Observability of Nonlinear Systems and Injectivity

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International Online Seminar on Interval Methods in Control Engineering
12.02.2021

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Introduction

Systems

$$\dot{x}(t) = f(x(t)), \quad x(0) = x^0$$
$$y(t) = h(x(t))$$

Definitions

y^1 corresponds to $x^0 = z^1$
y^2 corresponds to $x^0 = z^2$

Observability

For all $z^1 \neq z^2$ we have $y^1 \neq y^2$.

Statement of the Problem

- How to check observability?
- Find computational condition!
- Simpler problem: Check
  For all $z^1 \neq z^2 \in x^I$ we have $y^1 \neq y^2$.
- x^I interval vector

Lie Series

Taylor series expansion of y

$$y\left(t\right)=\sum_{k=0}^{\infty}\frac{t^{k}}{k!}\left(L_{f}^{k}h\right)\left(x^{0}\right)$$

Lie derivative

$$\begin{aligned}\left(L_{f}h\right)\left(x\right) &= \left(\frac{\partial h}{\partial x}\right)f(x)\\ \left(L_{f}^{0}h\right)\left(x\right) &= h(x)\\ \left(L_{f}^{l+1}h\right)\left(x\right) &= \left(L_{f}\ L_{f}^{l}h\right),l\geq 1\end{aligned}$$

f,h real analytic $\Rightarrow$ $y(t)$ real analytic.

Reformulation

$$y^i(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(L_f^k h\right)\left(z^i\right)$$

When is $y^1(t) = y^2(t)$?

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \left(L_f^k h\right)(z^1) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(L_f^k h\right)(z^2)$$

$$\Downarrow$$

$$\left(L_f^k h\right)(z^1) = \left(L_f^k h\right)(z^2), k = 0, 1, 2, \ldots$$

Observability mapping

$$F(x) := \begin{pmatrix} h(x) \\ (L_f h)(x) \\ \left(L_f^2 h\right)(x) \\ \vdots \end{pmatrix}$$

Injectivity Condition

$$F(z^1) = F(z^2) \Rightarrow z^1 = z^2 \text{ uniquely}$$

$$\Rightarrow \text{System is observable.}$$

- This is injectivity of F!
- First step: finite number of columns

Injectivity Condition

- $F(z^1) = F(z^2)$
  for $z^1, z^2 \in x^I \Rightarrow z^1 = z^2$ uniquely
- $F(z^1) - F(z^2) = 0$
- Idea: Apply mean value theorem

$$g\left(z^1\right) - g\left(z^2\right) = \frac{\partial g}{\partial x}(\xi)\left(z^1 - z^2\right),$$

g scalar, ξ between z^1 and z^2

- $g = \begin{pmatrix} g_1 \\ \vdots \\ g_p \end{pmatrix}, g_i\left(z^1\right) - g_i\left(z^2\right) = \frac{\partial g_i}{\partial x}\left(z^1 - z^2\right)$

$$g\left(z^1\right) - g\left(z^2\right) = \begin{pmatrix} \frac{\partial g_1}{\partial x}(\xi_1) \\ \vdots \\ \frac{\partial g_p}{\partial x}(\xi_p) \end{pmatrix}\left(z^1 - z^2\right)$$

$$M = \begin{pmatrix} \frac{\partial g_1}{\partial x}(\xi_1) \\ \vdots \\ \frac{\partial g_p}{\partial x}(\xi_p) \end{pmatrix}, M \neq \frac{\partial g}{\partial x}(\xi)$$

$M \in \frac{\partial g}{\partial x}\left(x^I\right) = M^I$ Interval matrix

M full rank $\Rightarrow z^1 - z^2 = 0 \Rightarrow g$ injective

Injectivity Condition

- Interval matrix M^I full rank $\Rightarrow$ M full rank
  $\Rightarrow$ F injective $\Rightarrow$ system observable on x^I
- M^I full rank?
- Apply result of Jiri Rohn (sufficient condition)
- In general NP hard
- Application in the talk by Thomas Paradowski

Thank you for your attention