Observability of Uncertain Nonlinear Systems Using Interval Analysis

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What to expect of an algorithm?

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Problem Formulation

nonlinear uncertain systems

$$\dot{x}(t) = f(x(t), \alpha), \quad x(0) = x_0$$

$$y(t) = h(x(t), \beta)$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^p$ as real-analytic functions, $x(t) \in D_x \subseteq \mathbb{R}^n$, system uncertainties $\alpha \in \mathbb{R}^a$ and $\beta \in \mathbb{R}^b$

Basic Approach

Linearize the system

$$A = \left(\frac{\partial f(x)}{\partial x}\right)\bigg|_{x=x_0}$$

$$C = \left(\frac{\partial h(x)}{\partial x}\right)\bigg|_{x=x_0}$$

Observability matrix

$$Q(x_0) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Kalman rank condition

If $\text{rank}(Q(x_0)) = n$ then the system is observable in x_0

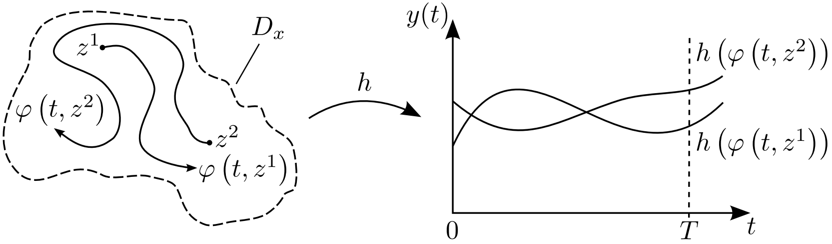
Definition: local observability of autonomous systems

If $\text{rank}(Q(x)) = n$ for all $x \in D_x$, then the system

$$\begin{aligned} \dot{x}(t) &= f(x(t)), \quad x(0) = x_0 \\ y(t) &= h(x(t)) \end{aligned}$$

is locally observable.

Measuring the Trajectory



Definition: distinguishability

Let $T > 0$. Two states $z^1, z^2 \in D_x$ are called **indistinguishable** on an interval $[0, T]$, if the corresponding trajectories of the outputs match, such that

$$\forall t \in [0, T]: \quad h\left(\varphi\left(t, z^1\right)\right) = h\left(\varphi\left(t, z^2\right)\right).$$

Otherwise the states are called **distinguishable**. [1]

Observability of Nonlinear Systems

Definition: Global Observability with distinguishability

A system defined on $D_x \subseteq \mathbb{R}^n$ is globally observable if and only if all pairs z^1 and z^2 are distinguishable. That is the sets

$$\mathcal{M} = \left\{ \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \in \mathbb{R}^{2n} \middle| h\left(\varphi\left(t, z^1\right)\right) = h\left(\varphi\left(t, z^2\right)\right) \right\},$$
$$\mathcal{D} = \left\{ \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \in \mathbb{R}^{2n} \middle| z^1 = z^2 \right\}$$

are identical. It holds $\mathcal{M} = \mathcal{D}$. [2]

Distinguishability with Lie-Derivative

Taylor series

$$y(t) = h(\varphi(t,x)) = \sum_{k=0}^{\infty} \frac{t^k}{k!} L_f^k h(x)$$

Lie-derivatives

$$\begin{aligned} L_f^0 h(x) &= h(x) \\ L_f^k h(x) &= \frac{\partial}{\partial x}\left(L_f^{k-1} h(x)\right) f(x) \end{aligned}$$

$$\begin{aligned} x &\mapsto h(\varphi(t,x)) && \mathbb{R}^n \mapsto \mathbb{R}^p \\ x &\mapsto \left(L_f^k h(x)\right)_{k=0}^{\infty} && \mathbb{R}^n \mapsto \mathbb{R}^\infty \end{aligned}$$

Observability function

$$q(x) = \begin{bmatrix} L_f^0 h(x) \\ L_f^1 h(x) \\ \vdots \\ L_f^{\kappa-1} h(x) \end{bmatrix}$$

Distinguishability

$$\begin{aligned} q\left(z^1\right) &= q\left(z^2\right) \\ z^1 &= z^2 \end{aligned}$$

Example System

Nonlinear system

$$f(x) = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$$
$$h(x) = x_1^2 + x_1$$

Lie-derivatives

$$L_f^0 h(x) = h(x)$$
$$L_f^k h(x) = \frac{\partial}{\partial x}\left(L_f^{k-1} h(x)\right) f(x)$$

The first two Lie-derivatives

$$L_f^0 h(x) = x_1^2 + x_1$$
$$L_f^1 h(x) = 2x_1 x_2 + x_2$$

Solution for $L_f^0 h(z^1) = L_f^0 h(z^2)$ and $L_f^1 h(z^1) = L_f^1 h(z^2)$

$$\begin{pmatrix} z_1^1 \\ z_2^1 \end{pmatrix} = \left\{ \begin{pmatrix} z_1^2 \\ z_2^2 \end{pmatrix}, \begin{pmatrix} -z_1^2 - 1 \\ -z_2^2 \end{pmatrix} \right\}$$

Example System

Nonlinear system

$$\begin{aligned} f(x) &= \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \\ h(x) &= x_1^2 + x_1 \end{aligned}$$

Lie-derivatives

$$\begin{aligned} L_f^0 h(x) &= h(x) \\ L_f^k h(x) &= \frac{\partial}{\partial x}\left(L_f^{k-1} h(x)\right) f(x) \end{aligned}$$

The first two Lie-derivatives

$$\begin{aligned} L_f^0 h(x) &= x_1^2 + x_1 \\ L_f^1 h(x) &= 2x_1 x_2 + x_2 \end{aligned}$$

Solution for $L_f^0 h(z^1) = L_f^0 h(z^2)$ and $L_f^1 h(z^1) = L_f^1 h(z^2)$

$$\begin{pmatrix} z_1^1 \\ z_2^1 \end{pmatrix} = \left\{ \begin{pmatrix} z_1^2 \\ z_2^2 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} \right\}$$

Calculating the next Lie-derivatives

$$L_f^2 h(x) = 2x_2^2 - 2x_1^2 - x_1$$

Example System

Nonlinear system

$$\begin{aligned} f(x) &= \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \\ h(x) &= x_1^2 + x_1 \end{aligned}$$

Lie-derivatives

$$\begin{aligned} L_f^0 h(x) &= h(x) \\ L_f^k h(x) &= \frac{\partial}{\partial x}\left(L_f^{k-1} h(x)\right)\ f(x) \end{aligned}$$

The first two Lie-derivatives

$$\begin{aligned} L_f^0 h(x) &= x_1^2 + x_1 \\ L_f^1 h(x) &= 2x_1x_2 + x_2 \end{aligned}$$

Solution for $L_f^0 h(z^1) = L_f^0 h(z^2)$ and $L_f^1 h(z^1) = L_f^1 h(z^2)$

$$\begin{pmatrix} z_1^1 \\ z_2^1 \end{pmatrix} = \left\{\begin{pmatrix} z_1^2 \\ z_2^2 \end{pmatrix}\right\}$$

Calculating the next Lie-derivatives

$$\begin{aligned} L_f^2 h(x) &= 2x_2^2 - 2x_1^2 - x_1 \\ L_f^3 h(x) &= -8x_1x_2 - x_2 \end{aligned}$$

Now it holds $\mathcal{M} = \mathcal{D}$. Therefore, the system is distinguishable and therefore globally observable.

Lie Derivatives Symbolically

Example nonlinear system

$$\begin{aligned} f(x) &= \begin{pmatrix} -x_2 + x_1\left(1 - x_1^2 - x_2^2\right) \\ x_1 + x_2\left(1 - x_1^2 - x_2^2\right) \end{pmatrix} \\ h(x) &= x_1 + x_2 \end{aligned}$$

Lie derivative $L_f^4 h(x)$

$$L_f^4 h(x) = (x_2 + x_1(x_1^2 + x_2^2 - 1))((x_2 + x_1(x_1^2 + x_2^2 - 1))(6x_2 - 2x_1 + 2(2x_1 + 2x_2)(2x_1x_2 - 1) + 2x_2(2x_1x_2 + x_1^2 + 3x_2^2) + 6x_1(2x_1x_2 + 3x_1^2 + x_2^2 - 2) + 2(6x_1 + 2x_2)(3x_1^2 + x_2^2 - 1)$$

$$+6x_1(x_1^2 + x_2^2 - 1) + 2x_2(x_1^2 + x_2^2 - 1)) - (x_1 - x_2(x_1^2 + x_2^2 - 1))(2x_2 - 2x_1 + (2x_1 + 6x_2)(2x_1x_2 - 1) + (6x_1 + 2x_2)(2x_1x_2 + 1) + 2x_1(2x_1x_2 + x_1^2 + 3x_2^2) + 2x_2(2x_1x_2 + 3x_1^2 + x_2^2 - 2)$$

$$+(2x_1 + 2x_2)(3x_1^2 + x_2^2 - 1) + (2x_1 + 2x_2)(x_1^2 + 3x_2^2 - 1) + 2x_1(x_1^2 + x_2^2 - 1) + 2x_2(x_1^2 + x_2^2 - 1)) + (2x_1x_2 - 1)((2x_1 + 2x_2)(x_2 + x_1(x_1^2 + x_2^2 - 1)) - (2x_1 + 6x_2)(x_1 - x_2(x_1^2 + x_2^2 - 1))$$

$$+(x_1^2 + 3x_2^2 - 1)(2x_1x_2 + x_1^2 + 3x_2^2) + (2x_1x_2 + 1)(2x_1x_2 + 3x_1^2 + x_2^2 - 2)) + (3x_1^2 + x_2^2 - 1)((6x_1 + 2x_2)(x_2 + x_1(x_1^2 + x_2^2 - 1)) - (2x_1 + 2x_2)(x_1 - x_2(x_1^2 + x_2^2 - 1))$$

$$+(3x_1^2 + x_2^2 - 1)(2x_1x_2 + 3x_1^2 + x_2^2 - 2) + (2x_1x_2 - 1)(2x_1x_2 + x_1^2 + 3x_2^2))) - (x_1 - x_2(x_1^2 + x_2^2 - 1))((x_2 + x_1(x_1^2 + x_2^2 - 1))(2x_2 - 2x_1 + (2x_1 + 6x_2)(2x_1x_2 - 1) + (6x_1 + 2x_2)(2x_1x_2 + 1)$$

$$+2x_1(2x_1x_2 + x_1^2 + 3x_2^2) + 2x_2(2x_1x_2 + 3x_1^2 + x_2^2 - 2) + (2x_1 + 2x_2)(3x_1^2 + x_2^2 - 1) + (2x_1 + 2x_2)(x_1^2 + 3x_2^2 - 1) + 2x_1(x_1^2 + x_2^2 - 1) + 2x_2(x_1^2 + x_2^2 - 1))$$

$$-(x_1 - x_2(x_1^2 + x_2^2 - 1))(2x_2 - 6x_1 + 2(2x_1 + 2x_2)(2x_1x_2 + 1) + 6x_2(2x_1x_2 + x_1^2 + 3x_2^2) + 2x_1(2x_1x_2 + 3x_1^2 + x_2^2 - 2) + 2(2x_1 + 6x_2)(x_1^2 + 3x_2^2 - 1) + 2x_1(x_1^2 + x_2^2 - 1) + 6x_2(x_1^2 + x_2^2 - 1))$$

$$+(2x_1x_2 + 1)((6x_1 + 2x_2)(x_2 + x_1(x_1^2 + x_2^2 - 1)) - (2x_1 + 2x_2)(x_1 - x_2(x_1^2 + x_2^2 - 1)) + (3x_1^2 + x_2^2 - 1)(2x_1x_2 + 3x_1^2 + x_2^2 - 2) + (2x_1x_2 - 1)(2x_1x_2 + x_1^2 + 3x_2^2)) + (x_1^2 + 3x_2^2 - 1)((2x_1 + 2x_2)$$

$$(x_2 + x_1(x_1^2 + x_2^2 - 1)) - (2x_1 + 6x_2)(x_1 - x_2(x_1^2 + x_2^2 - 1)) + (x_1^2 + 3x_2^2 - 1)(2x_1x_2 + x_1^2 + 3x_2^2) + (2x_1x_2 + 1)(2x_1x_2 + 3x_1^2 + x_2^2 - 2)))$$

Power Series to calculate the Lie-Derivatives

Base object

$$a(t) := \sum_{k=0}^{\infty} \mathfrak{a}_k \, t^k$$

Calculation rules

$$\begin{aligned}
(a \pm b)(t) &= \sum_{k=0}^{\infty} (\mathfrak{a}_k \pm \mathfrak{b}_k) \, t^k \\
(a \cdot b)(t) &= \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} \mathfrak{a}_l \cdot \mathfrak{b}_{k-l} \right) t^k \\
(a/b)(t) &= \sum_{k=0}^{\infty} \left(\frac{\mathfrak{a}_k}{\mathfrak{b}_0} - \frac{1}{\mathfrak{b}_0} \sum_{l=0}^{k-1} \frac{\mathfrak{a}_0}{\mathfrak{b}_0} \mathfrak{b}_{k-l} \right) t^k
\end{aligned}$$

It follows

$$x(t) = \sum_{k=0}^{\infty} \mathfrak{x}_k \, t^k \text{ and } f(x(t)) = \sum_{k=0}^{\infty} \mathfrak{f}_k \, t^k$$

Calculation of the coefficients

$$\mathfrak{x}_{k+1} = \frac{1}{k+1} \mathfrak{f}_k$$

with $\mathfrak{f}_k = f(\mathfrak{x}_0, \mathfrak{x}_1, \ldots, \mathfrak{x}_k)$.
Calculating the Lie-Derivatives

from $y(t) = \sum_{k=0}^{\infty} \mathfrak{y}_k \, t^k$ follows $\mathfrak{y}_k = \frac{1}{k!} L_f^k h(x_0)$

Example on Calculating the Lie-Derivatives with the Power Series

Nonlinear System

$$
\begin{aligned}
f(x) &= \begin{pmatrix} x_2 \\ -x_1 + x_1^2 x_2 \end{pmatrix} \\
h(x) &= x_1 + x_2^3
\end{aligned}
$$

with the initial value $x(0) = \mathfrak{x}_0 = (2 \;\; 4)^\mathsf{T}$.
The Lie-derivatives and solutions are

$$
\begin{aligned}
L_f^0 h(x) &= x_1 + x_2^3 &&= 66 \\
L_f^1 h(x) &= x_2 - 3x_1 x_2^2 + 3x_1^2 x_2^3 &&= 676
\end{aligned}
$$

Calculation with power series

$$\mathfrak{x}_1 = \frac{1}{0+1} f(\mathfrak{x}_0) = (4 \; 14)^\mathsf{T}$$

$$
\begin{aligned}
h(\mathfrak{x}_0, \mathfrak{x}_1) &= \mathfrak{x}_{1,1} + (\mathfrak{x}_{2,0}\mathfrak{x}_{2,0})\,\mathfrak{x}_{2,1} \\
&\quad + (\mathfrak{x}_{2,0}\mathfrak{x}_{2,1} + \mathfrak{x}_{2,1}\mathfrak{x}_{2,0})\,\mathfrak{x}_{2,0} \\
&= 676
\end{aligned}
$$

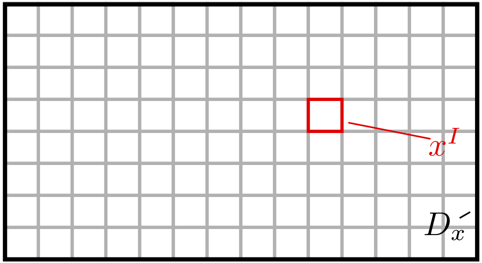
Using Interval Arithmetics

Definition: Interval

$$x^I = [\underline{x}\ \overline{x}]\,,\ \text{with } \underline{x} \leq \overline{x}$$

Distinguishability

$$q^I\left(x^I\right) \cap q^I\left(x_i^I\right) = \emptyset$$



Using Interval Arithmetics

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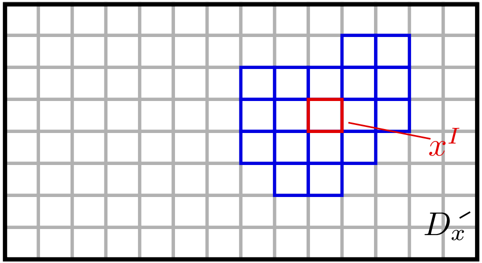

Using Interval Arithmetics

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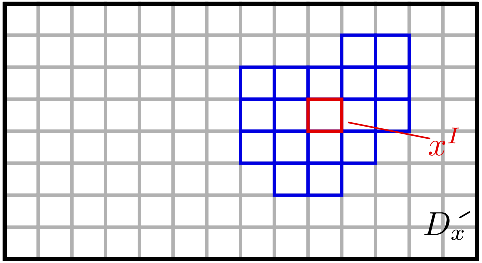
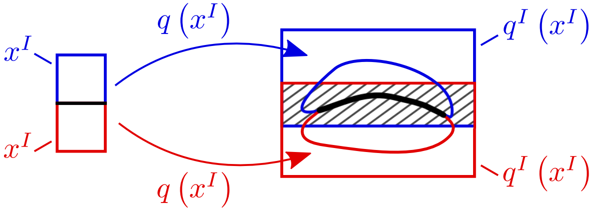
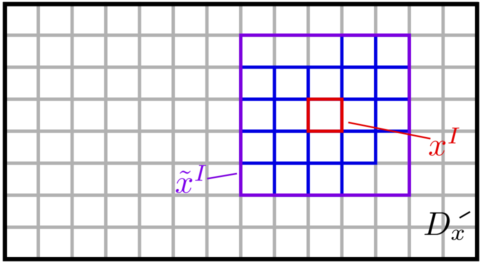




Global Problem Broken Down to an Local Problem

Definition: local observability of autonomous systems

If rank$(Q(x)) = n$ for all $x \in D_x$, then the nonlinear autonomous system is locally observable.



Interval observability matrix

$$Q^I\left(\tilde{x}^I\right) = \left.\frac{\partial q(x)}{\partial x}\right|_{x=\tilde{x}^I}$$

Local condition for observability

$$\text{Rang}\left(Q^I\left(\tilde{x}^I\right)\right) = n$$

Checking the Rank Condition

Theorem of Rohn [3]

$$\lambda_i^I\left(Q^I\right) \subseteq \left[\lambda_i\left(\text{mid}\left(Q^I\right)\right) - \rho\left(\text{rad}\left(Q^I\right)\right), \lambda_i\left(\text{mid}\left(Q^I\right)\right) + \rho\left(\text{rad}\left(Q^I\right)\right)\right]$$

ρ : spectral radius

Matrix $Q^I\left(\tilde{x}^I\right)$ of dimension $(n \times \kappa p)$

$$V^I := \begin{pmatrix} 0 & Q^I\left(\tilde{x}^I\right) \\ Q^I\left(\tilde{x}^I\right)^\mathsf{T} & 0 \end{pmatrix}$$

V^I of dimension $((n + \kappa p) \times (n + \kappa p))$

It follows:

$$\kappa p \geq n$$

Therefore, it is necessary to prove that $2n$ of

$$\lambda_i^I\left(V^I\right) \not\ni 0$$

for $i = 0, \ldots, n + \kappa p$.

Adaptation of the theorem of Rohn

$$\left\{\lambda\left(V^I\right) \middle| \left|\lambda\left(\text{mid}\left(V^I\right)\right)\right| - \rho\left(\text{rad}\left(V^I\right)\right) > 0\right\} = 2n$$

Deciding the Amount of Lie-Derivatives κ

Calculate always one Lie-derivative more and test the global and local condition for

- $\kappa - 1$, if $(\kappa - 1)p \geq n$
- κ
- $\kappa + 1$

Global condition

$$q^I\left(x^I\right) \cap q^I\left(x_i^I\right) = \emptyset$$

Local condition

$$\left\{\lambda\left(V^I\right) \middle| \left|\lambda\left(\text{mid}\left(V^I\right)\right)\right| - \rho\left(\text{rad}\left(V^I\right)\right) > 0\right\} = 2n$$

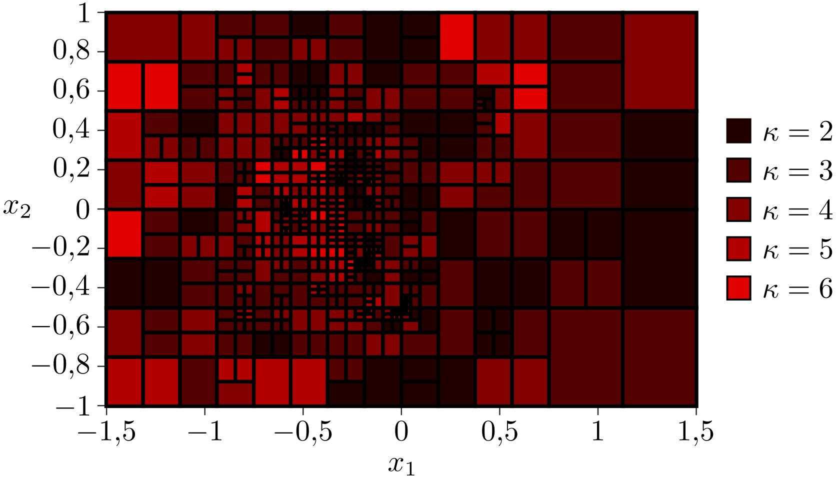
Example for Lie-Derivatives

Nonlinear system

$$f(x) = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$$
$$h(x) = x_1^2 + x_1$$

Analyzed on set

$$D_x = \begin{pmatrix} [-1{,}5 \quad 1{,}5] \\ [-1 \quad 1] \end{pmatrix}$$



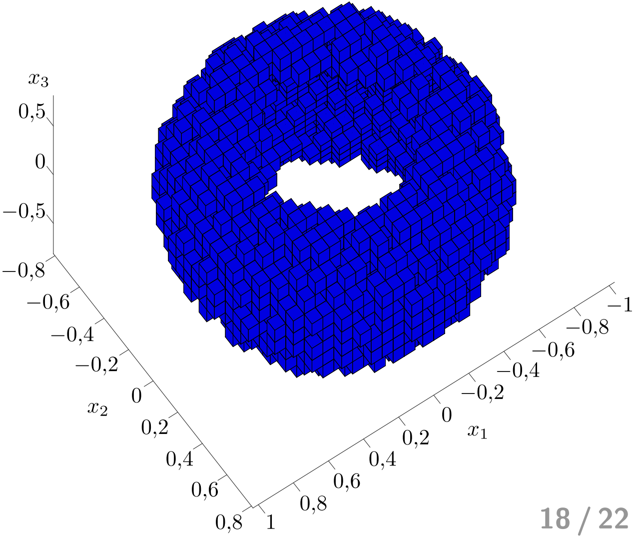
Example for an unobservable system [4] (second example)

Nonlinear system

$$\begin{aligned} f(x) &= \begin{pmatrix} -x_2 + x_1\left(1 - x_1^2 - x_2^2\right) \\ x_1 + x_2\left(1 - x_1^2 - x_2^2\right) \\ -x_3\left(x_1^2 + x_2^2\right) \end{pmatrix} \\ h(x) &= x_1^2 + x_2^2 + x_3^2 \end{aligned}$$

Analyzed on set

$$D_x = \begin{pmatrix} [-1 \quad 1] \\ [-1 \quad 1] \\ [-1 \quad 1] \end{pmatrix}$$



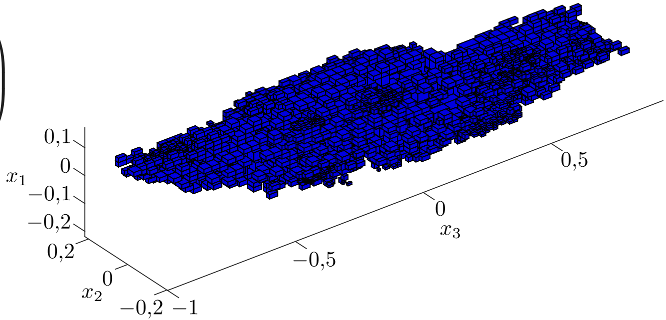
Example for an observable system [4] (first example)

Nonlinear system

$$\begin{aligned} f(x) &= \begin{pmatrix} -x_2 + x_1\left(1 - x_1^2 - x_2^2\right) \\ x_1 + x_2\left(1 - x_1^2 - x_2^2\right) \\ -x_3\left(x_1^2 + x_2^2\right) \end{pmatrix} \\ h(x) &= x_1 + x_2 + x_3 \end{aligned}$$

Analyzed on set

$$D_x = \begin{pmatrix} [-1 \quad 1] \\ [-1 \quad 1] \\ [-1 \quad 1] \end{pmatrix}$$



Considering uncertainties with an example from [5]

nonlinear uncertain systems

$$\dot{x}(t) = f(x(t), \alpha), \quad x(0) = x_0$$

$$y(t) = h(x(t), \beta)$$

Nonlinear system

$$f(x) = \begin{pmatrix} x_2 (x_2 + a) \\ x_2 \end{pmatrix}$$

$$h(x) = x_1$$

Analyzed on set

$$D_x = \begin{pmatrix} [-5 \;\; 5] \\ [-5 \;\; 5] \end{pmatrix}$$

With the uncertainty

$$a^I = [-1 \;\; 1]$$

All intervals for which observability has not yet been demonstrated lie within the interval uncertainty of

$$a^I = [-0{,}0859375 \;\; 0{,}0703125] .$$

Thank you for your attention

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[5] A. Vargas, J. Moreno, and M. Zeitz. Order extension of nonlinear systems for observer design under reduced observability properties. IFAC Proceedings Volumes, 35(1):333–338, 2002.