Interval Observer Design for Uncertain Discrete-Time Linear Switched Systems with Unknown Inputs

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Outline

Introduction

A switched systems represent a class of hybrid systems:

- a finite number of continuous subsystems (modes).
- a logical rule operates switching between subsystems.

Physical systems with switching features can be regarded as switched systems:

- power and electronics systems
- automated highway systems
- flight control systems
- network control systems,...

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- The problem of state vector estimation is very challenging and can be encountered in many applications.
- Several cases are encountred:
  - Models without uncertainties.
  - Models with uncertain parameters.
  - Uncertain parameters and unknown inputs
- Possible Solutions
  - Adaptive approaches
  - Robust approaches
  - Set-membership estimation / Interval observers.

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Given a system described by:

$$\begin{cases} \dot{x}(t) = f(x,u) \\ y(t) = h(x) \end{cases} \quad (1)$$

Definition

The dynamical system

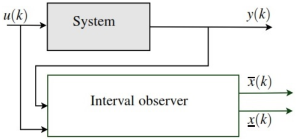
$$\begin{cases} \dot{z}(t) = \alpha(z,y,u) \\ \left[\underline{x}^T,\ \overline{x}^T\right]^T \end{cases} \quad (2)$$

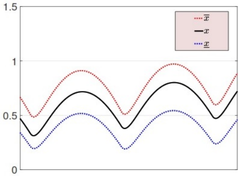
is an interval observer for (2) *if:*

$$\underline{x}(0) \le x(0) \le \overline{x}(0) \Rightarrow -\infty < \underline{x}(t) \le x(t) \le \overline{x}(t) < +\infty;\ \forall t \ge 0. \quad (3)$$

An interval observer:

- compute the set of admissible values.
- provide the lower and upper bounds of state vector.





Two conditions have to be verfied:

- Inclusion: $\underline{x}(t) \leq x(t) \leq \overline{x}(t), \ \forall t_0 \geq 0$
- Stability of estimation errors: $\underline{e} = x - \underline{x}$ and $\overline{e} = \overline{x} - x$

Consider the following discrete-time linear switched system:

$$\begin{cases} x(k+1) & = A_{\sigma_k} x(k) + B_{\sigma_k} u(k) + D_{\sigma_k} d(k) + \omega(k), \\ y_m(k) & = C_{\sigma_k} x(k) + v(k), \ \sigma_k \in \overline{1, N}, \ N \in \mathbb{N} \end{cases} \tag{4}$$

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- $x \in \mathbb{R}^n$: the state.
- $u \in \mathbb{R}^m$: the input.
- $y_m \in \mathbb{R}^p$: the output.
- $\omega \in \mathbb{R}^n$: the disturbances.
- $v \in \mathbb{R}^p$: the measurement noises.
- $d \in \mathbb{R}^l$: the unknown input.

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The objective is to design an interval observer:

1. A potential candidates to cope with such uncertainties.
2. a joint estimation of the state and the unknown input.
3. The given observer have to verify that:

$$\begin{matrix} \underline{x}(k) & \leq x(k) & \leq \overline{x}(k) \\ \underline{d}(k) & \leq d(k) & \leq \overline{d}(k) \end{matrix} \quad (5)$$

Two approaches are introduced:

- Classical approach based on decoupling the unknown input from the state vector:
  - Two changes of coordinates are used.
  - Estimate the bounds of the state vector x.
  - Computing two bounds for the unknown input vector d.
- New approach based on T-N-L method allowing the estimation of the state vector and the unknown input simultaneously.
  - new structure providing more design degrees of freedom.
  - relaxes the design conditions.

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Some assumptions are introduced

1. The state vector $x \in \mathbb{R}^n$ is bounded, i.e. $x \in \mathcal{L}_\infty^n$.
2. The switching signal $\sigma(k)$ is assumed to be known.
3. The state disturbance and the noise measurement are assumed to be bounded such that

$$-\overline{\omega} \leq \omega(k) \leq \overline{\omega},\ \forall k \geq 0, \quad (6)$$

$$-\overline{v} \leq v(k) \leq \overline{v},\ \forall k \geq 0, \quad (7)$$

   with $\overline{\omega} \in \mathbb{R}^n$ and $\overline{v} \in \mathbb{R}^p$.

4. The matrices C_{σ_k} and D_{σ_k} verify: $rank(C_{\sigma_k} D_{\sigma_k}) = rank(D_{\sigma_k}) = l, \forall \sigma_k \in \overline{1, N},\ l \leq p$

The stability is based on the following Lemma:

Lemma 1

Consider the discrete-time switched system $x(k+1) = f_{\sigma(k)}(\xi(k), \eta(k))$, $\sigma(k) \in \overline{1, N}$. Suppose that there exists $\mathcal{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \longrightarrow \mathbb{R}_+$, class $\mathcal{K}_\infty$ functions α_1, α_2, γ and constants $0 < \alpha < 1$, $\mu \geq 1$ such that $\forall \xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^l$, we have

$$\alpha_1(\|\xi\|) \leq V_{\sigma(k)}(\xi) \leq \alpha_2(\|\xi\|), \tag{8}$$

$$V_{\sigma(k)}(\xi(k+1)) - V_{\sigma(k)}(\xi(k)) \leq -\alpha V_{\sigma(k)}(\xi(k)) + \varrho(\|\eta\|), \tag{9}$$

and for each switching instant k_l, $l = 0, 1\ 2, 3, \ldots,$

$$V_{\sigma(k_l)}(\xi(k)) \leq \mu V_{\sigma(k_l - 1)}(\xi(k)). \tag{10}$$

Then the system $x(k+1) = f_{\sigma(k)}(\xi(k), \eta(k))$, $\sigma(k) \in \overline{1, N}$ is Input-to-State Stable (ISS) for any switching signal satisfying the average dwell time.

Given the change of coordinates $z = T_{\sigma_k} x$, the system (4) becomes

$$\begin{cases} z(k+1) = \tilde{A}_{\sigma_k} z(k) + \tilde{B}_{\sigma_k} u(k) + \tilde{D}_{\sigma_k} d(k) + \tilde{w}_{\sigma_k}(k), \\ y_m(k) = \tilde{C}_{\sigma_k} z(k) + v(k), \ \forall \sigma_k \in \overline{1, N}, \ N \in \mathbb{N}, \end{cases} \quad (11)$$

where

$$T_{\sigma_k} = \begin{bmatrix} D^*_{\sigma_k} \\ (C_{\sigma_k} D_{\sigma_k})^{\oplus} C_{\sigma_k} \end{bmatrix} \quad (12)$$

$D^*_{\sigma_k} D_{\sigma_k} = 0$ and $(C_{\sigma_k} D_{\sigma_k})^{\oplus}$ is the left pseudo-inverse of $(C_{\sigma_k} D_{\sigma_k})$.

$$\tilde{A}_{\sigma_k} = T_{\sigma_k} A_{\sigma_k} T^{-1}_{\sigma_k} = \begin{bmatrix} \tilde{A}_{1\sigma_k} & \tilde{A}_{2\sigma_k} \\ \tilde{A}_{3\sigma_k} & \tilde{A}_{4\sigma_k} \end{bmatrix}, \ \tilde{C}_{\sigma_k} = C_{\sigma_k} T^{-1}_{\sigma_k},$$

$$\tilde{B}_{\sigma_k} = T_{\sigma_k} B_{\sigma_k} = \begin{bmatrix} \tilde{B}_{1\sigma_k} \\ \tilde{B}_{2\sigma_k} \end{bmatrix}, \ \tilde{D}_{\sigma_k} = T_{\sigma_k} D_{\sigma_k} = \begin{bmatrix} 0 \\ I_l \end{bmatrix},$$

$$\tilde{\omega}_{\sigma_k}(k) = T_{\sigma_k}\omega(k) = \begin{bmatrix} \tilde{\omega}_{1\sigma_k}(k), \\ \tilde{\omega}_{2\sigma_k} \end{bmatrix}, \; z(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}, \; z_1 \in \mathbb{R}^{n-l}, \; z_2 \in \mathbb{R}^{l}.$$

We get two subsystems:

$$\begin{cases} z_1(k+1) = \tilde{A}_{1\sigma_k} z_1(k) + \tilde{A}_{2\sigma_k} z_2(k) + \tilde{B}_{1\sigma_k} u(k) + \tilde{\omega}_{1\sigma_k}(k) \\ z_2(k+1) = \tilde{A}_{3\sigma_k} z_1(k) + \tilde{A}_{4\sigma_k} z_2(k) + \tilde{B}_{2\sigma_k} u(k) + d(k) + \tilde{\omega}_{2\sigma_k}(k) \\ \tilde{y}_1(k) = \check{C}_{\sigma_k} z_1(k) + U_{1\sigma_k} v(k) \\ \tilde{y}_2(k) = z_2(k) + U_{2\sigma_k} v(k) \end{cases}, \tag{13}$$

$$\tilde{y}(k) = U_{\sigma_k} y_m(k), \quad \tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}, \tag{14}$$

$$U_{\sigma_k} = \begin{bmatrix} U_{1\sigma_k} \\ U_{2\sigma_k} \end{bmatrix} = \begin{bmatrix} (C_{\sigma_k} D_{\sigma_k})^{*} \\ (C_{\sigma_k} D_{\sigma_k})^{\oplus} \end{bmatrix}, \; \check{C}_{\sigma_k} = (C_{\sigma_k} D_{\sigma_k})^{*} C_{\sigma_k} (D^{*}_{\sigma_k})^{\oplus}$$

$$\tilde{\omega}_{\sigma_k}(k) = T_{\sigma_k}\omega(k) = \begin{bmatrix} \tilde{\omega}_{1\sigma_k}(k), \\ \tilde{\omega}_{2\sigma_k} \end{bmatrix}, \; z(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}, \; z_1 \in \mathbb{R}^{n-l}, \; z_2 \in \mathbb{R}^{l}.$$

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$\Rightarrow$ **Unknown input-free subsystem**

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(15)

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We get two subsystems:

$$\begin{cases} z_2(k+1) = \tilde{A}_{3\sigma_k} z_1(k) + \tilde{A}_{4\sigma_k} z_2(k) + \tilde{B}_{2\sigma_k} u(k) + \mathbf{d(k)} + \tilde{\omega}_{2\sigma_k}(k) \\ \tilde{y}_2(k) = z_2(k) + U_{2\sigma_k} v(k) \end{cases}, \tag{13}$$

⇒ **Unknown input-depending subsystem**

$$\tilde{y}(k) = U_{\sigma_k} y_m(k), \quad \tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}, \tag{14}$$

$$U_{\sigma_k} = \begin{bmatrix} U_{1\sigma_k} \\ U_{2\sigma_k} \end{bmatrix} = \begin{bmatrix} (C_{\sigma_k} D_{\sigma_k})^* \\ (C_{\sigma_k} D_{\sigma_k})^{\oplus} \end{bmatrix}, \; \check{C}_{\sigma_k} = (C_{\sigma_k} D_{\sigma_k})^* C_{\sigma_k} (D^*_{\sigma_k})^{\oplus}$$

The unknown free-susbsystem can be written:

$$\begin{cases} z_1(k+1) & = \tilde{A}_{1\sigma_k} z_1(k) + \tilde{A}_{2\sigma_k} \tilde{y}_2(k) + \tilde{B}_{1\sigma_k} u(k) + \tilde{\omega}_{1\sigma_k}(k) \\ & -\tilde{A}_{2\sigma_k} U_{2\sigma_k} v(k) \\ \tilde{y}_1(k) & = \check{C}_{\sigma_k} z_1(k) + U_{1\sigma_k} v(k). \end{cases} \quad (16)$$

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To design an interval observer for (16), two properties have to be satisfied:

- **Framer property** which is the notion of providing intervals in which the state variables stay.
- **Stability property** which cares the length of estimated intervals

$\Rightarrow$ The observer gains L_{σ_k} need to be chosen such that:

1. $\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k}$ are nonnegative.
2. The estimation errors are stable.

The solution:

- A nonsingular transformation $\beta_1 = Pz_1$ such that the matrices $P(\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})P^{-1}$ are nonnegative.
- The existence of a common transformation P for all $\sigma_k \in \overline{1, N}$ is not obvious, even impossible.
- The design, in the original coordinates of two conventional observers.

$$\begin{cases} \hat{z}_1^+(k+1) & = (\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})\hat{z}_1^+(k) + \tilde{B}_{1\sigma_k}u(k) + P_{\sigma_k}^{-1}|P_{\sigma_k}|\overline{\tilde{\omega}}_{1\sigma_k} \\ & +P_{\sigma_k}^{-1}\left[P_{\sigma_k}^+\left(\tilde{A}_{2\sigma_k}^+\overline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\underline{\tilde{y}}_2\right) - P_{\sigma_k}^-\left(\tilde{A}_{2\sigma_k}^+\underline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\overline{\tilde{y}}_2\right)\right] \\ & +P_{\sigma_k}^{-1}|P_{\sigma_k}||\tilde{A}_{2\sigma_k}U_{2\sigma_k}|\overline{v} + L_{\sigma_k}\tilde{y}_1 + P_{\sigma_k}^{-1}|P_{\sigma_k}||L_{\sigma_k}U_{1\sigma_k}|\overline{v}, \\ \hat{z}_1^-(k+1) & = (\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})\hat{z}_1^-(k) + \tilde{B}_{1\sigma_k}u(k) + P_{\sigma_k}^{-1}\left(-|P_{\sigma_k}|\right)\overline{\tilde{\omega}}_{1\sigma_k} \\ & +P_{\sigma_k}^{-1}\left[P_{\sigma_k}^+\left(\tilde{A}_{2\sigma_k}^+\underline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\overline{\tilde{y}}_2\right) - P_{\sigma_k}^-\left(\tilde{A}_{2\sigma_k}^+\overline{\tilde{y}}_2 - \tilde{A}_{2\sigma_k}^-\underline{\tilde{y}}_2\right)\right] \\ & -P_{\sigma_k}^{-1}|P_{\sigma_k}||\tilde{A}_{2\sigma_k}U_{2\sigma_k}|\overline{v} + L_{\sigma_k}\tilde{y}_1 - P_{\sigma_k}^{-1}|P_{\sigma_k}||L_{\sigma_k}U_{1\sigma_k}|\overline{v}, \end{cases} \tag{16}$$

- **Framer design**

Theorem 1

Let Assumptions 1-4 be satisfied and $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$. Given the nonsingular transformation matrices $P_{\sigma_k} \in \mathbb{R}^{(n-l)\times(n-l)}$ such that $P_{\sigma_k}(\tilde{A}_{1\sigma_k} - L_{\sigma_k}\check{C}_{\sigma_k})P_{\sigma_k}^{-1}$ are nonnegative and consider the suitably selected initial conditions

$$\begin{cases} \hat{z}_1^+(0) = P_{\sigma_k}^{-1}\left(P_{\sigma_k}^+\overline{z}_1(0) - P_{\sigma_k}^-\underline{z}_1(0)\right), \\ \hat{z}_1^-(0) = P_{\sigma_k}^{-1}\left(P_{\sigma_k}^+\underline{z}_1(0) - P_{\sigma_k}^-\overline{z}_1(0)\right), \end{cases} \quad (17)$$

where

$$\begin{cases} \overline{z}(0) = T_{\sigma_0}^+\overline{x}(0) - T_{\sigma_0}^-\underline{x}(0), \\ \underline{z}(0) = T_{\sigma_0}^+\underline{x}(0) - T_{\sigma_0}^-\overline{x}(0), \end{cases} \quad (18)$$

Theorem 1

Then, the bounds of the substate vector z_1 given by

$$\begin{cases} \overline{z}_1(k) = (P_{\sigma_k}^{-1})^+ P_{\sigma_k} \hat{\overline{z}}_1^+(k) - (P_{\sigma_k}^{-1})^- P_{\sigma_k} \hat{\underline{z}}_1^-(k), \\ \underline{z}_1(k) = (P_{\sigma_k}^{-1})^+ P_{\sigma_k} \hat{\underline{z}}_1^-(k) - (P_{\sigma_k}^{-1})^- P_{\sigma_k} \hat{\overline{z}}_1^+(k), \end{cases} \tag{19}$$

satisfy

$$\underline{z}_1(k) \leq z_1(k) \leq \overline{z}_1(k), \ \forall \ k \geq 0. \tag{20}$$

- **Stability conditions**

Theorem 2

Assume that the conditions of Theorem 1 are satisfied. If there exist positive scalars $\alpha_2 > \alpha_1 > 0$, $\gamma > 0$, $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, matrices W_{σ_l}, S_{σ_l} and diagonal positive definite matrices M_{σ_k} such that for $\sigma_{k,l} \in \overline{1, N}$ with $\sigma_k \neq \sigma_l$,

$$\begin{bmatrix} -(1-\alpha)M_{\sigma_k} & 0 & \tilde{A}_{1\sigma_k}^T M_{\sigma_k} - \check{C}_{\sigma_k}^T S_{\sigma_k} \\ 0 & -\gamma^2 I_n & M_{\sigma_k} \\ M_{\sigma_k}\tilde{A}_{1\sigma_k} - S_{\sigma_k}\check{C}_{\sigma_k} & M_{\sigma_k} & -M_{\sigma_k} \end{bmatrix} \preceq 0 \quad (21)$$

$$\alpha_1 I_n \leq M_{\sigma_k} \leq \alpha_2 I_n \quad (22)$$

Theorem 2

$$\begin{bmatrix} W_{\sigma_l} & M_{\sigma_k} \\ M_{\sigma_k} & M_{\sigma_k} \end{bmatrix} \succeq 0 \qquad (23)$$

then, the lower and upper observer errors are ISS and the framer (17)-(19) is an interval observer. In addition, the gains L_{σ_k}, given by $L_{\sigma_k} = M_{\sigma_k}^{-1} S_{\sigma_k}$

Based on the estimation of the state in the coordinates z_1, the bounds $\underline{x}$ and $\overline{x}$ are deduced in the following theorem.

Theorem 3

Let the assumptions of Theorem 1 and Theorem 2 hold, then

$$\underline{x}(k) \leq x(k) \leq \overline{x}(k), \ \forall \ k \geq 0 \tag{24}$$

Theorem 3

where:

$$\begin{cases} \overline{x}_1 & = T^{+}_{1\sigma_k}\overline{z}_1 - T^{-}_{1\sigma_k}\underline{z}_1 + T^{+}_{2\sigma_k}\overline{\tilde{y}}_2 - T^{-}_{2\sigma_k}\underline{\tilde{y}}_2 \\ & +(-T_{2\sigma_k}U_{2\sigma_k})^{+}\overline{v} + (-T_{2\sigma_k}U_{2\sigma_k})^{-}\overline{v} \\ \underline{x}_1 & = T^{+}_{1\sigma_k}\underline{z}_1 - T^{1}_{1\sigma_k}\overline{z}_1 + T^{+}_{2\sigma_k}\underline{\tilde{y}}_2 - T^{-}_{2\sigma_k}\overline{\tilde{y}}_2 \\ & -(-T_{2\sigma_k}U_{2\sigma_k})^{+}\overline{v} - (-T_{2\sigma_k}U_{2\sigma_k})^{-}\overline{v} \\ \overline{x}_2 & = T^{+}_{3\sigma_k}\overline{z}_1 - T^{1}_{3\sigma_k}\underline{z}_1 + T^{+}_{4\sigma_k}\overline{\tilde{y}}_2 - T^{-}_{4\sigma_k}\underline{\tilde{y}}_2 \\ & +(-T_{4\sigma_k}U_{2\sigma_k})^{+}\overline{v} + (-T_{4\sigma_k}U_{2\sigma_k})^{-}\overline{v} \\ \underline{x}_2 & = T^{+}_{3\sigma_k}\underline{z}_1 - T^{1}_{3\sigma_k}\overline{z}_1 + T^{+}_{4\sigma_k}\underline{\tilde{y}}_2 - T^{-}_{4\sigma_k}\overline{\tilde{y}}_2 \\ & -(-T_{4\sigma_k}U_{2\sigma_k})^{+}\overline{v} - (-T_{4\sigma_k}U_{2\sigma_k})^{-}\overline{v} \end{cases} \quad (25)$$

$$\underline{x}(k) = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}, \ \overline{x}(k) = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix}.$$

$$z_2(k+1) = U_{2\sigma_k} y_m(k+1) - U_{2\sigma_k} v(k+1) \tag{26}$$

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$$\begin{aligned} d(k) &= z_2(k+1) - \tilde{A}_{3\sigma_k} z_1(k) - \tilde{A}_{4\sigma_k} z_2(k) - \tilde{B}_{2\sigma_k} u(k) - \tilde{\omega}_{2\sigma_k}(k) \\ &= U_{2\sigma_k}\left[y_m(k+1) - v(k+1)\right] - \tilde{A}_{3\sigma_k} z_1(k) - A_{4\sigma_k} U_{2\sigma_k} y_m \\ &\quad + A_{4\sigma_k} U_{2\sigma_k} v(k) - \tilde{B}_{2\sigma_k} u(k) - \tilde{\omega}_{2\sigma_k}(k) \end{aligned} \qquad (27)$$

The upper and lower bounds of d:

$$\begin{cases} \overline{d}(k) & = \left[U^{+}_{2\sigma_k}\overline{\chi}(k+1) - U^{-}_{2\sigma_k}\underline{\chi}(k+1)\right] - \tilde{B}_{2\sigma_k}u(k) \\ & + \left[(-\tilde{A}_{3\sigma_k})^{+}\overline{z}_1(k) - (-\tilde{A}_{3\sigma_k})^{-}\underline{z}_1(k)\right] + \overline{\tilde{\omega}}_{2\sigma_k} \\ & + \left[(-A_{4\sigma_k}U_{2\sigma_k})^{+}\overline{y}_m(k) - (-A_{4\sigma_k}U_{2\sigma_k})^{-}\underline{y}_m(k)\right] + |A_{4\sigma_k}U_{2\sigma_k}|\overline{v}, \\ \underline{d}(k) & = \left[U^{+}_{2\sigma_k}\underline{\chi}(k+1) - U^{-}_{2\sigma_k}\overline{\chi}(k+1)\right] - \tilde{B}_{2\sigma_k}u(k) \\ & + \left[(-\tilde{A}_{3\sigma_k})\underline{z}_1(k) - (-\tilde{A}_{3\sigma_k})^{-}\overline{z}_1(k)\right] - \overline{\tilde{\omega}}_{2\sigma_k} \\ & + \left[(-A_{4\sigma_k}U_{2\sigma_k})^{+}\underline{y}_m(k) - (-A_{4\sigma_k}U_{2\sigma_k})^{-}\overline{y}_m(k)\right] - |A_{4\sigma_k}U_{2\sigma_k}|\overline{v}, \end{cases} \tag{28}$$

with $\chi(k) = y_m(k) - v(k)$. Where $\overline{\chi}(k)$ and $\underline{\chi}(k)$ are respectively upper and lower bound of $\chi(k)$

$$\begin{cases} \overline{\chi}(k) & = y_m(k) + \overline{v} \\ \underline{\chi}(k) & = y_m(k) - \overline{v} \end{cases}. \tag{29}$$

By augmenting unknown input $d(k)$ as a part of the state vector $\tilde{x}(k+1)$, the structural conditions for decoupling unknown input often used in litterature are relaxed.

$$\begin{cases} E_{\sigma_k}\tilde{x}(k+1) & = \tilde{A}_{\sigma_k}\tilde{x}(k) + \tilde{B}_{\sigma_k}u(k) + \tilde{I}\omega(k), \\ y(k) & = \tilde{C}_{\sigma_k}\tilde{x}(k) + v(k), \end{cases} \quad (30)$$

$$\tilde{x}(k+1) = \begin{bmatrix} x(k+1) \\ d(k) \end{bmatrix}, \tilde{x}(0) = \begin{bmatrix} x(0) \\ 0 \end{bmatrix},$$

$$E_{\sigma_k} = \begin{bmatrix} I & -D_{\sigma_k} \\ 0 & 0 \end{bmatrix}, \tilde{I} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$$\tilde{A}_{\sigma_k} = \begin{bmatrix} A_{\sigma_k} & 0 \\ 0 & 0 \end{bmatrix}, \tilde{B}_{\sigma_k} = \begin{bmatrix} B_{\sigma_k} \\ 0 \end{bmatrix}, \tilde{C}_{\sigma_k} = \begin{bmatrix} C_{\sigma_k} & 0 \end{bmatrix}.$$

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The goal: The interval observer of the augmented state $\tilde{x}(k+1)$:

- $\underline{\tilde{x}}(k)$
- $\overline{\tilde{x}}(k)$

such that

$$\underline{\tilde{x}}(k) \leq \tilde{x}(k) \leq \overline{\tilde{x}}(k), \forall k \in \mathbb{Z}_+, \quad (31)$$

The following framer candidate is considered

$$\begin{cases} \overline{\xi}(k+1) &= T_{\sigma_k}\tilde{A}_{\sigma_k}\overline{\tilde{x}}(k) + T_{\sigma_k}\tilde{B}_{\sigma_k}u(k) \\ & +L_{\sigma_k}(y(k) - \tilde{C}_{\sigma_k}\overline{\tilde{x}}(k)) + \Delta \\ \overline{\tilde{x}}(k) &= \overline{\xi}(k) + N_{\sigma_k}y(k) \\ \underline{\xi}(k+1) &= T_{\sigma_k}\tilde{A}_{\sigma_k}\underline{\tilde{x}}(k) + T_{\sigma_k}\tilde{B}_{\sigma_k}u(k) \\ & +L_{\sigma_k}(y(k) - \tilde{C}_{\sigma_k}\underline{\tilde{x}}(k)) - \Delta \\ \underline{\tilde{x}}(k) &= \underline{\xi}(k) + N_{\sigma_k}y(k) \end{cases} \quad (32)$$

$$\Delta = |T_{\sigma_k}\tilde{I}|\overline{\omega} + |L_{\sigma_k}|\overline{v} + |N_{\sigma_k}|\overline{v} \quad (33)$$

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such that

$$\underline{\tilde{x}}(k) \leq \tilde{x}(k) \leq \overline{\tilde{x}}(k), \forall k \in \mathbb{Z}_+, \quad (31)$$

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$$\begin{cases} \overline{\xi}(k+1) & = T_{\sigma_k}\tilde{A}_{\sigma_k}\overline{\tilde{x}}(k) + T_{\sigma_k}\tilde{B}_{\sigma_k}u(k) \\ & +L_{\sigma_k}(y(k) - \tilde{C}_{\sigma_k}\overline{\tilde{x}}(k)) + \Delta \\ \overline{\tilde{x}}(k) & = \overline{\xi}(k) + N_{\sigma_k}y(k) \\ \underline{\xi}(k+1) & = T_{\sigma_k}\tilde{A}_{\sigma_k}\underline{\tilde{x}}(k) + T_{\sigma_k}\tilde{B}_{\sigma_k}u(k) \\ & +L_{\sigma_k}(y(k) - \tilde{C}_{\sigma_k}\underline{\tilde{x}}(k)) - \Delta \\ \underline{\tilde{x}}(k) & = \underline{\xi}(k) + N_{\sigma_k}y(k) \end{cases} \quad (32)$$

$$\Delta = |T_{\sigma_k}\tilde{I}|\overline{\omega} + |L_{\sigma_k}|\overline{v} + |N_{\sigma_k}|\overline{v} \quad (33)$$

- L_{σ_k} : the observer gain associated to the σ_k-subsystem.
- The matrices T_{σ_k}, N_{σ_k} are computed:

$$T_{\sigma_k} E_{\sigma_k} + N_{\sigma_k} \tilde{C}_{\sigma_{k+1}} = I \quad (34)$$

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$$T_{\sigma_k} E_{\sigma_k} + N_{\sigma_k} \tilde{C}_{\sigma_{k+1}} = I \qquad (34)$$

The nonnegativity conditions: Framer design:

Theorem 4

Let Assumptions 1-4 hold, the lower bound $\underline{\tilde{x}}(k)$ and upper bound $\overline{\tilde{x}}(k)$ for the state $\tilde{x}(k)$ given by (32) satisfy (31), if (34) hold and $(T_{\sigma_k} \tilde{A}_{\sigma_k} - L_{\sigma_k} \tilde{C}_{\sigma_k}) \geq 0$, $\forall\ \sigma_k \in \overline{1, N}$ provided that
$\underline{\tilde{x}}_0 := \begin{bmatrix} \underline{x}(0) \\ \underline{d}(0) \end{bmatrix} \leq \tilde{x}(0) \leq \overline{\tilde{x}}_0 := \begin{bmatrix} \overline{x}(0) \\ \overline{d}(0) \end{bmatrix}$.

Proof.

- $\overline{e}(k) = \overline{\tilde{x}}(k) - \tilde{x}(k)$
- $\underline{e}(k) = \tilde{x}(k) - \underline{\tilde{x}}(k)$

The dynamic of the upper error follows

$$\begin{aligned} \overline{e}(k+1) \quad &= (T_{\sigma_k}\tilde{A}_{\sigma_k} - L_{\sigma_k}\tilde{C}_{\sigma_k})\overline{e}(k) + \Delta \\ &-T_{\sigma_k}\tilde{I}\omega(k) + L_{\sigma_k}v(k) + N_{\sigma_k}v(k+1). \end{aligned} \tag{35}$$

$$\Delta - T_{\sigma_k}\tilde{I}\omega(k) + L_{\sigma_k}v(k) + N_{\sigma_k}v(k+1) \geq 0 \tag{36}$$

From the fact that $\overline{e}(0) = \overline{\tilde{x}}(0) - \tilde{x}(0) \geq 0$, it follows that, for all $k \in \mathbb{Z}_+$, $\overline{e}(k) \geq 0$ and for the same reasons $\underline{e}(k) \geq 0$.

Interval observer design using H_∞ performance

Let us define the estimation error as follows

$$e(k) = \overline{e}(k) - \underline{e}(k) \tag{37}$$

Thus,

$$e(k+1) = (T_{\sigma_k}\tilde{A}_{\sigma_k} - L_{\sigma_k}\tilde{C}_{\sigma_k})e(k) + \Phi_{\sigma_k}\delta(k) \tag{38}$$

with

$$\delta(k) = \begin{bmatrix} -T_{\sigma_k}\tilde{I}\omega(k) \\ v(k) \\ v(k+1) \end{bmatrix} \tag{39}$$

and

$$\Phi_{\sigma_k} = 2\begin{bmatrix} I & L_{\sigma_k} & N_{\sigma_k} \end{bmatrix} \tag{40}$$

Interval observer design using H_∞ performance

Theorem 5

Assume that all assumptions of Theorem 4 hold. For given scalars $\gamma > 0$ and $0 < \alpha < 1$, if there exist positive scalars $\alpha_2 > \alpha_1 > 0$, a diagonal matrix $P_{\sigma_k} \in \mathbb{R}^{n\times n}$ such that $P_{\sigma_k} \succ 0$, $W_{\sigma_k} \in \mathbb{R}^{n\times n}$, $G_{\sigma_k} \in \mathbb{R}^{n\times p}$ and $H_{\sigma_k} \in \mathbb{R}^{n\times(n+p)}$ such that

$$P_{\sigma_k}\,\Theta^{\dagger}_{\sigma_k}\lambda_1\tilde{A}_{\sigma_k} + H_{\sigma_k}\psi_{\sigma_k}\lambda_1\tilde{A}_{\sigma_k} - G_{\sigma_k}\tilde{C}_{\sigma_k} \geq 0,\ \forall\sigma_k \in \overline{1,N} \tag{41}$$

$$\alpha_1 I \leq P_{\sigma_k} \leq \alpha_2 I,\ \forall\sigma_k \in \overline{1,N} \tag{42}$$

$$\begin{bmatrix} W_{\sigma_l} & P_{\sigma_k} \\ P_{\sigma_k} & P_{\sigma_k} \end{bmatrix} \succeq 0 \tag{43}$$

Interval observer design using H_∞ performance

Theorem 5

$$\begin{bmatrix} -(1-\alpha)P_{\sigma_k} & \star & \star & \star & \star \\ 0 & -\gamma^2 I & \star & \star & \star \\ 0 & 0 & -\gamma^2 I & \star & \star \\ 0 & 0 & 0 & -\gamma^2 I & \star \\ \kappa_{1\sigma_k} & 2P_{\sigma_k} & 2G_{\sigma_k} & 2\kappa_{2\sigma_k} & -P_{\sigma_k} \end{bmatrix} \preccurlyeq 0, \qquad (44)$$

with

$$W_{\sigma_l} = \mu P_{\sigma_l}, G_{\sigma_k} = P_{\sigma_k} L_{\sigma_k},\ H_{\sigma_k} = P_{\sigma_k} S_{\sigma_k},\ \forall \sigma_k,\ \sigma_l \in \overline{1, N}$$

$$\begin{aligned} \kappa_{1\sigma_k} &= P_{\sigma_k}\ \Theta^{\dagger}_{\sigma_k} \lambda_1 \tilde{A}_{\sigma_k} + H_{\sigma_k} \psi_{\sigma_k} \lambda_1 \tilde{A}_{\sigma_k} - G_{\sigma_k} \tilde{C}_{\sigma_k} \\ \kappa_{2\sigma_k} &= P_{\sigma_k}\ \Theta^{\dagger}_{\sigma_k} \lambda_2 + H_{\sigma_k} \psi_{\sigma_k} \lambda_2,\ \forall \sigma_k \in \overline{1, N} \end{aligned}$$

Interval observer design using H_∞ performance

Theorem 5

Then, (32) is an interval observer for (4). Moreover, the optimal observer gain matrix

$$L_{\sigma_k} = P_{\sigma_k}^{-1} G_{\sigma_k}, \ \forall \sigma_k \in \overline{1, N} \tag{45}$$

Given the system (4) with 3 modes ($N = 3$) where

$$A_1 = \begin{bmatrix} 0.55 & 0.5 & 0.7 \\ 0 & 0.8 & 0.5 \\ 0 & 0 & 0.4 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 \\ 0.5 \\ 0.7 \end{bmatrix}, \ C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \ D_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

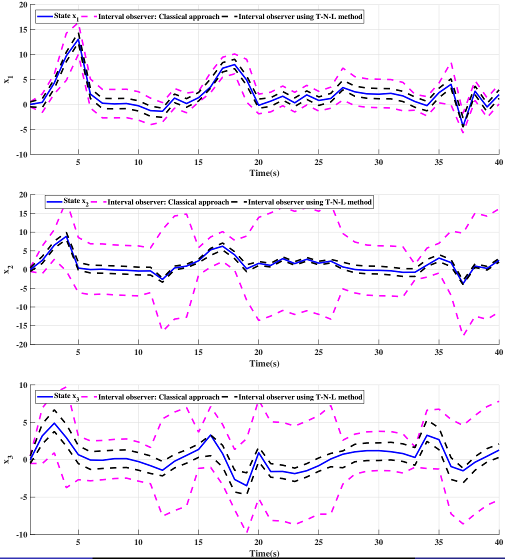
$$A_2 = \begin{bmatrix} -0.44 & -0.4 & -0.56 \\ 0 & -0.64 & -0.4 \\ 0 & 0 & -0.32 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0.4 \\ 0.6 \\ 0 \end{bmatrix}, \ C_2 = \begin{bmatrix} 1.01 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 1 \\ 0 \\ 4.73 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0.1 & 1 & 1 \\ 0 & .2 & -0.5 \\ 0 & 0 & 0.2 \end{bmatrix}, \; B_3 = \begin{bmatrix} 0.1 \\ 0.0 \\ 0.1 \end{bmatrix}, \; C_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \; D_3 = \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}$$

We have as the following conditions $\mid w(k) \mid \leq \overline{w}$ with $\overline{w} = \begin{bmatrix} 0.06 & 0.06 & 0.06 \end{bmatrix}$, and $\mid v(k) \mid \leq \overline{v}$ with $\overline{v} = \begin{bmatrix} 0.06 & 0.06 \end{bmatrix}$.
The unknown input is given as $d(k) = 0.5 \sin(0.5k)$.

For the the state vector, we get:



For the unknown input, we have:

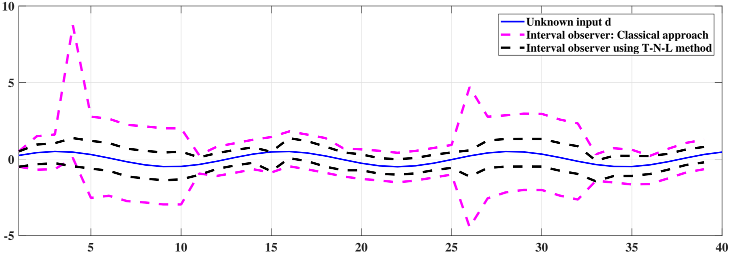


Figure 2: The unknown input

- Two methods to design interval observer for switched systems in presence of unknown input are presented.
- Sufficient conditions for the stability of the interval observer are derived in terms of LMIs.
- The effectiveness of the proposed approaches are shown on a numerical example.

The End