{\theta}_{c,1}) \bigcap \mathcal{P}_2(\boldsymbol{T}_2, \boldsymbol{\theta}_{c,2})$$

$$\boldsymbol{T}_\cap = 2\boldsymbol{T}_1(\boldsymbol{T}_1 + \boldsymbol{T}_2)^{-1}\boldsymbol{T}_2$$

$$\boldsymbol{\theta}_{c,\cap} = \boldsymbol{T}_2(\boldsymbol{T}_1 + \boldsymbol{T}_2)^{-1}\boldsymbol{\theta}_{c,1} + \boldsymbol{T}_1(\boldsymbol{T}_1 + \boldsymbol{T}_2)^{-1}\boldsymbol{\theta}_{c,2}$$

Our overapproximation proposition

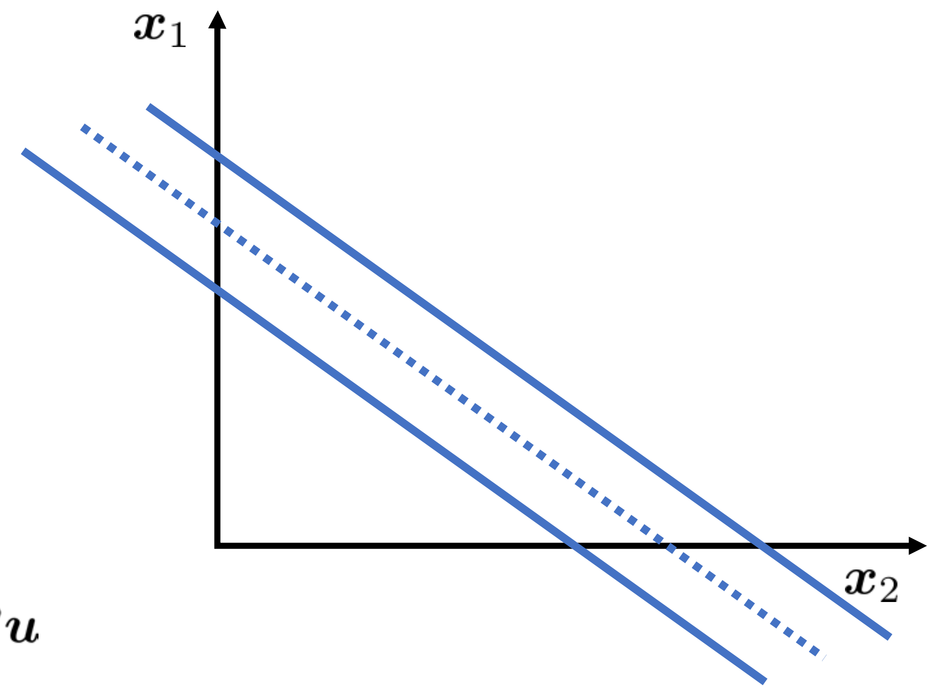


Strips

$$\mathcal{S}(\boldsymbol{p}, c) := \{\boldsymbol{x} | \; |\boldsymbol{p}^\intercal \boldsymbol{x} - c| \leq 1\}$$

$$\mathcal{H}^+ = \{\boldsymbol{x} \in \mathbb{R}^n | \; \boldsymbol{p}^\intercal \boldsymbol{x} = c + 1\}$$

$$\mathcal{H}^- = \{\boldsymbol{x} \in \mathbb{R}^n | \; \boldsymbol{p}^\intercal \boldsymbol{x} = c - 1\}$$



Strip Propagation

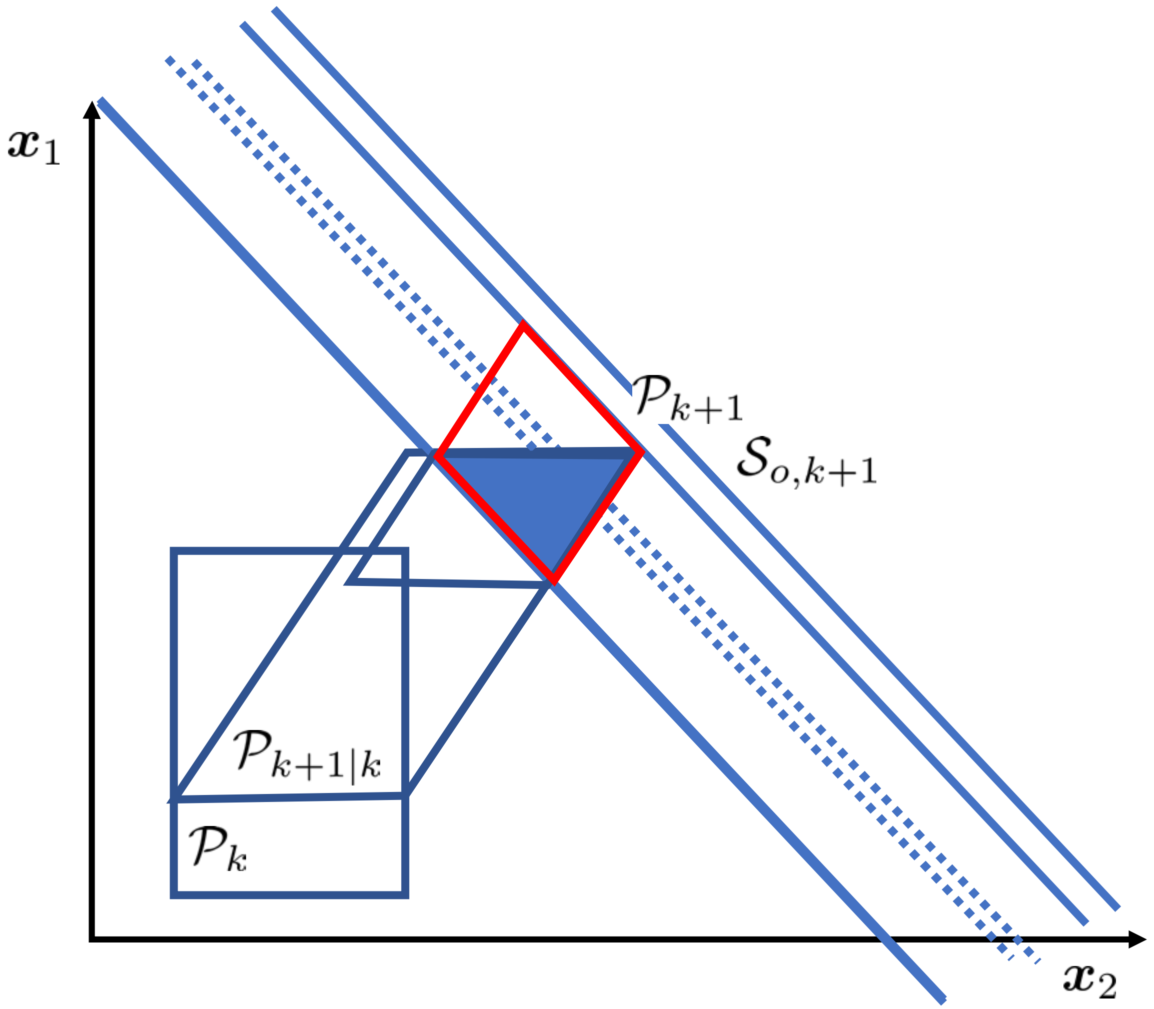
$$\boldsymbol{p}_{k+1} := \left(\boldsymbol{p}_k^\intercal \boldsymbol{A}^{-1}\right)^\intercal, \quad c_{k+1} := c_k + \boldsymbol{p}_k^\intercal \boldsymbol{A}^{-1} \boldsymbol{B}\boldsymbol{u}.$$

proof:

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u} \quad \rightarrow \quad \boldsymbol{x}_k = \boldsymbol{A}^{-1}\boldsymbol{x}_{k+1} - \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{u}$$

$$\begin{aligned} \mathcal{S}_k(\boldsymbol{p}_k, c_k) &:= \{\boldsymbol{x}_k | \; |\boldsymbol{p}_k^\intercal \boldsymbol{x}_k - c_k| \leq 1\} \\ &\rightarrow \{\boldsymbol{x}_k | \; |\boldsymbol{p}_k^\intercal (\boldsymbol{A}^{-1}\boldsymbol{x}_{k+1} - \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{u}) - c_k| \leq 1\} \\ &\rightarrow \{\boldsymbol{x}_{k+1} | \; |\underbrace{\boldsymbol{p}_k^\intercal \boldsymbol{A}^{-1}} \boldsymbol{x}_{k+1} - \underbrace{(p_k^\intercal \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{u} + c_k)}| \leq 1\} \end{aligned}$$

SSE with Parallelotopes (ROPO)

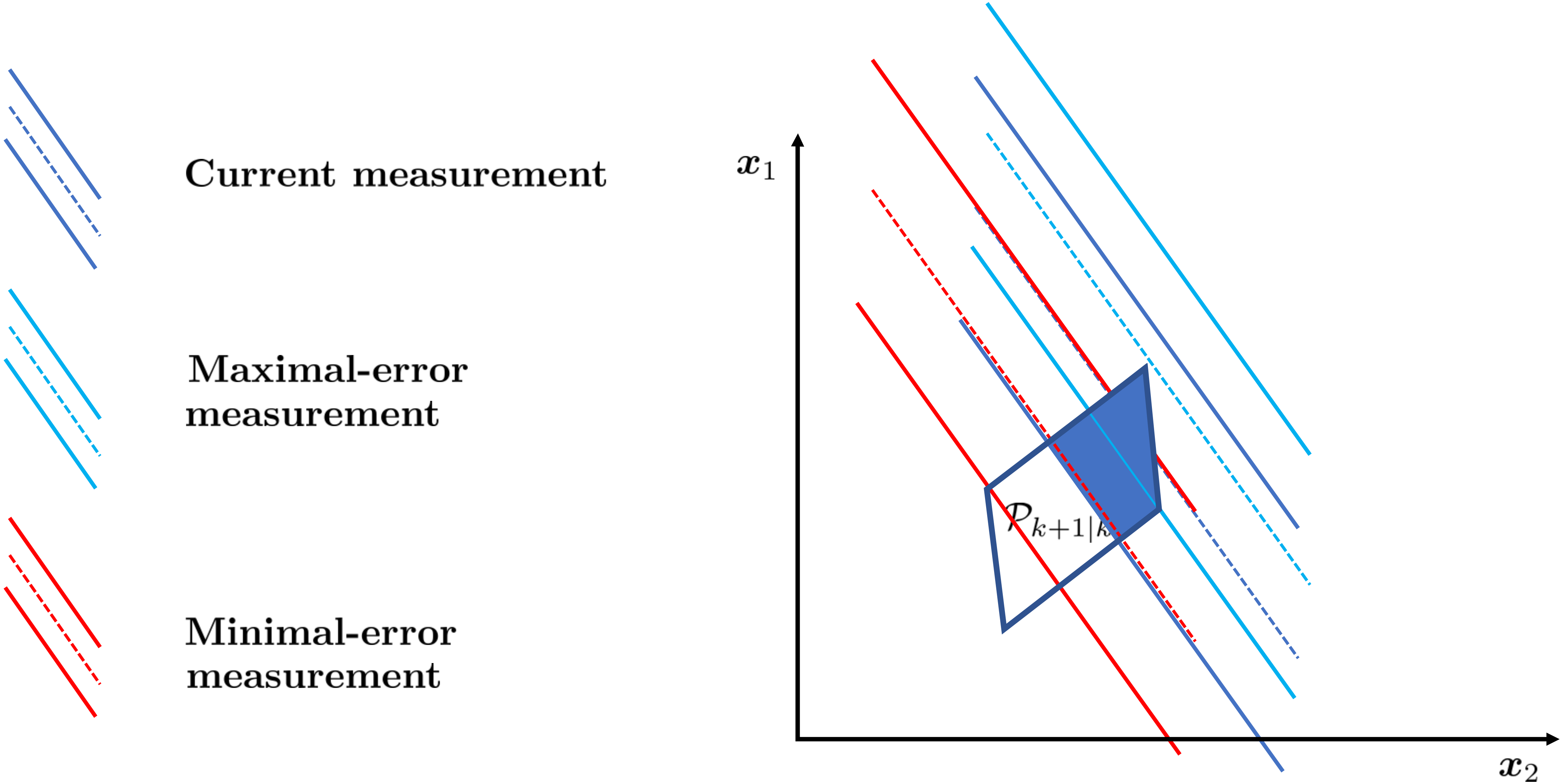


- Build the output set.
- Propagate the parallelotope.
- Find a parallelotope that enclose the intersection.

Recursive Optimal Parallelotopic Outbounding

- ROPO algorithm is developed in three steps.
  - Step 1: Reduced Strip
  - Step 2: Reduced Parallelotope
  - Step 3: Choose the minimum volume parallelotope.

Vicino, A., & Zappa, G. (1996). **Sequential approximation of feasible parameter sets for identification with set membership uncertainty.** IEEE Transactions on Automatic Control, 41(6), 774-785.



Valero, C. E., & Paulen, R. (2019). **Effective Recursive Set-membership State Estimation for Robust Linear MPC**. DYCOPS Conference, IFAC-Papers OnLine, 52(1), 486-491.

Problems

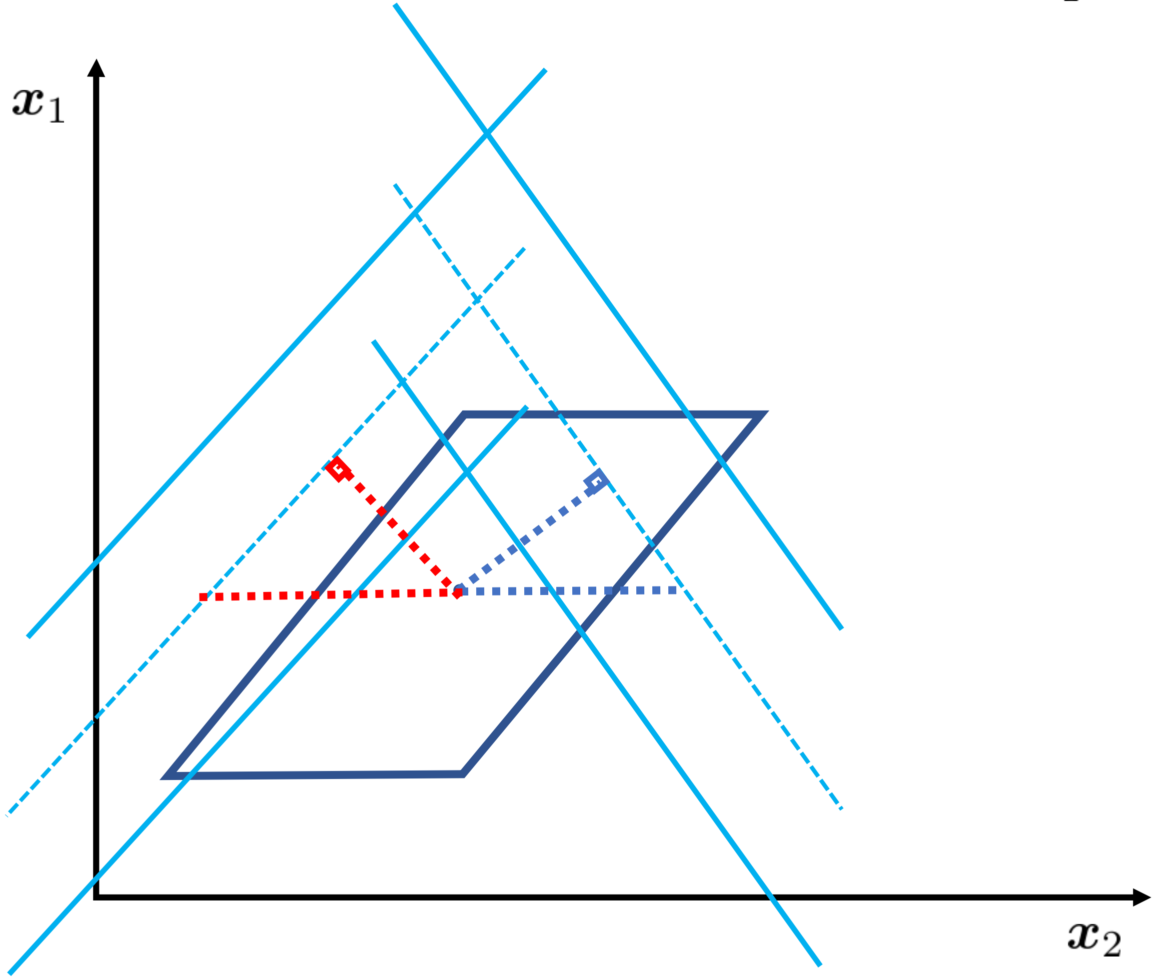
How to determine the best index criteria to find the minimum and maximum strip?

- Distance between a point and a plane.

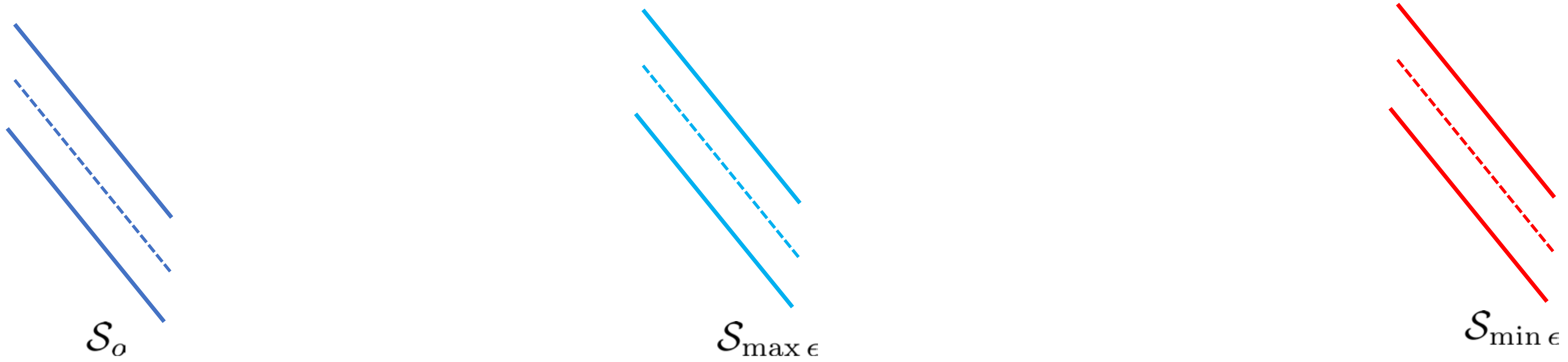
$$e := \frac{(\boldsymbol{p}^\intercal \boldsymbol{\theta}_c - c)}{\|\boldsymbol{p}\|_2}$$

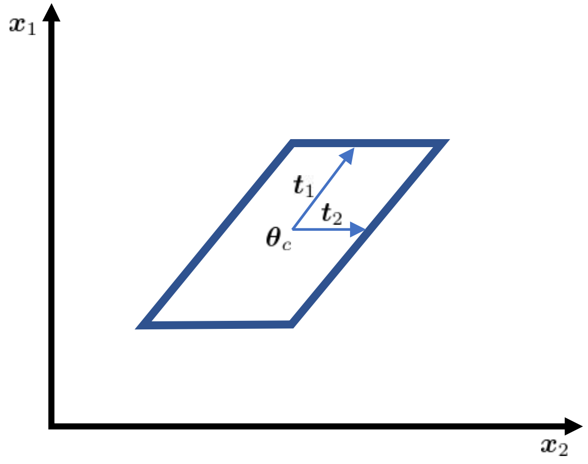
- Projection between a point and a plane.

$$e := -C\frac{\boldsymbol{p}^\intercal \boldsymbol{\theta}_c}{\|\boldsymbol{p}^\intercal\|_2}\boldsymbol{p}$$



ROPO Extreme





- Keep strips with extremal measurements.
- Propagate the parallelotope and the extremal measurement strips through the linear dynamics.
- Perform estimation taking the new measurement information and the extremal measurements.
- Update the extremal-measurements strips based on the prediction error.

Valero, C. E., & Paulen, R. (2019). **Set-Theoretic State Estimation for Multi-output Systems using Block and Sequential Approaches**. In 2019 22nd International Conference on Process Control (PC19) (pp. 268-273). IEEE.

Simulation Studies

Case 1: SSE with Parallelotope

$$\boldsymbol{A} := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{B} := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \boldsymbol{E} = \boldsymbol{\eta} := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \boldsymbol{C} := \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \boldsymbol{F} := 1$$

Constraints

$$-1 \leq u \leq 1$$

$$x_1 \geq 0$$

$$y_m = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{\varepsilon}$$

$$\mid \boldsymbol{\varepsilon} \mid \leq 1$$

Initial conditions

$$x_0 := \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

$$\mathcal{P}_0 := \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \boldsymbol{v} + \begin{bmatrix} 21 \\ 1 \end{bmatrix} \middle| \, \|\boldsymbol{v}\|_\infty \leq 1 \right\}$$

$$\boldsymbol{Q} = \boldsymbol{I}, R = 1 \times 10^{-6}, N_p = 10, N_u = N_p$$

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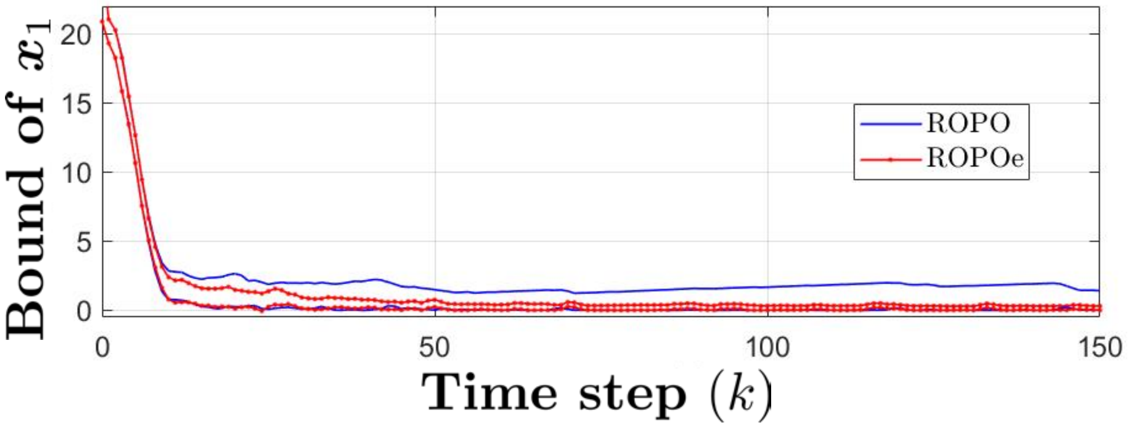
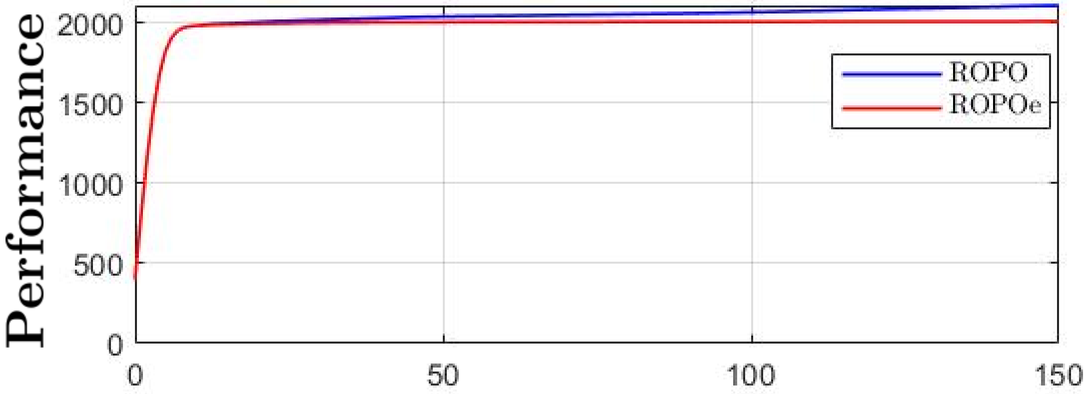
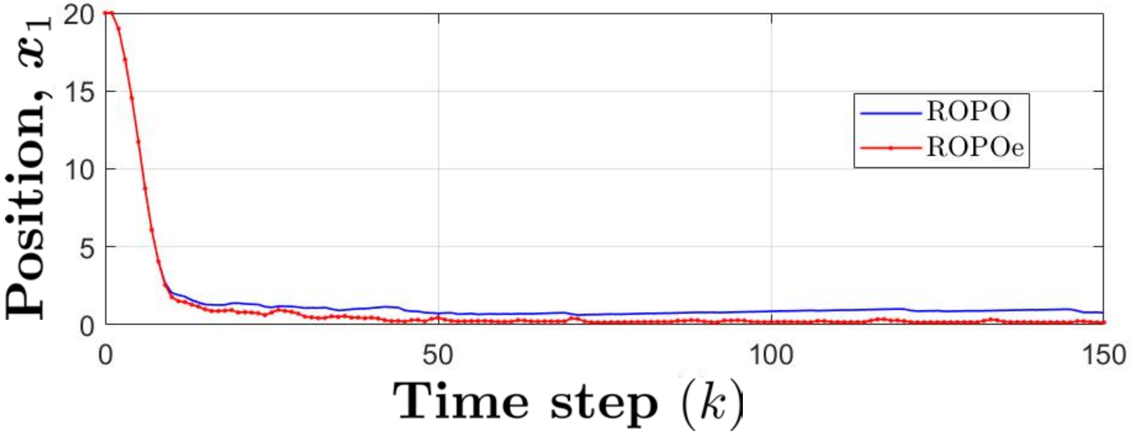
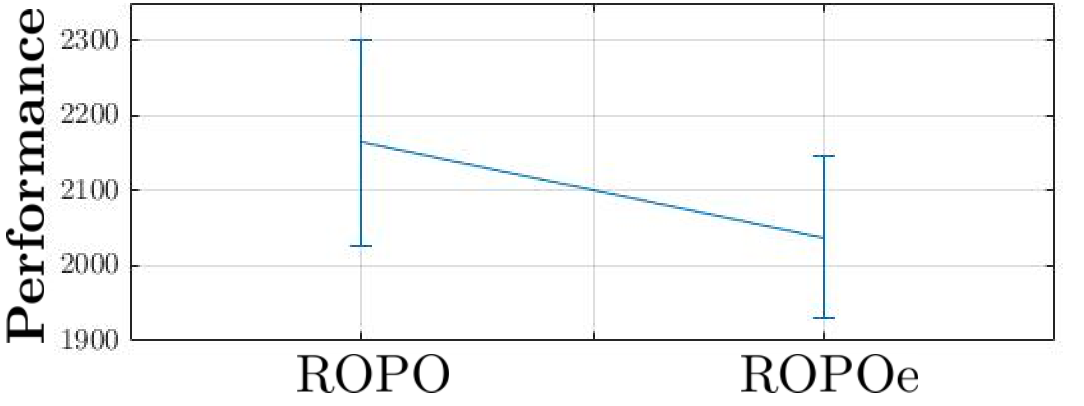
Simulation Studies









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Conclusions

- It is possible to enhance SSE taking into account past measurements of the system.

ROPO with extremal measurements(ROPOe)

- The propagation of one strip according to the dynamic of the plant.

$$\boldsymbol{p}_{k+1} := \left(\boldsymbol{p}_k^\intercal \boldsymbol{A}^{-1}\right)^\intercal, \quad c_{k+1} := c_k + \boldsymbol{p}_k^\intercal \boldsymbol{A}^{-1} \boldsymbol{B} \boldsymbol{u}.$$

- The studied concepts apply to other bounded-error estimators.

Zonotopic Kalman Filter, Zonotopes, etc.