Enclosing optimal trajectories using spatio-temporal constrained zonotopes

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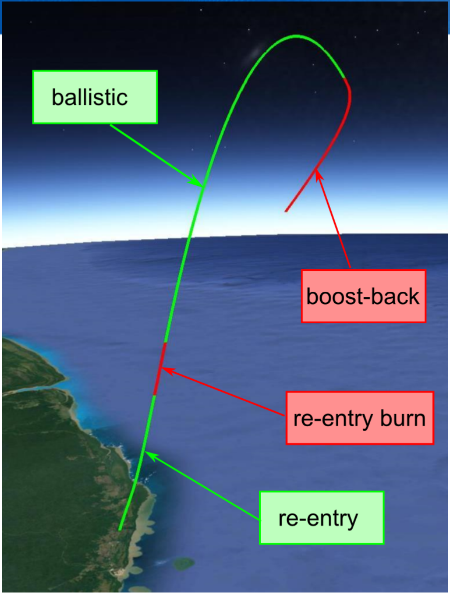
Motivation

Robust control!

Motivation



Guiding a launch vehicle = Optimal Control Problem (OCP)

OCP formulation

$$\min_{u(\cdot)} \int_0^{t_f} \ell(y(t), u(t), \xi) dt$$

$$\text{s.t.} \begin{cases} \dot{y}(t) = f(y(t), u(t), \xi), \\ y(0) = y_0, \\ y(t_f) \in \mathcal{Y}_f, \\ t_f \text{ is free.} \end{cases}$$

Model not exact!
Depends on

- parameters ξ,
- initial state y_0.

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Motivation

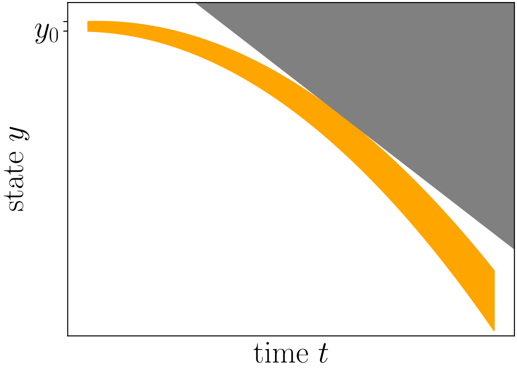
Hypothesis:
bounded uncertainties on parameters and initial state.
$\xi \in [\xi]$ and $y_0 \in [y_0]$

Dynamics with uncertainties

$$\begin{cases} \dot{y} \in [f](y, u, [\xi]) \\ y(0) \in [y_0] \end{cases}$$

Goal: enclose optimal trajectories, assess risks

Problem: control = infinite dimensional unknown



Orange: Possible trajectories of a falling ball with uncertainties
Grey: unsafe set

Outline

1. Caracterization of Optimal trajectories
2. Enclosing trajectories
   - Validated methods
   - Using spatio temporal zonotopes
3. Enclosing intial co-state and switch times
   - Backward propagation of constraints
   - Inflate & Contract

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Characterization of optimal trajectories

Pontryagin's Maximum Principle (PMP)

If $(y(\cdot), u(\cdot))$ = solution of the OCP, then $\exists p(\cdot)$ s.t.

$$\begin{aligned} &\dot{p}(t) = -\frac{\partial H}{\partial y}(y(t), p(t), u(t)) \\ &\forall t \in [0, t_f], u(t) \in \arg\min_{v \in \mathcal{U}} H(y(t), p(t), v), \\ &C(t_f, y(t_f), p(t_f)) = 0, \end{aligned}$$

with $H(y, p, u) = \ell(y, u) + p \cdot f(y, u)$.

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with $H(y, p, u) = \ell(y, u) + p \cdot f(y, u)$.

$\Longrightarrow$ solution of OCP = trajectory of system $\dot{x}(t) = g(x(t), \xi)$ with:

$$x = \begin{pmatrix} t \\ y \\ p \end{pmatrix}, g = \begin{pmatrix} 1 \\ f(y, \arg\min H, \xi) \\ -\frac{\partial H}{\partial y}(y, p, \arg\min H, \xi) \end{pmatrix}$$

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$\arg\min H \to$ hybrid dynamics

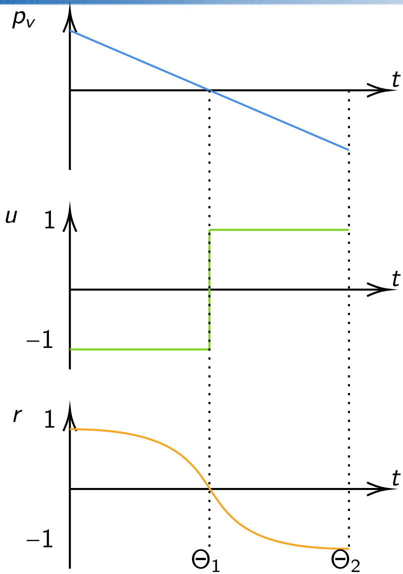
A simple example

Double Integrator in minimal time

$$\min_{u(t)\in[-1,1]} \int_0^{t_f} 1dt,$$
$$\begin{cases} \dot{r} = v, \\ \dot{v} = \xi u, \end{cases}$$
$$r(0) = 1, r(t_f) = -1,$$
$$v(0) = v(t_f) = 0.$$

PMP $\implies \exists p_r(.), p_v(.)$ s.t.

$$H(y,p,u) = 1 + p_r v + p_v \xi u,$$
$$\begin{cases} \dot{p}_r = 0, \\ \dot{p}_v = -p_r, \end{cases}$$
$$u(t) = \arg\min H = \begin{cases} -1 & \text{if } p_v > 0, \\ 1 & \text{if } p_v < 0. \end{cases}$$



Characterization of optimal trajectories

Analysis of hybrid system → find/postulate trajectory structure
→ switched system with constraints.

Optimal trajectories are caracterized by:

$$\begin{cases} \dot{x}(t) = g_n(x(t), \xi), & \forall t \in [\Theta_{n-1}, \Theta_n] \\ x(0) = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix}, & \end{cases}$$

with constraints

$$C_n(x(\Theta_n)) = 0, \forall n \in 1..N,$$

Variables :

- initial co-state $p_0 \in \mathbb{R}^n$,
- transition times $0 < \Theta_1 < \ldots < \Theta_N = t_f$.

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How do we enclose the solutions?

Outline

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Validated methods

Principle:

1. enclose results in sets : $[\pi] = [3.14, 3.15]$
2. replace function f with inclusion function $[f]$ s.t.

$$[f]([a]) \supseteq \{f(a) | \forall a \in [a]\}.$$

Notation: $[f]$ = any inclusion function. It may input and output zonotopes.

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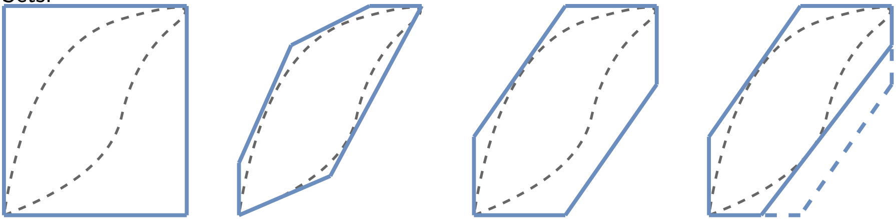
Sets:



Interval vector — Polytope — Zonotope — Constrained zonotope

Cheap but not precise — Precise but expensive — Middle ground

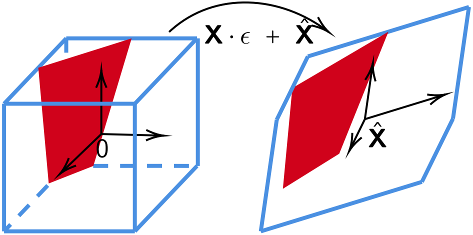
What is a constrained zonotope?

Zonotope:

$$\mathbb{X} = [\mathbf{X}, \hat{\mathbf{X}}] = \left\{ \mathbf{X} \cdot \epsilon + \hat{\mathbf{X}} : \epsilon \in [-1, 1]^{d_\epsilon} \right\}$$

Constrained zonotope:

$$\mathbb{X}^{\mathbb{A}} = [\mathbf{X}, \hat{\mathbf{X}}, \mathbf{A}, \hat{\mathbf{A}}] = \left\{ \mathbf{X} \cdot \epsilon + \hat{\mathbf{X}} : \epsilon \in [-1, 1]^{d_\epsilon}, \mathbf{A} \cdot \epsilon + \hat{\mathbf{A}} = 0 \right\}$$



Why use constrained zonotopes?

Let a zonotope $\mathbb{X} \subset \mathbb{R}^d$, let $c : \mathbb{R}^d \to \mathbb{R}$. We have :

$$\mathcal{X} = \{x \in \mathbb{X} : c(x) = 0\} \subset \mathbb{X}^{\mathbb{A}},$$

with $\mathbb{A} = [c](\mathbb{X})$.

Proof: principle of symbolic zonotope

$$\implies \forall x \in \mathbb{X}, \exists \epsilon \in [-1,1]^{d_\epsilon} \text{ s.t. } \begin{cases} x & = \mathbf{X} \cdot \epsilon + \hat{\mathbf{X}}, \\ c(x) & = \mathbf{A} \cdot \epsilon + \hat{\mathbf{A}}. \end{cases}$$

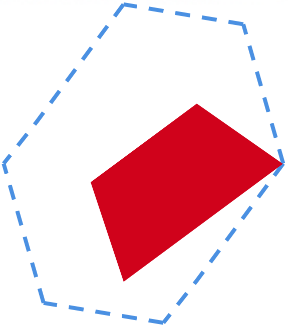
$$\implies \forall x \in \mathcal{X}, \exists \epsilon \in [-1,1]^{d_\epsilon} \text{ s.t. } \begin{cases} \hat{\mathbf{X}} + \mathbf{X} \cdot \epsilon = & x, \\ \hat{\mathbf{A}} + \mathbf{A} \cdot \epsilon = & 0. \end{cases}$$

$$\implies \forall x \in \mathcal{X}, x \in \mathbb{X}^{\mathbb{A}}.$$

Non symbolic zonotope → use zonotope $\left[\begin{pmatrix} \mathbf{X} \\ \mathbf{A} \end{pmatrix}, \begin{pmatrix} \hat{\mathbf{X}} \\ \hat{\mathbf{A}} \end{pmatrix}\right]$[1].

[1] see [Scott et al., 2016]

How to display constrained zonotopes?



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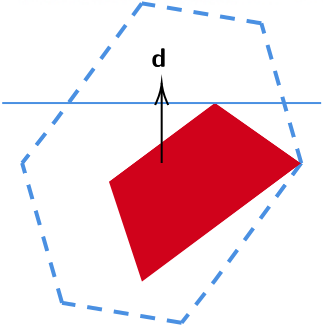

Bound $\mathbb{X}^{\mathbb{A}}$

solve LP:

$$\max_{\epsilon \in [-1,1]^{d_\epsilon}} \mathbf{d}^T \cdot (\mathbf{X} \cdot \epsilon + \hat{\mathbf{X}})$$
$$\text{s.t. } \mathbf{A} \cdot \epsilon + \hat{\mathbf{A}} = 0$$

→ bounding plane

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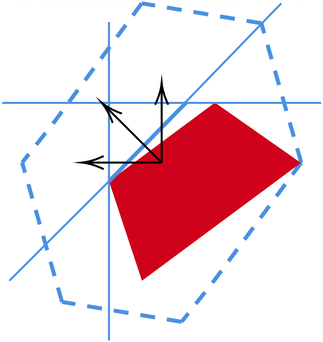

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$\rightarrow$ bounding plane

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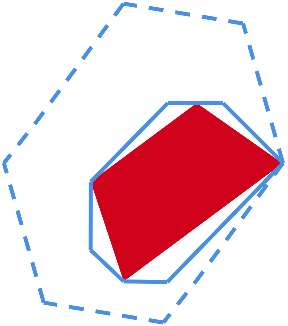

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→ bounding plane

Pros of constrained zonotopes:

- straightforward representation of $\{x \in \mathbb{X} : c(x) = 0\}$,
- easily embedded in zonotope based algorithms.

Cons:

- must solve LPs to get an explicit representation.

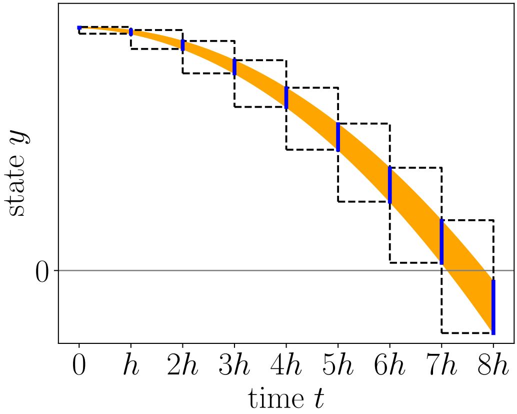
Validated simulation

Let an uncontrolled system:

$$\begin{cases} \dot{x} \in [g](x, [\xi]) \\ x(0) \in [x_0] \end{cases}$$

Validated simulation = enclosure in a sequence of Picard boxes (dashed) and zonotopes (blue).

DynIbex = C++ library with validated Runge Kutta methods and zonotopes.



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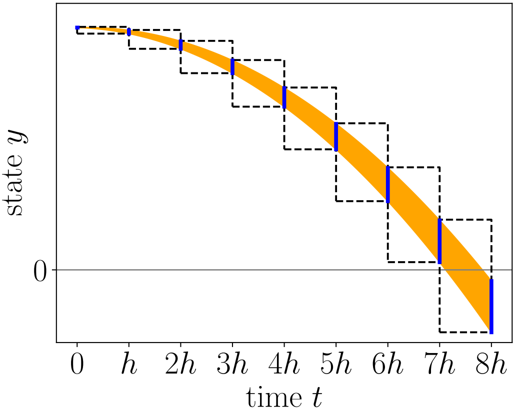
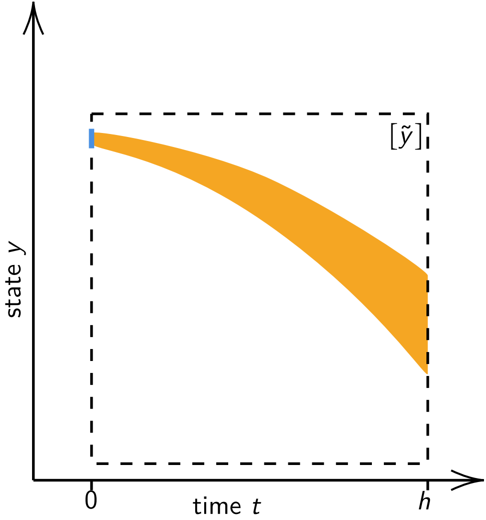


How to enclose the event in a single zonotope?

Building spatio temporal zonotopes with validated Taylor

$[\tilde{y}]$= Picard box that encloses all trajectories over time range $[0, h]$



Building spatio temporal zonotopes with validated Taylor

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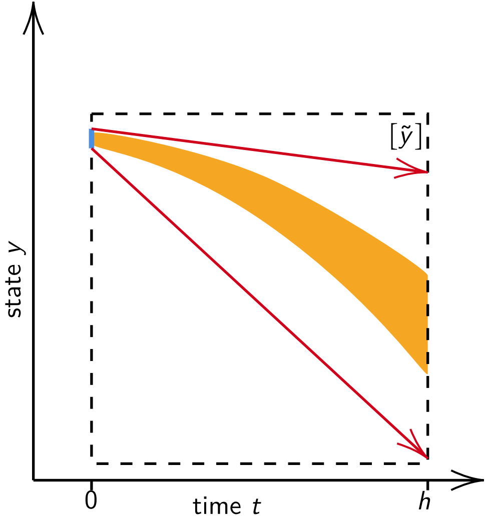
Subsequent derivatives of f over $[\tilde{y}]$ are enclosed



Building spatio temporal zonotopes with validated Taylor

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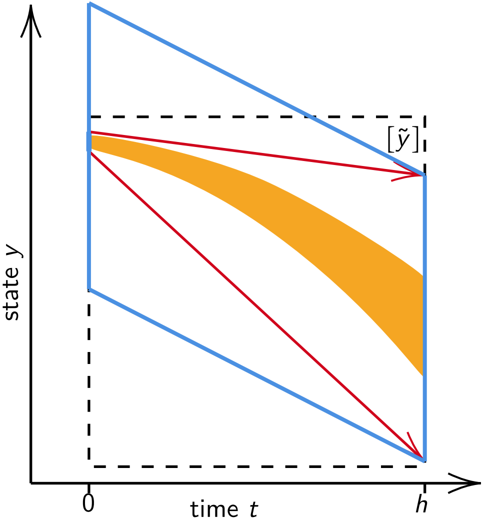
A validated Taylor interpolation yields a zonotope enclosing trajectories over time range $[0, h]$

Spatio temporal zonotopes = state + time coordinates



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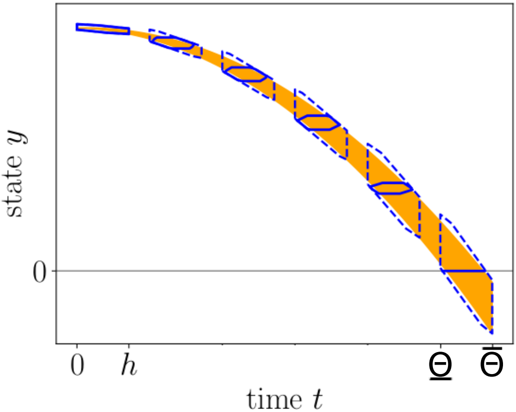
Constrained spatio temporal zonotopes

Let a variable transition time $\Theta \in [\underline{\Theta}, \overline{\Theta}]$.

1. take $h = \overline{\Theta} - \underline{\Theta}$,
2. enclose trajectories over $[\underline{\Theta}, \overline{\Theta}]$ in a zonotope,
3. add $C(x(\Theta)) = 0$ as constraints,
4. propagate constraints backward with guaranteed linearization.

Problem : how do we know bounds $\underline{\Theta}$ and $\overline{\Theta}$?



Dashed: spatio temporal zonotopes
Plain: zonotopes + optimality condition

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Outline

Backward propagation of constraints

Define flow $\Phi_{0,\tau}(p_0, \Theta)$ = integrating system:

$$\begin{cases} \dot{x}(t) = g_n(x(t), \xi), & \forall t \in [\Theta_{n-1}, \Theta_n] \\ x(0) = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix}, & \end{cases}$$

until time τ.

Optimality condition = constraint functions:

$$C_n(\Phi_{0,\Theta_n}(p_0, \Theta)) = 0, \forall n \in 1..N,$$

Goal: apply validated Taylor at order 0:

$$\forall x \in \mathbb{X}, f(x) \in f(\hat{\mathbf{X}}) + [\nabla f](\mathbb{X}) \cdot (\mathbb{X} - \hat{\mathbf{X}}),$$

with $f = C_n \circ \Phi_{0,\tau}$.

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Backward propagation of constraints

$\nabla\Phi_{0,\tau} = R_{0,\tau}$ solution of:

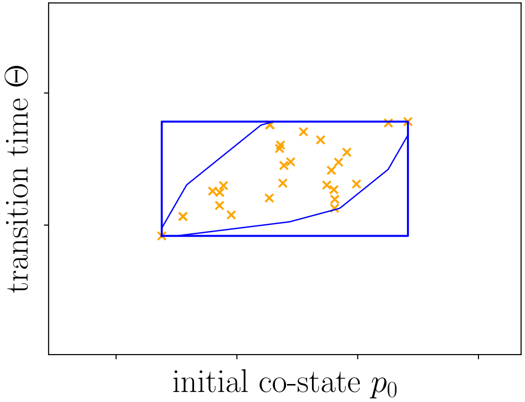
$$\begin{cases} \dot{x}(t) & = g(x(t)) \\ \dot{R}_{0,t}(x_0) & = \dfrac{\partial g}{\partial x}(x(t)) \cdot R_{0,t}(x_0) \\ x(0) & = x_0 \\ R_{0,0}(x_0) & = I_n. \end{cases}$$

Validated simulation $\rightarrow [\nabla\Phi_{0,\tau}]$
$\rightarrow [C_n \circ \Phi_{0,\tau}]$
$\rightarrow \mathbb{X}^{\mathbb{A}}$ with $\mathbb{X}$ enclosure of p_0 or Θ, $\mathbb{A} = [C_n \circ \Phi_{0,\tau}](\mathbb{X})$

Constraint are propagated backward to time 0.

Enclosing variables with an inflate & contract method

Problem : need an enclosure of the variables.



Inflate & contract method:

Start with a box enclosing numerical solutions, inflate it until it contains all solutions.

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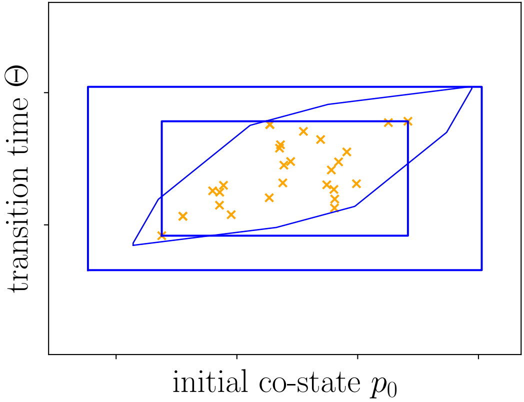

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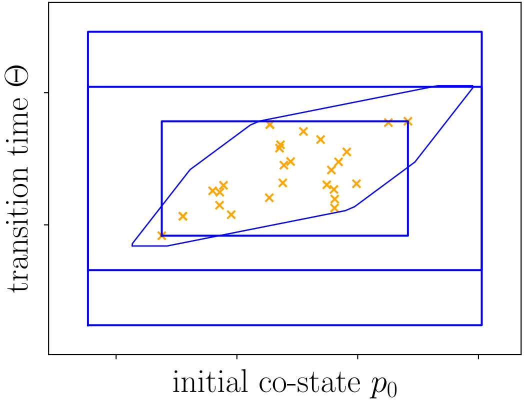

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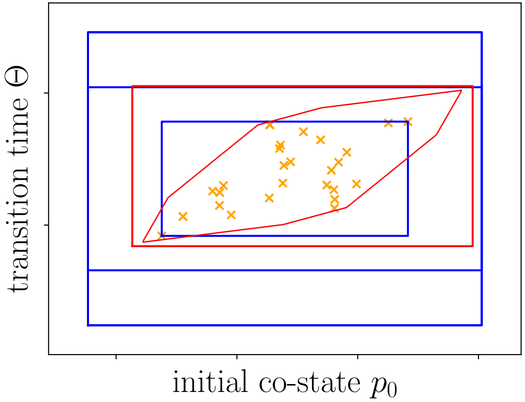

Inflate & contract method:

Start with a box enclosing numerical solutions, inflate it until it contains all solutions.

Contract the box with fixed point iterations.

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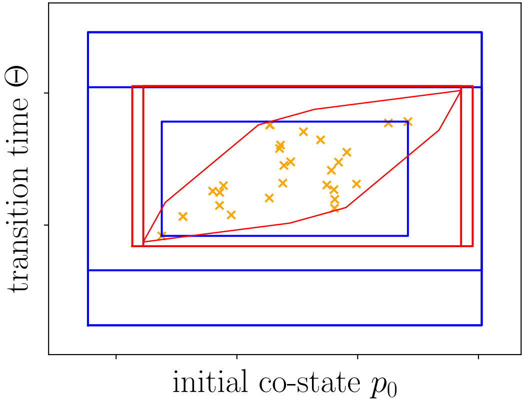

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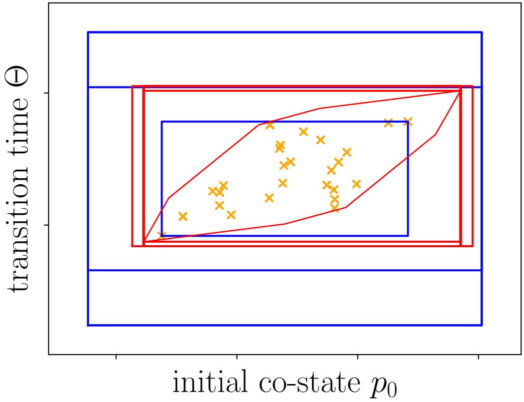


Inflate & contract method:

Start with a box enclosing numerical solutions, inflate it until it contains all solutions.

Contract the box with fixed point iterations.

→ enclosure of all variables

→ self started method

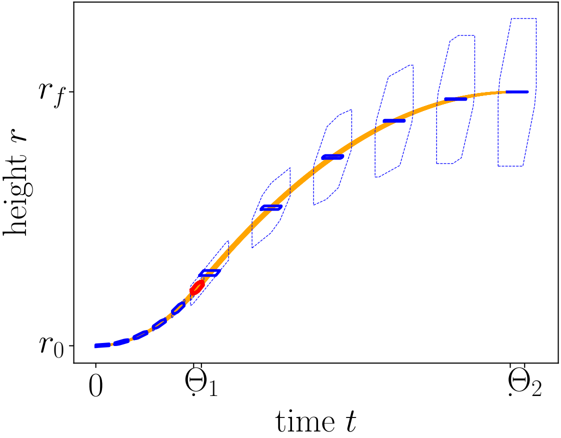
Back to aerospace problems

Consider a simple take-off problem:

Goddard's problem

$$\min_{u(\cdot)} \int_0^{t_f} |u| dt$$

$$\text{s.t.} \begin{cases} \dot{r}(t) = v, \\ \dot{v}(t) = -\frac{G}{r^2} + \frac{Cu}{m}, \\ \dot{m}(t) = -b|u|, \\ y(0) = y_0 \\ r(t_f) = r_f, \\ v(t_f), m(t_f), t_f \\ \quad \text{are free.} \end{cases}$$



Orange: trajectories for various values of the parameters. They are enclosed as intended.

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Conclusion

Our method:

1. OCP → uncontrolled switched system,
2. enclose system at transition time with spatio temporal zonotopes,
3. add optimality conditions as constraints, propagate them backward,
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We find an enclosure of optimal trajectories.

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Future works:

- more complex aerospace problems,
- decrease the over approximation.

Thank you for your attention

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