# Robust output feedback MPC using interval observers

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# Outline

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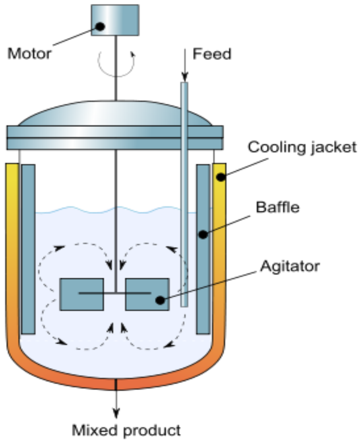
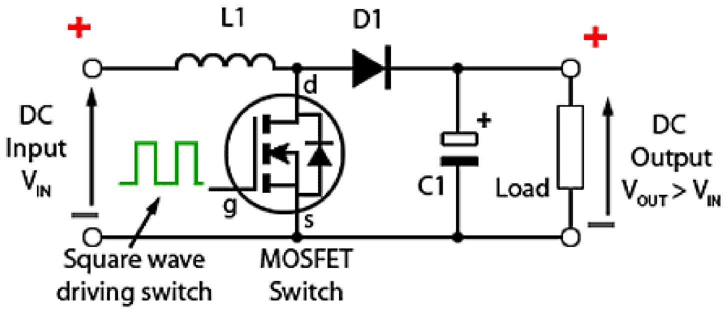
# Motivation

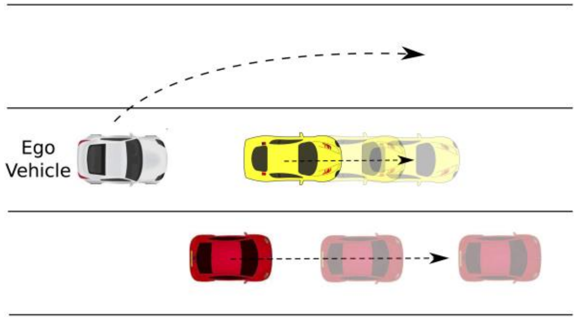
- Constrained systems are recurrent: physical limitations, performance and safety;




Chemical reactors [Wikipedia]




Power electronics [Elprocus]




Vehicle control [MPC and VDL Labs]

- Usual feedback solutions based on Lyapunov methods often fail to ensure constraint satisfaction → Model Predictive Control

# Motivation

- What about *robustness*?
  - Model uncertainties and noises $\rightarrow$ discrepancies between prediction and real system;
  - Unavailable states $\rightarrow$ state estimation;
  - How to ensure constraint satisfaction and feasibility?

Classical solutions: Tubes (rigid, homothetic), error set-membership estimation, moving-horizon estimation (MHE), minmax optimization, multi-stage MPC, ...

# Motivation

- What about *robustness*?

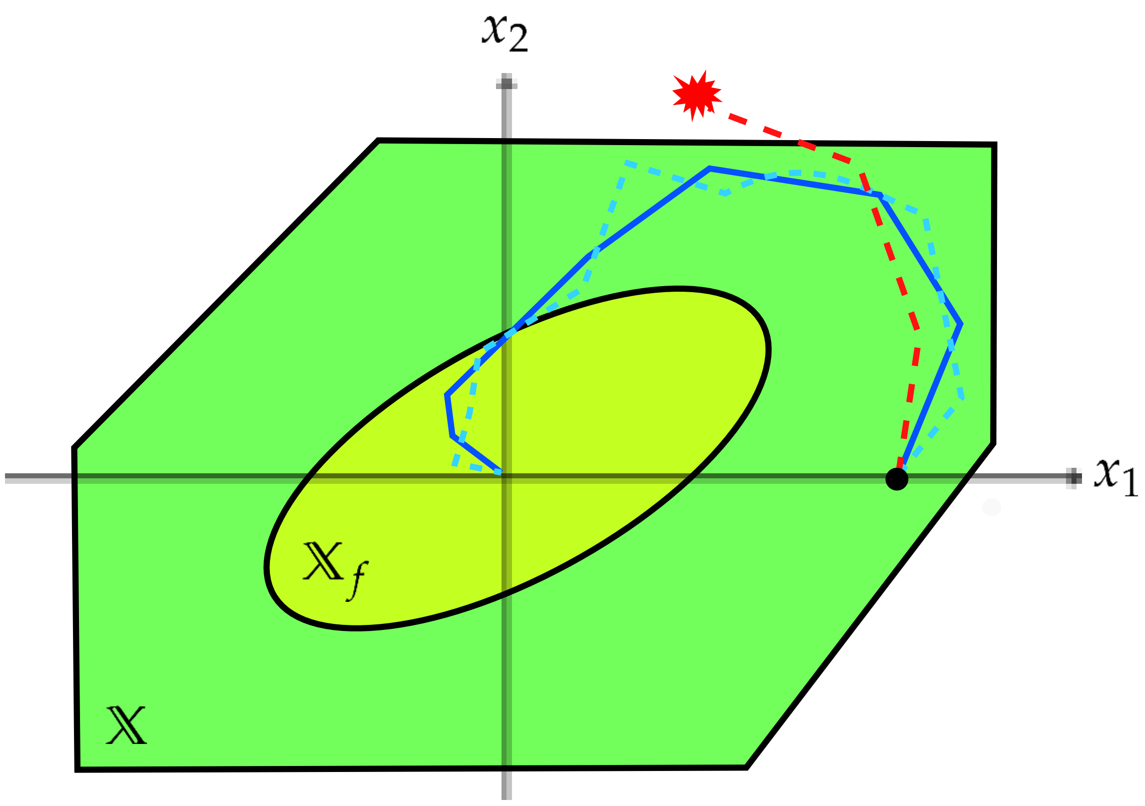



Illustration of loss of feasibility due to uncertainty

# Outline

# Problem statement

Consider the following discrete-time LPV system:

$$\begin{aligned} x_{k+1} &= A(\theta_k)x_k + B(\theta_k)u_k + w_k \\ y_k &= Cx_k + v_k \end{aligned} \tag{1}$$

where $x_k$ is the state vector, $u_k$ is the control input, $y_k$ is the measurement vector, $w_k$ and $v_k$ are process and measurement noises, respectively.

Assumption 1: The additive perturbations $w_k \in [\underline{w}_k, \overline{w}_k]$ and $v_k \in [\underline{v}_k, \overline{v}_k]$ for all $k \in \mathbb{Z}_+$, where $\underline{w}, \overline{w} \in \ell_\infty^n$ and $\underline{v}, \overline{v} \in \ell_\infty^p$ are known signals. The scheduling parameter is unmeasured, but takes values in a known bounded set $\Theta$.

Assumption 2: Initial conditions of (1) are bounded such as $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, for some known $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$.

# Problem statement

Assumption 3: There exist matrices $A_0 \in \mathbb{R}^{n\times n}, B_0 \in \mathbb{R}^{n\times m}$ and $\Delta A_i \in \mathbb{R}^{n\times n}$,$\Delta B_i \in \mathbb{R}^{n\times m}$, $i = 1,\ldots,\nu$ for some $\nu \in \mathbb{Z}_+$, such that the following relations are satisfied for all $\theta \in \Theta$:

$$A(\theta) = A_0 + \sum_{i=1}^{\nu} \lambda_i(\theta)\Delta A_i, \quad B(\theta) = B_0 + \sum_{i=1}^{\nu} \lambda_i(\theta)\Delta B_i,$$

$$\sum_{i=1}^{\nu} \lambda_i(\theta) = 1, \quad \lambda_i(\theta) \in [0,1].$$

Assumption 4: Let $C \geq 0$.

# Problem statement

Problem 1 *(Robust constrained control)* Design an output feedback control that stabilizes (1) while respecting the following constraints

$$x_k \in \mathbb{X}, \quad u_k \in \mathbb{U}, \quad \forall k \in \mathbb{Z}_+$$

having $\mathbb{X}$ and $\mathbb{U}$ as known convex bounded sets, for any possible realization of disturbances $w_k$ and $v_k$, and of the scheduling parameter $\theta_k$.

# Preliminaries

For our developments, we will need the following lemmas:

Lemma 1: [Efimov et al. 2013] Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \le x \le \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$. Then,

(1) if $A \in \mathbb{R}^{m\times n}$ is a constant matrix, then

$$A^+\underline{x} - A^-\overline{x} \le Ax \le A^+\overline{x} - A^-\underline{x} \tag{2}$$

(2) if $A \in \mathbb{R}^{m\times n}$ is a matrix variable and $\underline{A} \le A \le \overline{A}$ for some $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, then

$$\underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- \le Ax \le \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^- \tag{3}$$

# Preliminaries

Lemma 2: [Smith, 1995] For $A \in \mathbb{R}_+^{n \times n}$, the system

$$x_{k+1} = Ax_k + \omega_k,\ \omega : \mathbb{Z}_+ \to \mathbb{R}_+^n,\ \omega \in \mathcal{L}_\infty^n,\ k \in \mathbb{Z}_+$$

has a non-negative solution $x_k \in \mathbb{R}_+^n$ for all $k \in \mathbb{Z}_+$ provided that $x_0 \geq 0$.

Lemma 3: [Farina and Rinaldi, 2000] A matrix $A \in \mathbb{R}_+^{n \times n}$ is Schur stable iff there exists a diagonal matrix $P \in \mathbb{R}^{n \times n}$, $P > 0$, such that $A^\top PA - P \prec 0$.

# Outline

# Interval estimators

An *interval observer* is a two-point set-membership estimator, with stability guarantees. Under *cooperativity conditions*, they produce the following bounds:

$$\underline{x}_k \leq x_k \leq \overline{x}_k$$

**Main idea**: use the relation above to check constraints, since

$$[\underline{x}_k, \overline{x}_k] \subset \mathbb{X} \implies x_k \in \mathbb{X}.$$

**Main features**: low computation complexity and ease of design (LMIs).

# Interval estimators

Using the measurement $y_k$:

$$x_{k+1} = (A_0 - LC)x_k + \sum_{i=1}^{\nu} \lambda_i(\theta)\Delta A_i x_k + Ly_k + (B_0 + \sum_{i=1}^{\nu} \lambda_i(\theta)\Delta B_i)u_k - Lv_k + w_k$$

the following IO can be proposed:

$$\begin{aligned} \overline{x}_{k+1} &= (A_0 - L_o C)\overline{x}_k + \Delta A_+ \overline{x}_k^+ + \Delta A_- \underline{x}_k^- + B_0 u_k + \Delta B u_k^+ + L_o y_k - L_o^+ \underline{v}_k + L_o^- \overline{v}_k + \overline{w}_k \\ \underline{x}_{k+1} &= (A_0 - L_o C)\underline{x}_k - \Delta A_+ \underline{x}_k^- - \Delta A_- \overline{x}_k^+ + B_0 u_k - \Delta B u_k^- + L_o y_k - L_o^+ \overline{v}_k + L_o^- \underline{v}_k + \underline{w}_k \end{aligned} \quad (4)$$

where $L_o$ is the observer gain to be designed. Define the observer estimation errors $e_k = x_k - \underline{x}_k$ and $e_k = \overline{x}_k - x_k$.

Lemma 4: Let assumptions 1–3 be satisfied. Then, provided that $A_0 - L_o C$ is non-negative, the estimation errors are non-negative, *i.e.*, $\underline{e}_k, \overline{e}_k \geq 0$ for all $k > 0$.

# Interval estimators

In order to derive stability conditions for IO (4), let us rewrite it as:

$$\chi_{k+1} = \left(\mathcal{A}_0 - \tilde{L}_o C_1\right)\chi_k + \mathcal{A}_+ \chi_k^+ + \mathcal{A}_- \chi_k^- + \delta_k$$

where $\mathcal{A}_0 = \operatorname{diag}(A_0, A_0) \in \mathbb{R}^{2n \times 2n}$, $\tilde{L}_o = \operatorname{diag}(L_o, L_o) \in \mathbb{R}^{2n \times 2p}$, $C_1 = \operatorname{diag}(C, C) \in \mathbb{R}^{2p \times 2n}$, $\delta_k = \operatorname{vec}(\overline{\delta}_k, \underline{\delta}_k)$, and

$$\mathcal{A}_+ = \begin{bmatrix} \Delta A_+ & 0 \\ -\Delta A_- & 0 \end{bmatrix}, \quad \mathcal{A}_- = \begin{bmatrix} 0 & \Delta A_- \\ 0 & -\Delta A_+ \end{bmatrix},$$

$$\overline{\delta}_k = B_0 u_k + \Delta B u_k^+ + L_o y_k - L_o^+ \underline{v}_k + L_o^- \overline{v}_k + \overline{w}_k,$$

$$\underline{\delta}_k = B_0 u_k - \Delta B u_k^- + L_o y_k - L_o^+ \overline{v}_k + L_o^- \underline{v}_k + \underline{w}_k.$$

# Interval estimators

The next result verifies stability:

Theorem 1: Let assumptions 1–3 be satisfied. If there exist diagonal matrices $\tilde{P}$,$Q_1$,$Q_2$,$Q_3$, $\Omega_+$,$\Omega_-$, $\Psi \in \mathbb{R}^{2n\times 2n}$, matrices $\Gamma \in \mathbb{R}^{2n\times 2n}$ and $\tilde{U} \in \mathbb{R}^{2n\times p}$, such that the following LMIs are verified:

$$\tilde{P}\mathcal{A}_0 - \tilde{U}C_1 \geq 0$$

$$\begin{bmatrix} \tilde{P} - Q_1 & -\Omega_+ & -\Omega_- & 0 & \mathcal{A}_0^\top \tilde{P} - C_1^\top \tilde{U}^\top \\ \star & -Q_2 & -\Psi & 0 & \mathcal{A}_+^\top \tilde{P} \\ \star & \star & -Q_3 & 0 & \mathcal{A}_-^\top \tilde{P} \\ \star & \star & \star & \Gamma & \tilde{P} \\ \star & \star & \star & \star & \tilde{P} \end{bmatrix} \succeq 0$$

$$\tilde{P} > 0, \quad \Gamma \succ 0, \quad Q_1, Q_2, Q_3, \Omega_+, \Omega_- \geq 0,$$

$$Q_1 + \min\{Q_2, Q_3\} + 2\min\{\Omega_+, \Omega_-\} > 0$$

then system (4) with a gain $L_o = P^{-1}U$ is an IO for system (1), *i.e.*, relation $\underline{x}_k \leq x_k \leq \overline{x}_k$ is satisfied for all $k \in \mathbb{Z}_+$ and, in addition, $\chi \in \ell_\infty^{2n}$ provided that $\delta \in \ell_\infty^{2n}$.

# Interval estimators

To better illustrate the developments of this section, consider the following prototype model:

$$x_{k+1} = \begin{bmatrix} 0.5 & 0.6 + \theta_k \\ \theta_k & 0.3 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + w_k$$

$$y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k + v_k$$

$$\mathbb{W} = [-0.1, 0.1] \times [-0.1, 0.1], \quad \mathbb{V} = [-0.1, 0.1], \quad \text{and } \Theta = [0, -0.3].$$

Interpolating functions $\lambda_1 = \frac{\theta_k - \underline{\theta}_k}{\overline{\theta}_k - \underline{\theta}_k}$ and $\lambda_2 = \frac{\overline{\theta}_k - \theta_k}{\overline{\theta}_k - \underline{\theta}_k}$.

# Interval estimators

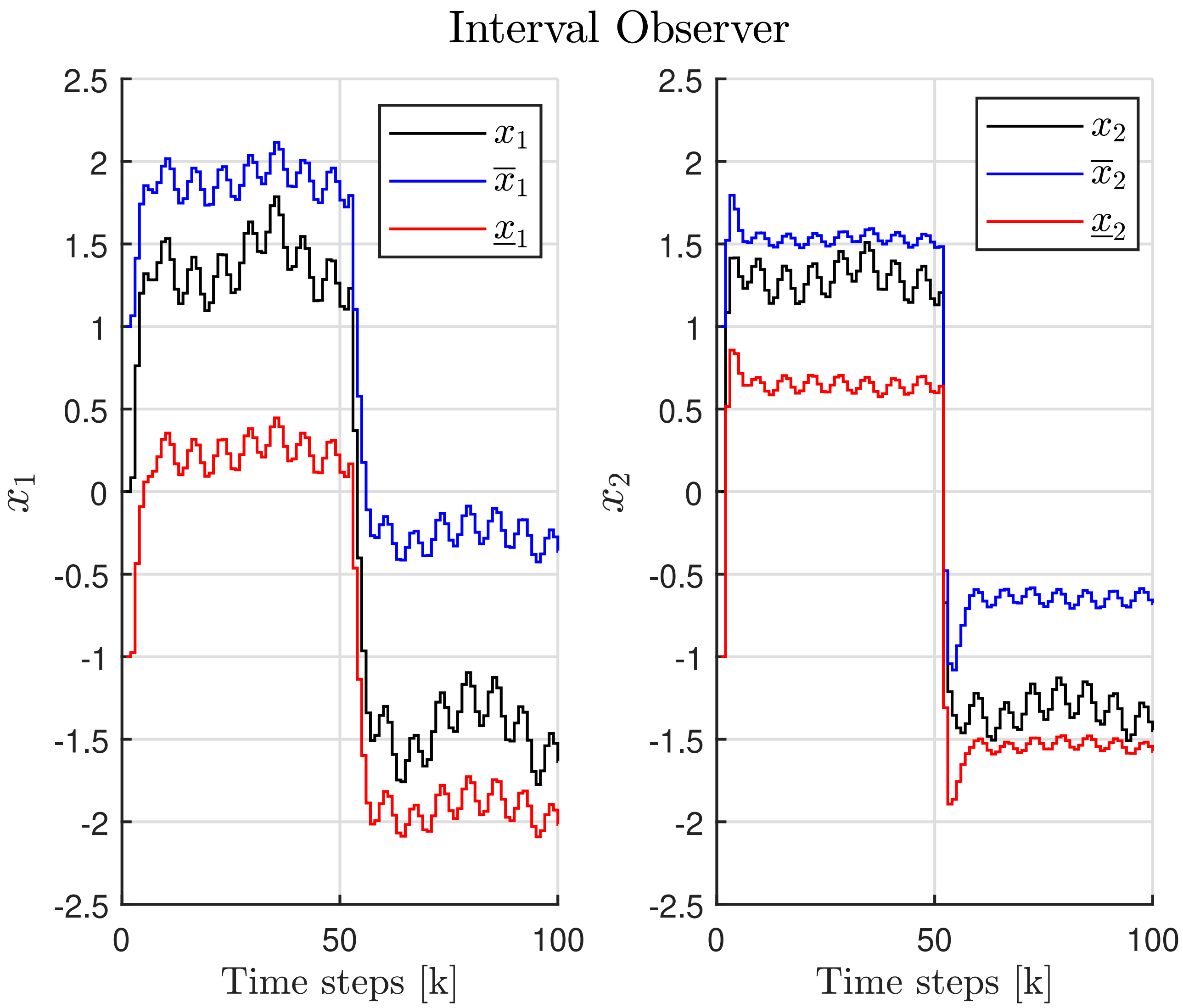
### Simulate the IO

$u_k = 1$, for $k = [0, \ldots 49]$

$u_k = -1$, for $k = [50, \ldots 100]$

$\theta_k = -|0.3\sin(0.1k)|$

$w_k = 0.1\sin(k), \;\; v_k = 0.1\sin(k)$



# Interval estimators

As seen in (4), the IO requires the measurement $y_k \rightarrow$ unsuitable for prediction.

**Solution**: propose an *interval predictor* $\rightarrow$ an open-loop *framer*, *i.e.*, independent of $y_k$.

Recalling that $y_k = Cx_k + v_k$, we can write the following relation under Lemma 1 and Assumption 4:

$$L_p^+ C\underline{z}_k - L_p^- C\overline{z}_k \leq L_p C z_k \leq L_p^+ C\overline{z}_k - L_p^- C\underline{z}_k. \quad (5)$$

then the terms $L_p y_k - L_p v_k = L_p C x_k$ can be replaced by the bounding relations above.

# Interval estimators

The proposed IP:

$$\begin{aligned}
\overline{z}_{k+1} &= (A_0 - L_p C)\overline{z}_k + \Delta A_+ \overline{z}_k^+ + \Delta A_- \underline{z}_k^- + L_p^+ C \overline{z}_k - L_p^- C \underline{z}_k + B_0 u_k + \Delta B u_k^+ + \overline{w}_k \\
\underline{z}_{k+1} &= (A_0 - L_p C)\underline{z}_k - \Delta A_+ \underline{z}_k^+ - \Delta A_- \overline{z}_k^- + L_p^+ C \underline{z}_k - L_p^- C \overline{z}_k + B_0 u_k - \Delta B u_k^- + \underline{w}_k
\end{aligned} \tag{6}$$

Define the prediction estimation errors $\underline{\epsilon}_k = x_k - \underline{z}_k$ and $\overline{\epsilon}_k = \overline{z}_k - x_k$.

Lemma 5: Let assumptions 1–4 be satisfied. Then, provided that $A_0 - L_p C$ is non-negative, the prediction errors are non-negative, *i.e.*, $\underline{\epsilon}_k, \overline{\epsilon}_k \geq 0$ for all $k \in \mathbb{Z}_+$.

# Interval estimators

In order to derive stability conditions for IP (6), let us rewrite it as:

$$\mathcal{Z}_{k+1} = \left(\mathcal{A}_0 + \tilde{L}_p C_2\right) \mathcal{Z}_k + \mathcal{A}_+ \mathcal{Z}_k^+ + \mathcal{A}_- \mathcal{Z}_k^- + \varrho_k,$$

where $\mathcal{A}_0$, $\mathcal{A}_+$ and $\mathcal{A}_-$ are the same as for IO (4), $\tilde{L}_p = \mathrm{diag}(L_p^-, L_p^-) \in \mathbb{R}^{2n \times 2p}$, $\varrho_k = \mathrm{vec}(\overline{\rho}_k, \underline{\rho}_k)$ and

$$C_2 = \begin{bmatrix} C & -C \\ -C & C \end{bmatrix},$$

$$\overline{\rho}_k = B_0 u_k + \Delta B u_k^+ + \overline{w}_k, \quad \underline{\rho}_k = B_0 u_k - \Delta B u_k^- + \underline{w}_k.$$

# Interval estimators

Theorem 2: Let assumptions 1–4 be satisfied. If there exist diagonal matrices $\tilde{P}_2$, $Q_1, Q_2, Q_3$, $\Omega_+, \Omega_-, \Psi$, $\Gamma \in \mathbb{R}^{2n\times 2n}$ and $U^+, U^- \in \mathbb{R}^{n\times p}$, such that

$$\tilde{P}_2\mathcal{A}_0 - \tilde{U}^+ C_1 + \tilde{U}^- C_1 \geq 0$$

$$\begin{bmatrix} \tilde{P}_2 - Q_1 & -\Omega_+ & -\Omega_- & 0 & (\tilde{P}_2\mathcal{A}_0 + \tilde{U}^- C_2)^\top \\ \star & -Q_2 & -\Psi & 0 & (\tilde{P}_2\mathcal{A}_+)^\top \\ \star & \star & -Q_3 & 0 & (\tilde{P}_2\mathcal{A}_-)^\top \\ \star & \star & \star & \Gamma & \tilde{P}_2 \\ \star & \star & \star & \star & \tilde{P}_2 \end{bmatrix} \succeq 0$$

$$Q_1, Q_2, Q_3, \Omega_+, \Omega_-, U^+, U^- \geq 0, \quad \Gamma > 0, \quad P_2 > 0$$

$$\tilde{P}_2 = \mathrm{diag}(P_2, P_2), \quad \tilde{U}^+ = \mathrm{diag}(U^+, U^+), \quad \tilde{U}^- = \mathrm{diag}(U^-, U^-),$$

$$Q = Q_1 + \min\{Q_2, Q_3\} + 2\min\{\Omega_+, \Omega_-\} > 0$$

then (6) with gains $L_p^- = P_2^{-1}U^-$ and $L_p^+ = P_2^{-1}U^+$ is an IP for system (1), *i.e.*, $\underline{z}_k \leq x_k \leq \overline{z}_k$ holds for all $k \in \mathbb{Z}_+$, and (6) is ISS with respect to the input $\varrho \in \ell_\infty^{2n}$.

# Interval estimators

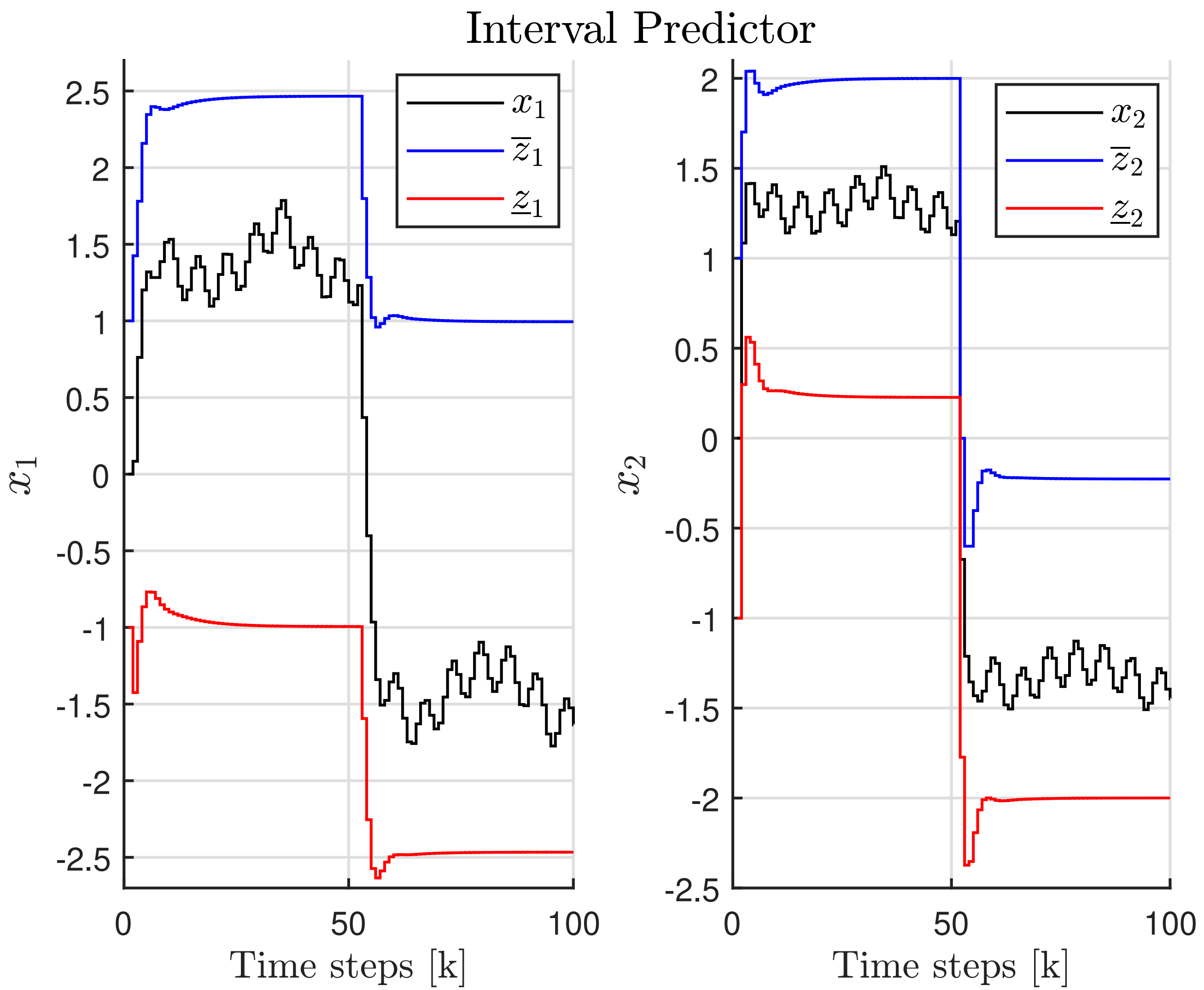
Simulate the IP

$u_k = 1, \text{ for } k = [0, \ldots 49]$

$u_k = -1, \text{ for } k = [50, \ldots 100]$

$\theta_k = -|0.3 \sin(0.1k)|$

$w_k = 0.1 \sin(k), \;\; v_k = 0.1 \sin(k)$



# Outline

# Recall on MPC

How to prove stability $\rightarrow$ stabilizing ingredients:

- Terminal set $\mathbb{X}_f$: the set that the endpoint of the prediction must reach;
- Terminal gain $\kappa_f$: there exists a stabilizing controller;
- Terminal cost $V_f$.

# Recall on MPC

How to prove stability → stabilizing ingredients. Recall the classic axioms of Mayne et al.:

**Definition 1** *The stabilizing ingredients are such that the following axioms are verified:*

1. $\mathbb{X}_f \subset \mathbb{X}$, *closed and* $0 \in \mathbb{X}_f$*: the state constraint is satisfied in* $\mathbb{X}_f$*;*
2. $\kappa_f(x) \in \mathbb{U}$, $\forall x \in \mathbb{X}_f$*: the control constraint is satisfied in* $\mathbb{X}_f$*;*
3. $f(x, \kappa_f(x)) \in \mathbb{X}_f$, $\forall x \in \mathbb{X}_f$*:* $\mathbb{X}_f$ *is positively invariant under* $\kappa_f(x)$*;*
4. $[V_f + \ell](x, \kappa_f(x)) \leq 0$, $\forall x \in \mathbb{X}_f$*:* $V_f$ *is a local Lyapunov function.*

# IP: Control design

How to design a feedback controller for the IP? Let us consider:

$$u_k = K\mathcal{Z}_k + K_+\mathcal{Z}_k^+ + K_-\mathcal{Z}_k^- + R\mathcal{W} \quad (7)$$

where $\mathcal{W}_k = \text{vec}(\underline{w}_k, \overline{w}_k)$. This control leads to the following closed-loop:

$$\mathcal{Z}_{k+1} = \mathcal{K}\mathcal{Z}_k + \mathcal{K}_+\mathcal{Z}_k^+ + \mathcal{K}_-\mathcal{Z}_k^- + \tilde{D}\mathcal{W} \quad (8)$$

where $\mathcal{K} = \mathcal{A}_0 + \tilde{L}_p C_2 + \mathcal{B}_0 K$, $\mathcal{K}_* = \mathcal{A}_* + \mathcal{B}_0 K_*$ $\tilde{D} = \mathbb{I}_{2n} + \mathcal{B}_0 R$ and $\mathcal{B}_0 = [B_0^\top, B_0^\top]$.

# IP: Control design

This brings us to the following result:

Theorem 3: Let assumptions 1–4 be satisfied. If there exist matrices $P$, $Q_1$, $Q_2$, $Q_3$, $\Gamma$, $\Omega_+$, $\Omega_-$, $\Psi \in \mathbb{R}^{2n\times 2n}$ and $W_1$, $W_2$, $W_3$, $W_4 \in \mathbb{R}^{m\times 2n}$ such that

$$\begin{bmatrix} P-Q_1 & -\Omega_+ & -\Omega_- & 0 & W_1^\top \mathcal{B}_0^\top + PD_z^\top \\ \star & -Q_2 & -\Psi & 0 & W_2^\top \mathcal{B}_0^\top + P\mathcal{A}_+^\top \\ \star & \star & -Q_3 & 0 & W_3^\top \mathcal{B}_0^\top + P\mathcal{A}_-^\top \\ \star & \star & \star & \Gamma & W_4^\top \mathcal{B}_0^\top + P \\ \star & \star & \star & \star & P \end{bmatrix} \succ 0$$

$$P > 0, \quad \Gamma > 0, \quad Q_1, Q_2, Q_3, \Omega_+, \Omega_- \geq 0,$$

$$Q = Q_1 + \min\{Q_2, Q_3\} + 2\min\{\Omega_+, \Omega_-\} > 0,$$

then IP (6) under control (7) with gains $K = W_1P^{-1}, K_+ = W_2P^{-1}, K_- = W_3P^{-1}, R = W_4P^{-1}$ is ISS with respect to the inputs $\mathcal{W} \in \ell_\infty^{2n}$.

# IP: Control design

How to ensure that $u_k \in \mathbb{U}$?

Corollary 1: Let there exist symmetric and positive definite matrices $S \in \mathbb{R}^{m\times m}$ and $Z \in \mathbb{R}^{2n\times 2n}$ such that $\mathbb{U} = \{u \in \mathbb{R}^m : u^\top S u \leq 1\}$ and $\mathcal{W}_k \in \{\mathcal{W} \in \mathbb{R}^{2n} : \mathcal{W}^\top Z \mathcal{W} \leq 1\}$, and the conditions of Theorem 4 be satisfied with additional inequalities:

$$\frac{\eta}{\alpha\kappa}\Gamma \leq \min\{\kappa^{-1}Z, P\},\; P \geq \kappa Z^{-1},$$

$$\begin{bmatrix} \frac{\eta}{3}P & 0 & 0 & W_1^\top + W_2^\top \\ 0 & \frac{\eta}{3}P & 0 & W_3^\top - W_1^\top \\ 0 & 0 & \frac{\kappa}{3}P & W_4^\top \\ W_1 + W_2 & W_3 - W_1 & W_4 & S^{-1} \end{bmatrix} \geq 0$$

for some constants $\eta > 0$ and $\kappa > 0$, then control (7) satisfies the constraint $u_k \in \mathbb{U}$ for all $\mathcal{Z}_k \in \mathbb{X}_f \times \mathbb{X}_f$.

# The predictive controller

Determine $\mathcal{S}_n = \{s_0, \ldots, s_{N-1}\}$ solving the OCP

$$\mathcal{S}_N^k := \arg\min_{\mathcal{S}_N} V_N(\mathcal{Z}_{k,0}, \ldots, \mathcal{Z}_{k,N}, \mathcal{S}_N)$$

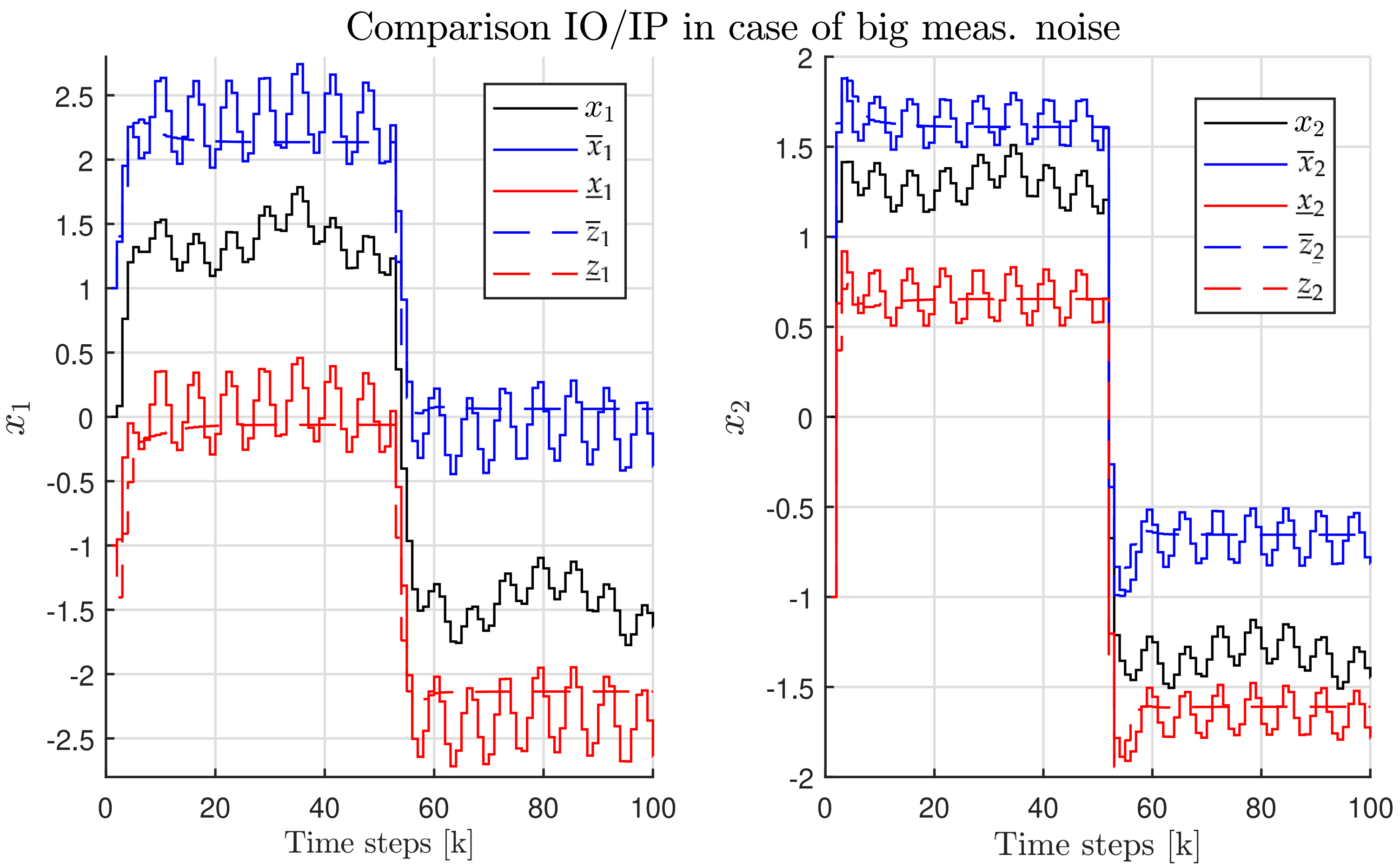
with a cost function $V_N(\mathcal{Z}_{k,0}, \ldots, \mathcal{Z}_{k,N}, \mathcal{S}_N) = V_f(\mathcal{Z}_{k,N}) + \sum_{i=0}^{N-1} \ell(\mathcal{Z}_{k,i}, s_i)$.

under the following constraints:

| | | |
|---|---|---|
| $\underline{z}_{k,0} = \min\{\overline{x}_k, \overline{z}_{k-1,1}\}, \quad \overline{z}_{k,0} = \max\{\underline{x}_k, \underline{z}_{k-1,1}\}$ | (9a) | $\rightarrow$ intialization |
| $\mathcal{Z}_{k,i+1}$ computed by X | (9b) | $\rightarrow$ prediction using the IP |
| $\mathcal{Z}_{k,i+1} \subset \mathbb{X} \times \mathbb{X}, \quad s_i \subset \mathbb{U},$ | (9c) | $\rightarrow$ state and input constraint |
| $\mathcal{Z}_{k,N} \in \mathbb{X}_f \times \mathbb{X}_f$ | (9d) | $\rightarrow$ terminal constraint |

# The predictive controller

Why initialize using information from both IO and IP? Let $\mathbb{V} = [-0.5, 0.5]$.



# The predictive controller

**Algorithm 1:** IO-MPC

**Offline:** Solve LMIs, estimate $\mathbb{X}_f$ and select $\Psi_1 = P^{-1}$, $\Psi_2 \leq \frac{\alpha}{2} P^{-1}$ and $\Psi_3 \leq \frac{\alpha}{8} P^{-1}$.

**Input:** Initial conditions $\underline{x}_0, \overline{x}_0$ and prediction horizon $N$.

**Online:**

1. **for** each decision instant $k \in \mathbb{Z}_+$ **do**
2. Measure $y_k$ and update IO (4).
3. Initialize IP (6).
4. Solve OCP (17) under constraints (9a)-(9d).
5. Assign $u_k = s_0^k$ and apply to the system.
6. **end for**

# The predictive controller

Theorem 4: Let $[\underline{x}_0, \overline{x}_0] \subset \mathbb{X}$ and assumptions 1–4 be satisfied with $[\underline{w}_{k+1}, \overline{w}_{k+1}] \subseteq [\underline{w}_k, \overline{w}_k]$ for all $k \in \mathbb{Z}_+$. Then, following Algorithm 1, the closed-loop system composed by (1), (4) and (6) has the following features:

1. Recursive feasibility of reaching the terminal set in $N$ steps;
2. ISS of dynamics (8) in $\mathbb{X}_f$ and practical ISS for (1);
3. Constraint satisfaction.

# The LTI and the TD case

The same ideas were applied to linear time-invariant (LTI) and time-delayed systems (TD):

$$
\begin{aligned}
x_{k+1} &= A_0 x_k + A_1 x_{k-h} + B u_k + w_k, \quad k \in \mathbb{Z}_+ \\
x_k &= \phi_k, \quad k \in [-h, \dots, 0] \\
y_k &= C x_k + v_k
\end{aligned}
$$

Main differences:

- Optimization of gains made through the interval width $\delta x_k = \overline{x}_k - \underline{x}_k$.
- Control design made regarding the interval center $x_k^\star = \frac{\overline{x}_k + \underline{x}_k}{2}$.
- For the TD case, the Lyapunov-Krasovskii framework is required;

# Complexity

One of the main advantages of using IO/IP is their fixed complexity.

Assume that the number of hyperplanes needed to define $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{X}_f$ depends linearly on $n$, and that $m = n$. Therefore, the worst-case number of variables for solving the constrained OCP is $10Nn$ ($8Nn$ for the linear cases).

# Outline

# Numerical example (LPV)

Recall the LPV prototype example:

$$x_{k+1} = \begin{bmatrix} 0.5 & 0.6+\theta_k \\ \theta_k & 0.3 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + w_k$$

$$y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k + v_k$$

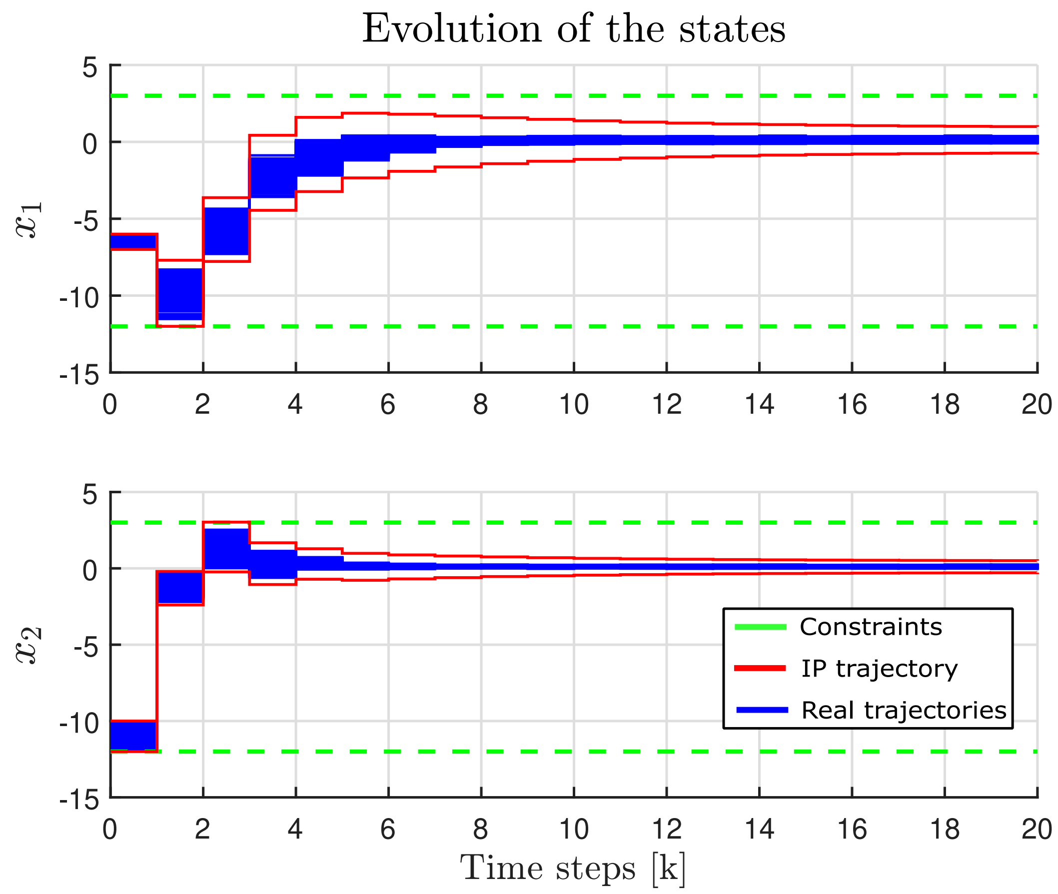
Constraints: $\mathbb{X} = [-12, 3] \times [-12, 3]$, $\mathbb{U} = [-2, 2]$

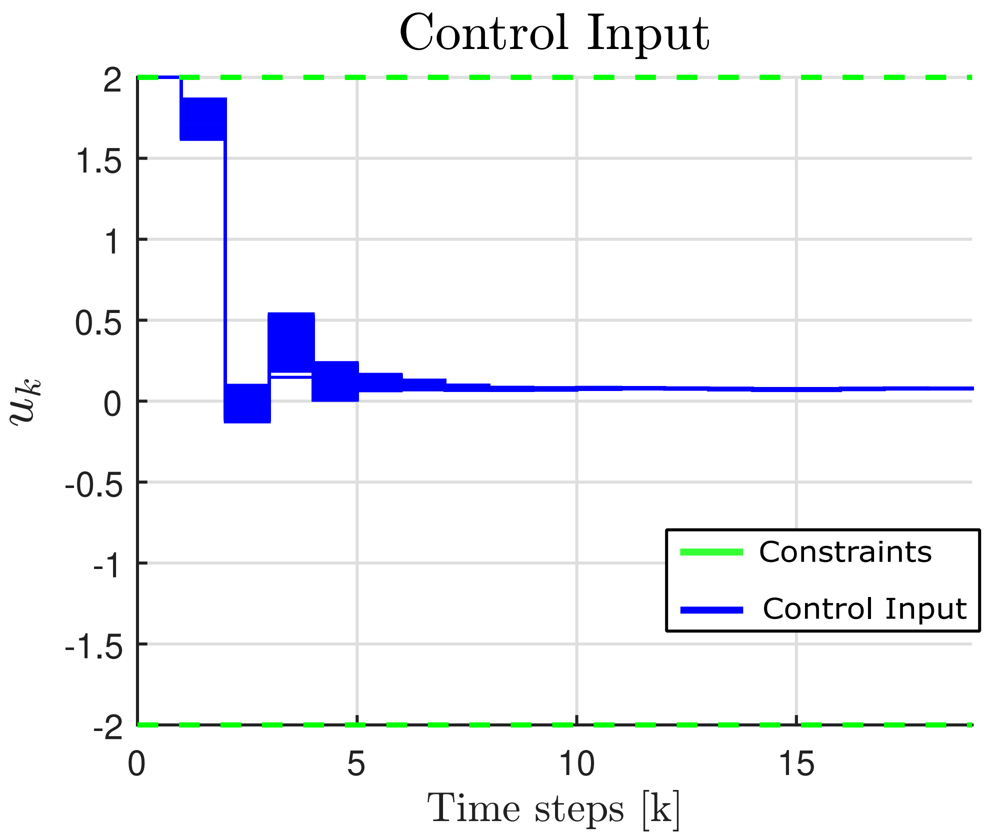
Disturbances: $\mathbb{W} = [-0.1, 0.1]^2$, $\mathbb{V} = [-0.1, 0.1]$

Interpolating functions $\lambda_1 = \frac{\theta_k - \underline{\theta}_k}{\overline{\theta}_k - \underline{\theta}_k}$ and $\lambda_1 = \frac{\overline{\theta}_k - \theta_k}{\overline{\theta}_k - \underline{\theta}_k}$, $\quad \Theta = [-0.1, 0.1]$

Select $\underline{x}_0 = \text{vec}(-7, -12)$ and $\overline{x}_0 = \text{vec}(-6, -10)$. Prediction horizon $N = 20$, simulation time span $T = 20$ steps $\times$ 100 runs.

# Numerical example (LPV)

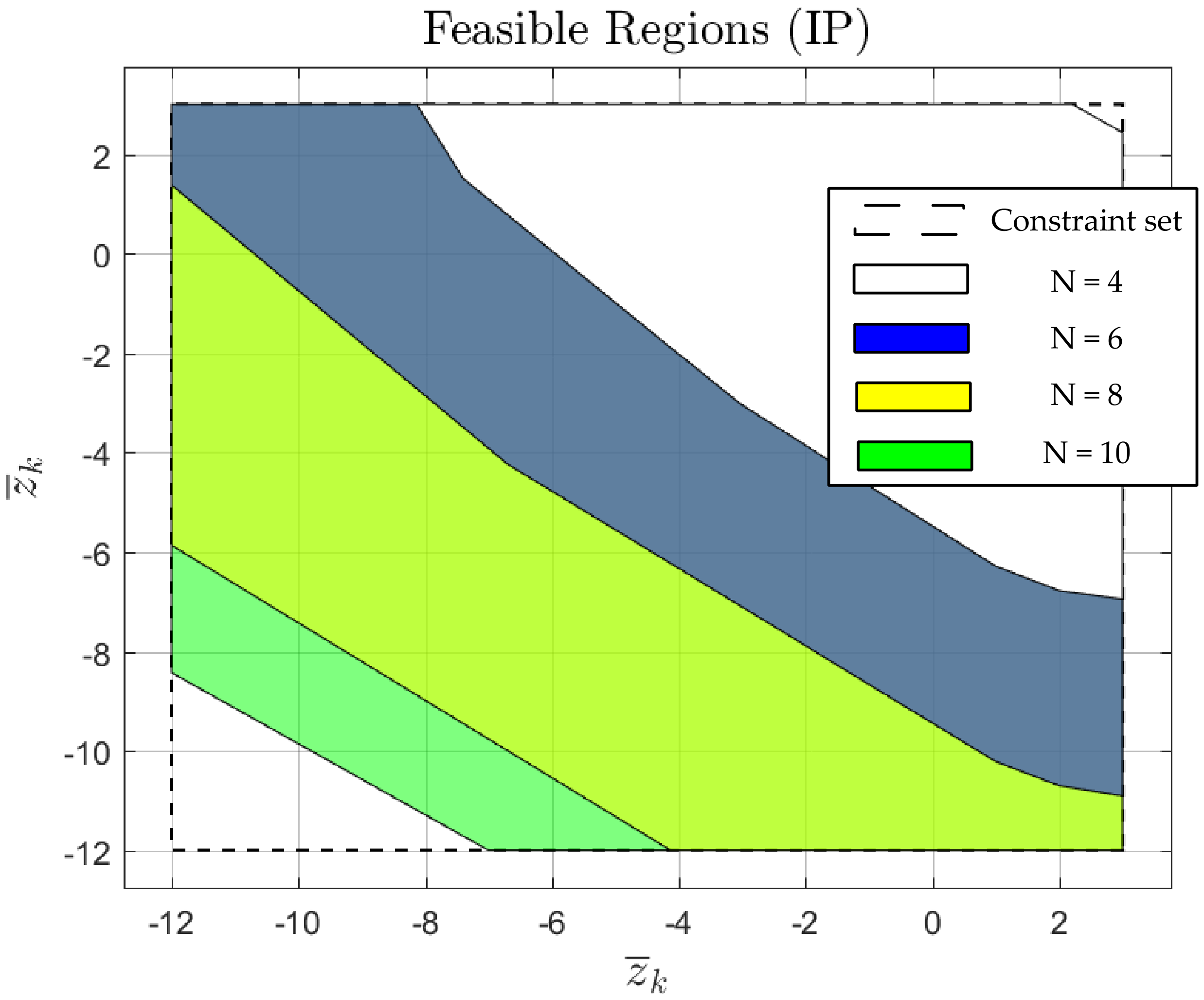


# Numerical example (LPV)

Solver: *fmincon* (active set method)

For $N = 10$, computation time $0.22 \pm 0.0313$ second/step with a maximum of $0.7725$ second.



# Numerical example (LTI)

Consider the (linearized) CSTR model, given by the following matrices:

$$A = \begin{bmatrix} 0.745 & -0.002 \\ 5.610 & 0.780 \end{bmatrix}, \; B = \begin{bmatrix} 5.6 \times 10^{-6} \\ 0.464 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

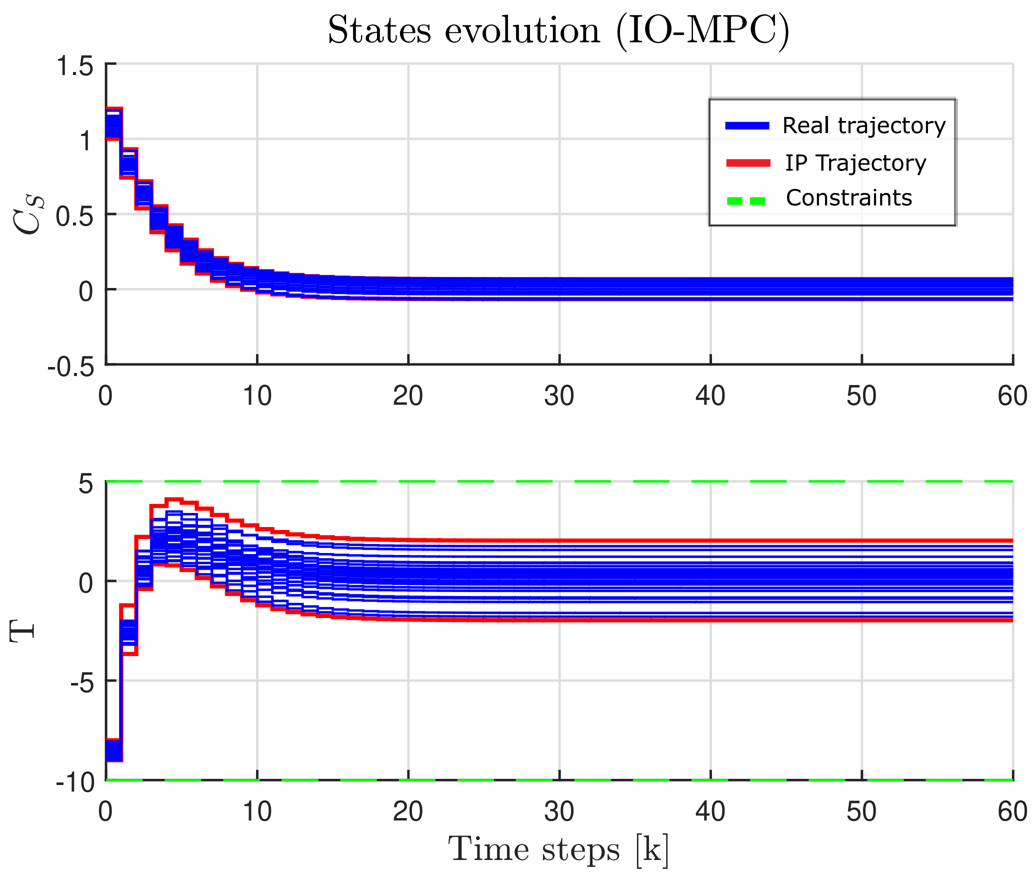
Constraints: $\mathbb{X} = [-2, 2] \times [-10, 5]$ and $\mathbb{U} = [-4.5, 4.5]$

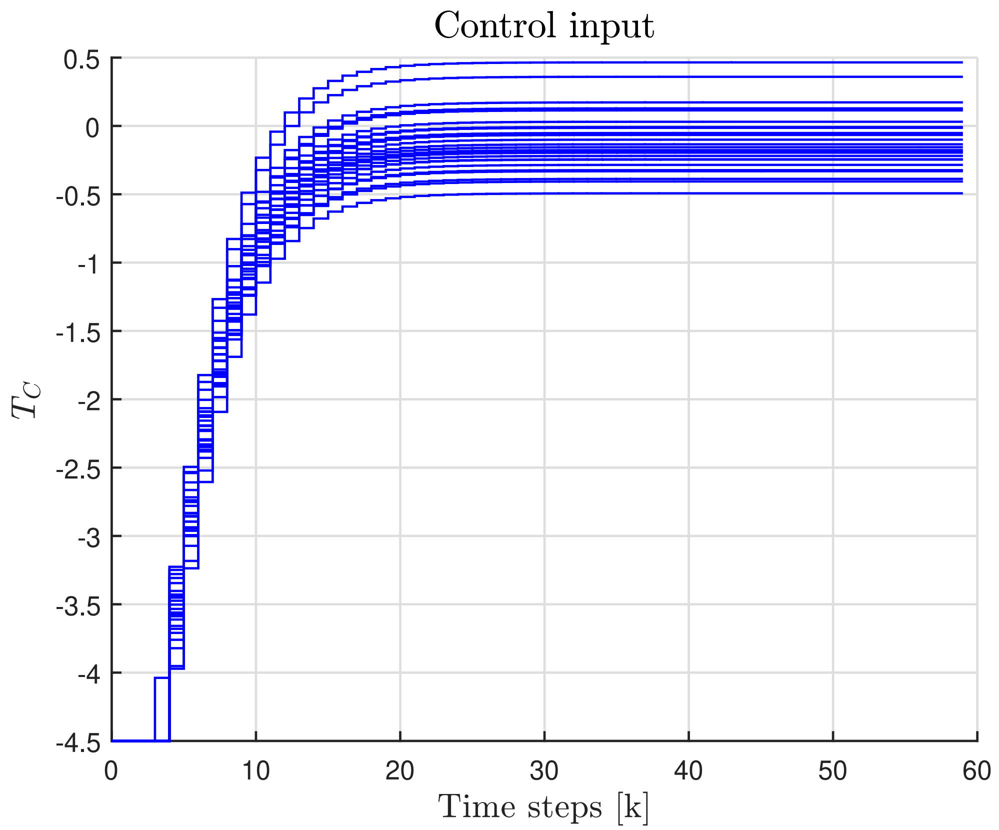
Disturbances: $\mathbb{W} = [-0.02, 0.02] \times [-0.2, 0.2]$ and $\mathbb{V} = [-0.3, 0.3]$

For a later comparison, the Tube-MPC from [Mayne et al, 2009] will be implemented, taking an LQR controller for its design with matrices $Q_{LQ} = 0.1\mathbb{I}_2$ and $R_{LQ} = 0.1$.

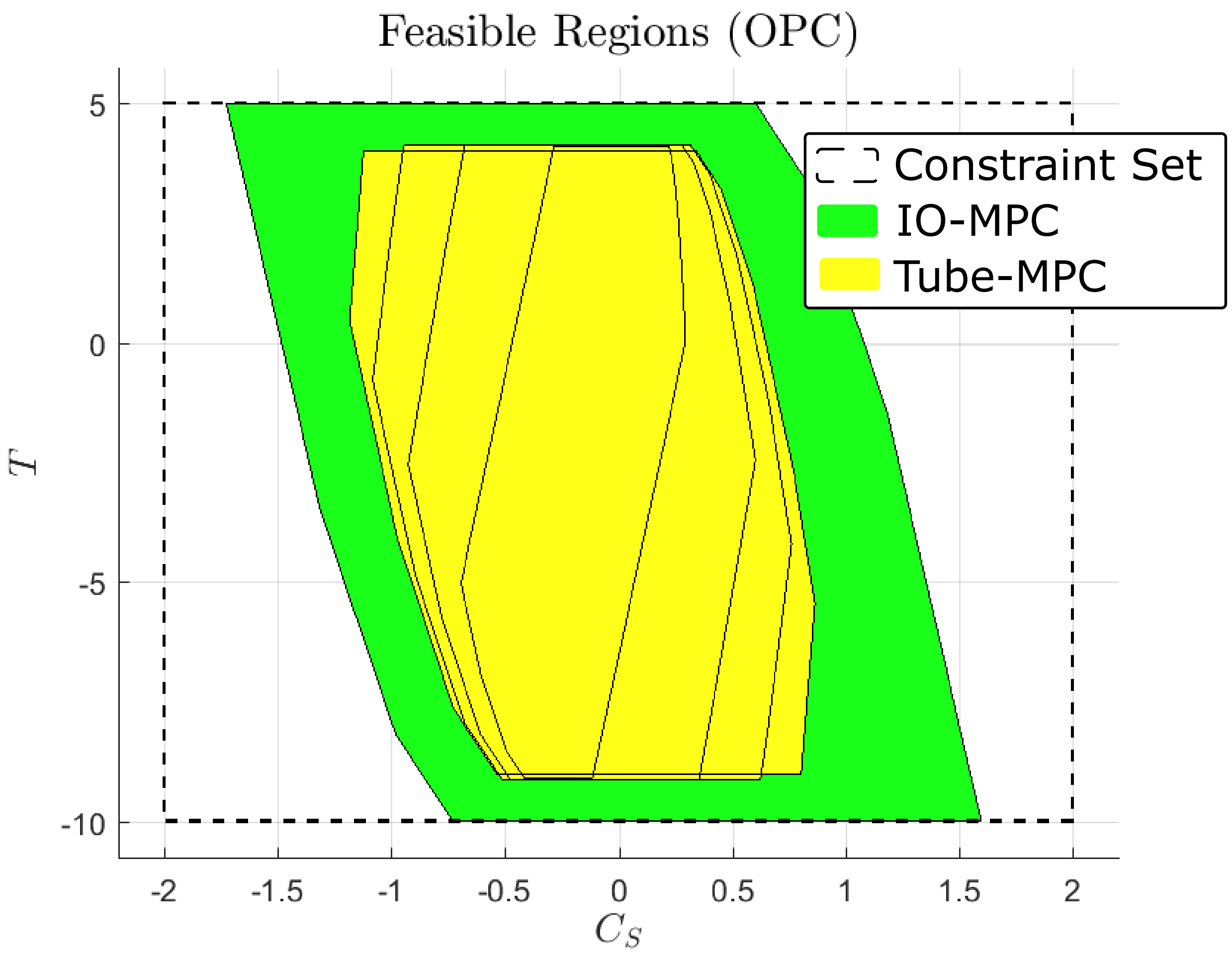
Solver: *quadprog*, computation time: $0.0032 \pm 0.0021$ second/step, maximum of 0.1358.

# Numerical example (LTI)



# Numerical example (LTI)



# Numerical example (TD)

Consider the following TD system:

$$x_{k+1} = \begin{bmatrix} 0.5 & -0.1 \\ 0.5 & 0.2 \end{bmatrix} x_k + \begin{bmatrix} 0.1 & -0.3 \\ 0 & -0.1 \end{bmatrix} x_{k-h} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + w_k$$

$$y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k + v_k$$

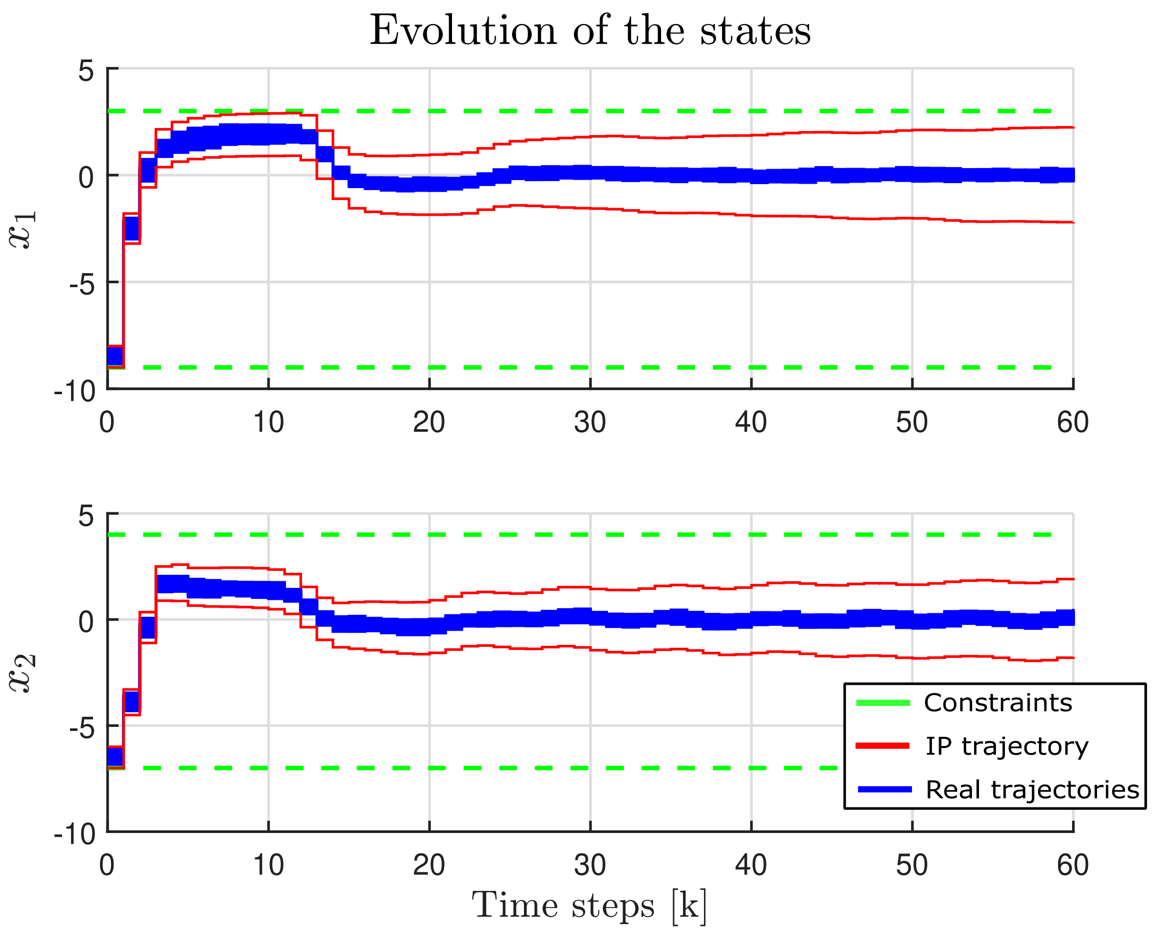
Constraints: $\mathbb{X} = [-9, 3] \times [-7, 4]$ and $\mathbb{U} = [-1, 1]$

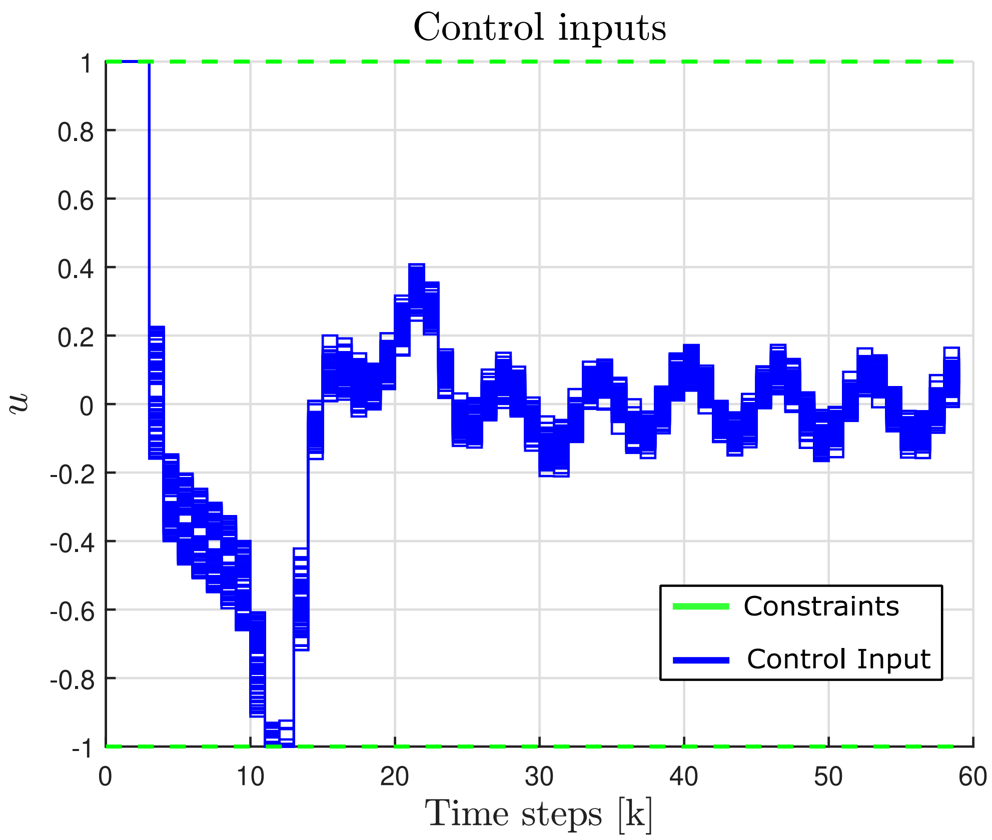
Disturbances: $\mathbb{W} = [-0.2, 0.2]^2$ and $\mathbb{V} = [-0.5, 0.5]$

and a known time-delay $h = 10$.

Solver: *quadprog*, computation time: $0.0032 \pm 0.0021$ second/step, maximum of $0.1358$.

# Numerical example (TD)



# Conclusions & perspectives

Conclusions:

- Developed new interval estimators for LTI, LPV and TD systems, as well as their respective state feedback controllers;
- Proposed new robust output feedback MPC algorithms;
- Illustrated the methodologies with numerical experiments;
- Advantages: low fixed complexity, ease of design, low conservativeness.

Perspectives:

- Enhance the interval estimators and the proposed MPC algorithms aiming to reduce conservativeness;
- Test their efficiency in practical scenarios.

# Thank you for your attention ☺

Feel free to ask questions or contact me by e-mail: alex.dos-reis-de-souza@inria.fr