Set-based State Estimation of Nonlinear Discrete-time Systems Using Constrained Zonotopes

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MECHATRONICS, CONTROL, AND ROBOTICS

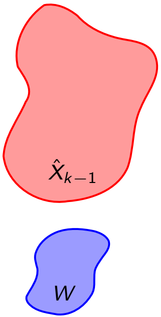
Outline

Outline

Set-based state estimation [Alamo et al., 2005]

$$\begin{cases} \mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1}) \\ \mathbf{y}_k = \mathbf{g}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{v}_k) \end{cases}$$

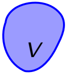
Previous states



Bounded uncertainties

Measurement

$\mathbf{y}_k$



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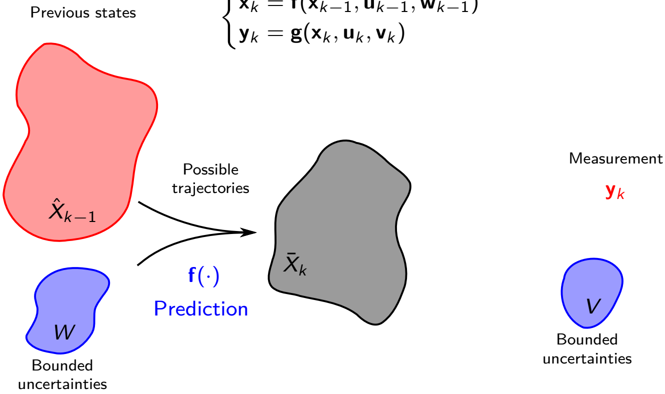

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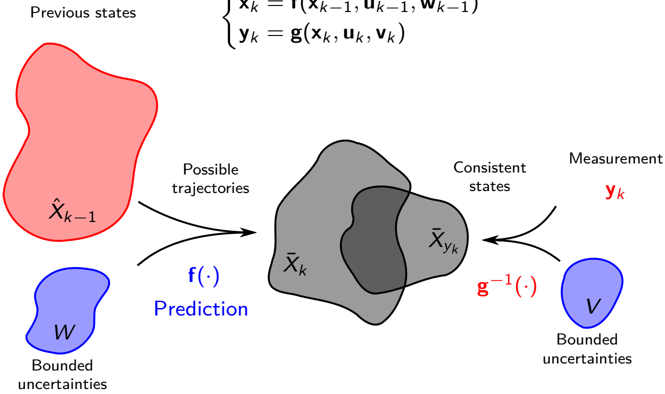

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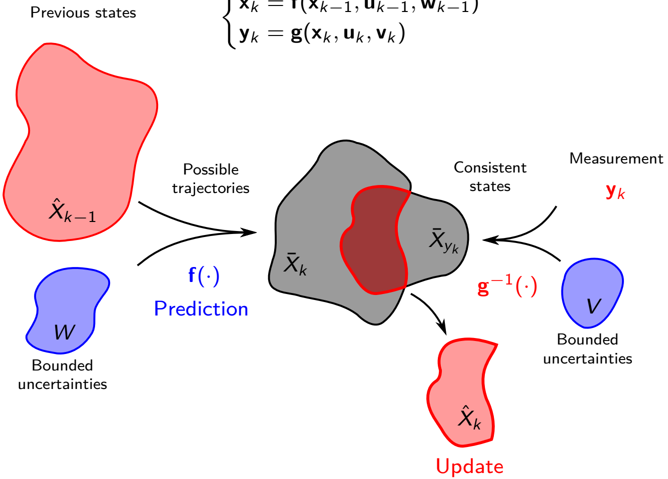
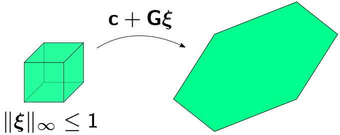


Zonotopes [Kühn, 1998]

Zonotope

$$Z \triangleq \{\mathbf{c} + \mathbf{G}\xi : \|\xi\|_\infty \leq 1\}$$
$$= \{\mathbf{G}, \mathbf{c}\}$$

- $\mathbf{c} \rightarrow$ center
- $\mathbf{G} \rightarrow$ generators



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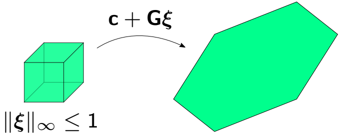

Operations

- Linear mapping

$$\mathbf{R}Z \triangleq \{\mathbf{Rz} : \mathbf{z} \in Z\} = \{\mathbf{RG}_z, \mathbf{Rc}_z\}$$

- Minkowski sum

$$Z \oplus W \triangleq \{\mathbf{z} + \mathbf{w} : \mathbf{z} \in Z, \mathbf{w} \in W\} = \{[\mathbf{G}_z \;\; \mathbf{G}_w], \mathbf{c}_z + \mathbf{c}_w\}$$

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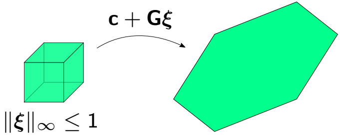


Operations

- Linear mapping

$$\mathbf{R}Z \triangleq \{\mathbf{R}\mathbf{z} : \mathbf{z} \in Z\} = \{\mathbf{R}\mathbf{G}_z, \mathbf{R}\mathbf{c}_z\}$$

- Minkowski sum

$$Z \oplus W \triangleq \{\mathbf{z} + \mathbf{w} : \mathbf{z} \in Z, \mathbf{w} \in W\} = \{[\mathbf{G}_z \ \ \mathbf{G}_w], \mathbf{c}_z + \mathbf{c}_w\}$$

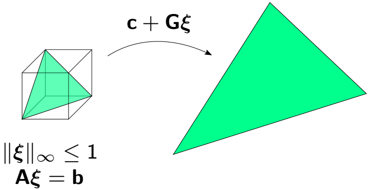
- Intersection: not a zonotope!
  - Conservative intersection with strips [Alamo et al., 2005, Bravo et al., 2006, Alamo et al., 2008]

Constrained zonotopes [Scott et al., 2016]

Constrained zonotope

$$Z \triangleq \{\mathbf{c} + \mathbf{G}\boldsymbol{\xi} : \|\boldsymbol{\xi}\|_\infty \leq 1, \mathbf{A}\boldsymbol{\xi} = \mathbf{b}\}$$
$$= \{\mathbf{G}, \mathbf{c}, \mathbf{A}, \mathbf{b}\}$$

- $\mathbf{c} \rightarrow$ center $\quad \mathbf{G} \rightarrow$ generators
- $\mathbf{A}\xi = \mathbf{b} \rightarrow$ constraints



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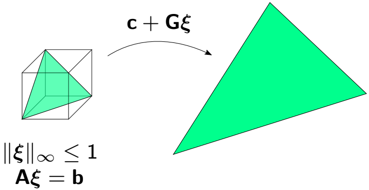

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$$\mathbf{R}Z \triangleq \{\mathbf{R}\mathbf{z} : \mathbf{z} \in Z\} = \{\mathbf{R}\mathbf{G}_z, \mathbf{R}\mathbf{c}_z, \mathbf{A}_z, \mathbf{b}_z\}$$

- Minkowski sum

$$Z \oplus W \triangleq \{\mathbf{z} + \mathbf{w} : \mathbf{z} \in Z, \mathbf{w} \in W\} = \left\{ \begin{bmatrix} \mathbf{G}_z & \mathbf{G}_w \end{bmatrix}, \mathbf{c}_z + \mathbf{c}_w, \begin{bmatrix} \mathbf{A}_z & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix}, \begin{bmatrix} \mathbf{b}_z \\ \mathbf{b}_w \end{bmatrix} \right\}$$

- Generalized intersection

$$Z \cap_{\mathbf{R}} Y \triangleq \{\mathbf{z} \in Z : \mathbf{R}\mathbf{z} \in Y\} = \left\{ \begin{bmatrix} \mathbf{G}_z & \mathbf{0} \end{bmatrix}, \mathbf{c}_z, \begin{bmatrix} \mathbf{A}_z & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_y \\ \mathbf{R}\mathbf{G}_z & -\mathbf{G}_y \end{bmatrix}, \begin{bmatrix} \mathbf{b}_z \\ \mathbf{b}_y \\ \mathbf{c}_y - \mathbf{R}\mathbf{c}_z \end{bmatrix} \right\}$$

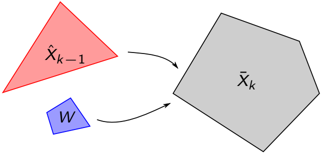
Linear state estimation [Scott et al., 2016]

- $\mathbf{x}_k = \mathbf{A}\mathbf{x}_{k-1} + \mathbf{B}_u\mathbf{u}_{k-1} + \mathbf{B}_w\mathbf{w}_{k-1}, \quad \mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{D}_u\mathbf{u}_k + \mathbf{D}_v\mathbf{v}_k$
- $\mathbf{w}_{k-1} \in W, \quad \mathbf{v}_k \in V$

Prediction step

$$\bar{X}_k = \mathbf{A}\hat{X}_{k-1} \oplus \mathbf{B}_u\mathbf{u}_{k-1} \oplus \mathbf{B}_w W$$

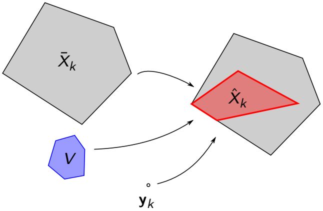
- Linear mapping
- Minkowski sum



Update step

$$\hat{X}_k = \bar{X}_k \cap_C ((\mathbf{y}_k - \mathbf{D}_u\mathbf{u}_k) \oplus (-\mathbf{D}_v V))$$

- Generalized intersection



Complexity reduction

Complexity increase

- Set operations increase the number of generators (and constraints)
- Enclose a zonotope (constrained zonotope) by another one with lower complexity

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Zonotopes

- Generator reduction ($\mathbf{G}$)
- Various methods [Combastel, 2003, Girard, 2005, Althoff et al., 2010, Yang and Scott, 2018]
- Relatively low cost

Constrained zonotopes

- Generator reduction ($\mathbf{G}$): requires lifting $\rightarrow$ higher dimensional set
- Constraint elimination ($\mathbf{A}\xi = \mathbf{b}$) [Scott et al., 2016]
- Relatively high cost

Outline

Problem formulation and algorithm

Discrete-time Systems with Nonlinear Dynamics

$$\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1})$$
$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{D}_u\mathbf{u}_k + \mathbf{D}_v\mathbf{v}_k$$

Bounds (convex polytopic sets)

- $\mathbf{w}_k \in W_k$
- $\mathbf{v}_k \in V_k$
- $\mathbf{x}_0 \in \bar{X}_0$

Problem formulation and algorithm

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Bounds (convex polytopic sets)

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Prediction-update algorithm

- Prediction step: $\bar{X}_k \supseteq \{\mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1}) : \mathbf{x}_{k-1} \in \hat{X}_{k-1},\ \mathbf{w}_{k-1} \in W_{k-1}\}$
- Update step: $\hat{X}_k \supseteq \{\mathbf{x}_k \in \bar{X}_k : \mathbf{C}\mathbf{x}_k + \mathbf{D}_u\mathbf{u}_k + \mathbf{D}_v\mathbf{v}_k = \mathbf{y}_k,\ \mathbf{v}_k \in V_k\}$

Published in

- [Rego et al., 2020a] Rego, B. S., Raffo, G. V., Scott, J. K., & Raimondo, D. M. (2020a). Guaranteed methods based on constrained zonotopes for set-valued state estimation of nonlinear discrete-time systems. *Automatica*, 111, 108614.

Mean value extension (CZMV)

First proposal

- Mean Value Theorem
- Zonotopes [Alamo et al., 2005] $\rightarrow$ Constrained zonotopes

- $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^{n_w} \rightarrow \mathbb{R}^n$ continuously differentiable
- $X \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^{n_w}$ constrained zonotopes
- Approximation point $\gamma_x \in X$
- Interval matrix $\mathbf{J}_x \supseteq \nabla_x^T \mathbf{f}(X, W)$
- For any $(\mathbf{x}, \mathbf{w}) \in X \times W$, there exist $\hat{\mathbf{J}}_x \in \mathbf{J}_x$ such that

$$\mathbf{f}(\mathbf{x}, \mathbf{w}) = \mathbf{f}(\gamma_x, \mathbf{w}) + \hat{\mathbf{J}}_x(\mathbf{x} - \gamma_x)$$

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Implication

$$\mathbf{f}(\mathbf{x}, \mathbf{w}) \in \mathbf{f}(\gamma_x, \mathbf{w}) \oplus \underbrace{\mathbf{J}_x(X - \gamma_x)}_{\text{How to calculate?}}$$

Mean value extension (CZMV)

New operator

- Zonotope inclusion [Alamo et al., 2005] $\rightarrow$ CZ-inclusion

CZ-inclusion

- Constrained zonotope $Z = \{\mathbf{G}, \mathbf{c}, \mathbf{A}, \mathbf{b}\}$, interval matrix $\mathbf{J}$
- Zonotope $\bar{Z} = \{\bar{\mathbf{G}}, \bar{\mathbf{c}}\} \supseteq Z$

Then:

$$\mathbf{J}X \subseteq \triangleleft(\mathbf{J}, X) \triangleq \text{mid}(\mathbf{J})X \oplus \mathbf{P}B_\infty^n$$

$$P_{ii} = \frac{1}{2}\text{diam}(m_i) + \frac{1}{2}\sum_{j=1}^{\bar{n}_g}\sum_{k=1}^{n}\text{diam}(J_{ik})|\bar{M}_{kj}|, \qquad \mathbf{m} \supseteq (\mathbf{J} - \text{mid}(\mathbf{J}))\bar{\mathbf{c}}$$

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Proposal

- $\bar{Z} \supseteq Z$ obtained by eliminating the constraints in Z [Scott et al., 2016]

Mean value extension (CZMV)

Prediction step (mean value extension, CZ)

- Let $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ be of class $\mathcal{C}^1$
- Constrained zonotopes $\hat{X}_{k-1}$ and W, choose a $\gamma_x \in \hat{X}_{k-1}$
- Constrained zonotope Z_w satisfying $\mathbf{f}(\gamma_x, \mathbf{u}_{k-1}, W) \subseteq Z_w$
- Interval matrix $\mathbf{J}_x$ satisfying $\nabla_x^T \mathbf{f}(\hat{X}_{k-1}, \mathbf{u}_{k-1}, W) \subseteq \mathbf{J}_x$

Then:

$$\mathbf{f}(\hat{X}_{k-1}, \mathbf{u}_{k-1}, W) \subseteq Z_w \oplus \triangleleft \left(\mathbf{J}_x, \hat{X}_{k-1} - \gamma_x\right) = \bar{X}_k$$

- Enclosure $\bar{X}_k$ is a constrained zonotope

Update step (linear)

$$\hat{X}_k = \bar{X}_k \cap_{\mathbf{C}} ((\mathbf{y}_k - \mathbf{D}_u \mathbf{u}_k) \oplus (-\mathbf{D}_v V))$$

First-order Taylor extension (CZFO)

Second proposal

- First order Taylor expansion
- Zonotopes [Combastel, 2005] → Constrained zonotopes

Prediction step (first-order Taylor extension, CZ)

- Let $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ be of class $\mathcal{C}^2$
- Constrained zonotope $Z = \hat{X}_{k-1} \times W = \{\mathbf{G}, \mathbf{c}, \mathbf{A}, \mathbf{b}\} \subset \mathbb{R}^{n+n_w}$
- Choose $(\gamma_x, \gamma_w) = \gamma_z \in Z$
- Interval matrices $\mathbf{Q}^{[q]} \supseteq \mathbf{H}_z f_q(\hat{X}_{k-1}, \mathbf{u}_{k-1}, W)$, $\tilde{\mathbf{Q}}^{[q]} \supseteq \mathbf{G}^T \mathbf{Q}^{[q]} \mathbf{G}$, $\mathbf{L}_{q,:} \supseteq (\mathbf{c} - \gamma_z)^T \mathbf{Q}^{[q]}$
- $\tilde{\mathbf{c}}$, $\tilde{\mathbf{G}}$, $\tilde{\mathbf{G}}_{\mathbf{v}}$, $\tilde{\mathbf{A}}$, $\tilde{\mathbf{b}}$

Then:

$$\mathbf{f}(\hat{X}_{k-1}, \mathbf{u}_{k-1}, W) \subseteq \mathbf{f}(\gamma_x, \mathbf{u}_{k-1}, \gamma_w) \oplus \nabla_z^T \mathbf{f}(\gamma_x, \mathbf{u}_{k-1}, \gamma_w)(Z - \gamma_z) \oplus R = \bar{X}_k$$

$$R = \tilde{\mathbf{c}} \oplus [\tilde{\mathbf{G}} \;\; \tilde{\mathbf{G}}_{\mathbf{v}}] B_\infty(\tilde{\mathbf{A}}, \tilde{\mathbf{b}}) \oplus \triangleleft(\mathbf{L}, (\mathbf{c} - \gamma_z) \oplus 2\mathbf{G} B_\infty(\mathbf{A}, \mathbf{b}))$$

- Enclosure $\bar{X}_k$ is a constrained zonotope, $\quad \hat{X}_k = \bar{X}_k \cap_{\mathbf{C}} ((\mathbf{y}_k - \mathbf{D}_u \mathbf{u}_k) \oplus (-\mathbf{D}_v V))$

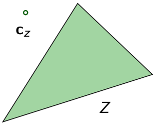
Selection of approximation point

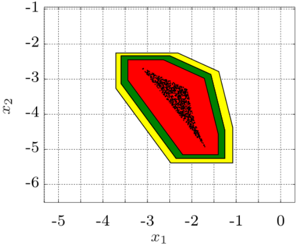
Mean value extension

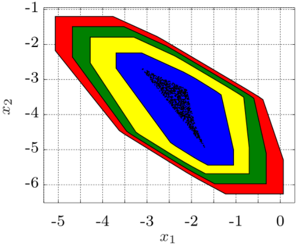
- Zonotope $\{\bar{\mathbf{G}}, \bar{\mathbf{c}}\} \supseteq \hat{X}_{k-1} - \gamma_x$ in $\triangleleft(\mathbf{J}_x, \hat{X}_{k-1} - \gamma_x)$
- Optimal choice: $\min \|\text{diam}(\mathbf{J}_x - \text{diam}(\mathbf{J}_x)\bar{\mathbf{c}})\|_1$ (linear programming)

First-order Taylor extension

- γ_z as close as possible to the center of Z
- When $\mathbf{c}_z \notin Z$: Move the center of Z to a $\bar{\gamma}_z \in Z$ (equivalent CG-rep, linear programming)







Example: Quadrotor UAV

System equations

$$\dot{x} = u_0$$
$$\dot{y} = v_0$$
$$\dot{z} = w_0$$
$$\dot{u}_0 = \frac{1}{m}(\cos\psi \sin\theta \cos\phi + \sin\psi \sin\phi)U_1 + \frac{1}{m}D_x$$
$$\dot{v}_0 = \frac{1}{m}(\sin\psi \sin\theta \cos\phi - \cos\psi \sin\phi)U_1 + \frac{1}{m}D_y$$
$$\dot{w}_0 = -g + \frac{1}{m}(\cos\theta \cos\phi)U_1 + \frac{1}{m}D_z$$
$$\dot{\phi} = p + q\sin\phi \tan\theta + r\cos\phi \tan\theta,$$
$$\dot{\theta} = q\cos\phi - r\sin\phi$$
$$\dot{\psi} = q\sin\phi \sec\theta + r\cos\phi \sec\theta,$$
$$\dot{p} = \frac{I_{yy} - I_{zz}}{I_{xx}} qr + \frac{l}{I_{xx}} U_2$$
$$\dot{q} = \frac{I_{zz} - I_{xx}}{I_{yy}} pr + \frac{l}{I_{yy}} U_3$$
$$\dot{r} = \frac{I_{xx} - I_{yy}}{I_{zz}} pq + \frac{1}{I_{zz}} U_4$$

Measurement

Sensor	Variables	Noise bounds
GPS	$\{x, y\}$	± 0.15 m
Barometer	$\{z\}$	± 0.51 m
IMU	$\{\phi, \theta, \psi\}$	$\pm 2.618 \cdot 10^{-3}$ rad
	$\{p, q, r\}$	$\pm 16.558 \cdot 10^{-3}$ rad/s

Bounds

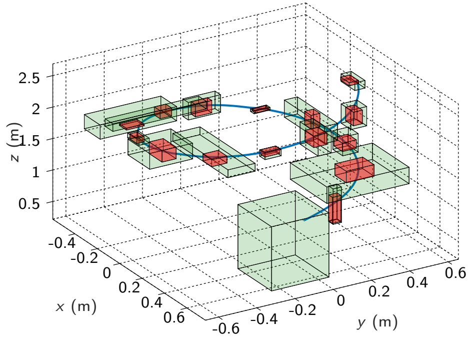
$$X_0 = \{\mathbf{G}_0, \mathbf{0}\}, \quad \|\boldsymbol{d}\|_\infty \leq 1$$

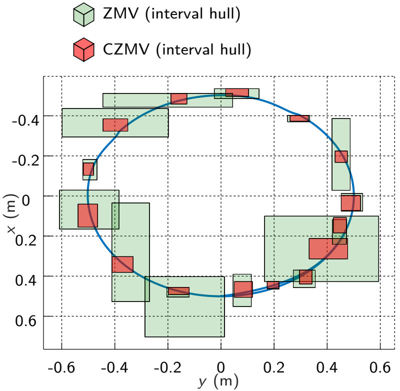
$$\mathbf{G}_0 = \operatorname{diag}\left(2, 2, 2, 1, 1, 1, \frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{12}, \frac{\pi}{12}, \frac{\pi}{12}\right)$$

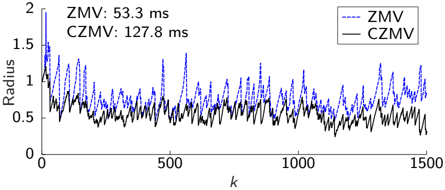
Complexity

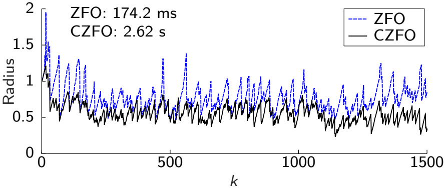
- 40 generators
- 12 constraints

Results: Quadrotor UAV









Outline

Problem formulation

Discrete-time Systems with Nonlinear Measurement and Invariants

$$\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1})$$
$$\mathbf{y}_k = \mathbf{g}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{v}_k)$$

Bounds (convex polytopic sets)

- $\mathbf{x}_0 \in X_0$
- $\mathbf{w}_k \in W_k$
- $\mathbf{v}_k \in V_k$

Invariants

$$\mathbf{h}(\mathbf{x}_k) = \mathbf{0}, \ \forall k \geq 0$$

Examples

- Mechanical systems: holonomic and/or non-holonomic constraints
- Attitude estimation: rotation matrices, quaternions

Published in

- [Rego et al., 2021d] Rego, B. S., Scott, J. K., Raimondo, D. M., & Raffo, G. V. (2021a). Set-valued state estimation of nonlinear discrete-time systems with nonlinear invariants based on constrained zonotopes. *Automatica*, 129, 109638.

Proposed algorithm

Prediction-update-consistency algorithm

- Prediction step: $\bar{X}_k \supseteq \{\mathbf{f}(\mathbf{x}, \mathbf{u}_{k-1}, \mathbf{w}) : \mathbf{x} \in \hat{X}_{k-1},\ \mathbf{w} \in W_{k-1}\}$
- Update step: $\hat{X}_k \supseteq \{\mathbf{x} \in \bar{X}_k : \mathbf{g}(\mathbf{x}, \mathbf{u}_k, \mathbf{v}) = \mathbf{y}_k,\ \mathbf{v} \in V_k\}$
- Consistency step: $\tilde{X}_k \supseteq \{\mathbf{x} \in \hat{X}_k : \mathbf{h}(\mathbf{x}) = \mathbf{0}\}$

Proposed algorithm

Prediction-update-consistency algorithm

- Prediction step: $\bar{X}_k \supseteq \{\mathbf{f}(\mathbf{x}, \mathbf{u}_{k-1}, \mathbf{w}) : \mathbf{x} \in \hat{X}_{k-1},\ \mathbf{w} \in W_{k-1}\}$
- Update step: $\hat{X}_k \supseteq \{\mathbf{x} \in \bar{X}_k : \mathbf{g}(\mathbf{x}, \mathbf{u}_k, \mathbf{v}) = \mathbf{y}_k,\ \mathbf{v} \in V_k\}$
- Consistency step: $\tilde{X}_k \supseteq \{\mathbf{x} \in \hat{X}_k : \mathbf{h}(\mathbf{x}) = \mathbf{0}\}$

Prediction step

- Based on Rego et al. [2020a], but with $(\gamma_x, \gamma_w) \in \square\tilde{X}_{k-1} \times \square W$
  - Valid enclosures
  - Allows for more possible approximation points
  - Can lead to more accurate results

Update: Mean value extension

Update step (mean value extension, CZ)

- Nonlinear mapping $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_v} \to \mathbb{R}^{n_y}$ is of class $\mathcal{C}^1$
- Constrained zonotopes $\bar{X}_k$ and V, choose one $\gamma_x \in \square\bar{X}_k$
- Constrained zonotope $Z_v \subset \mathbb{R}^{n_y}$ satisfying $-\mathbf{g}(\gamma_x, \mathbf{u}_k, V) \subseteq Z_v$
- Real matrix $\tilde{\mathbf{J}} \in \mathbb{R}^{n_y \times n}$

Then:

$$\{\mathbf{x} \in \bar{X}_k : \mathbf{g}(\mathbf{x}, \mathbf{u}_k, \mathbf{v}) = \mathbf{y},\ \mathbf{v} \in V\} \subseteq \hat{X}_k = \bar{X}_k \cap_{\mathbf{C}} Y, \quad \mathbf{C} \triangleq \tilde{\mathbf{J}}$$

$$Y = (\mathbf{y} + \tilde{\mathbf{J}}\gamma_x) \oplus Z_v \oplus \lhd(\tilde{\mathbf{J}} - \mathbf{J}, \bar{X} - \gamma_x)$$

- Generalized intersection: less conservative

Selection of $\tilde{\mathbf{J}}$

- $\tilde{\mathbf{J}} = \mathrm{mid}(\mathbf{J})$

Update: First-order Taylor extension

Update step (first-order Taylor extension, CZ)

- Nonlinear mapping $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_v} \to \mathbb{R}^{n_y}$ is of class $\mathcal{C}^2$
- Constrained zonotopes $\bar{X}_k$, V, choose one $\gamma_z = (\gamma_x, \gamma_v) \in \square\bar{X}_k \times \square V \triangleq \square Z$
- Interval matrices $\mathbf{Q}^{[q]} \supseteq \mathbf{H}_z g_q(\square\bar{X}_k, \mathbf{u}_k, \square V)$, $\tilde{\mathbf{Q}}^{[q]} \supseteq \mathbf{G}^T\mathbf{Q}^{[q]}\mathbf{G}$, $\mathbf{L}_{q,:} \supseteq (\mathbf{c}_z - \gamma_z)^T\mathbf{Q}^{[q]}$
- $\tilde{\mathbf{c}}$, $\tilde{\mathbf{G}}$, $\tilde{\mathbf{G}}_\mathbf{v}$, $\tilde{\mathbf{A}}$, $\tilde{\mathbf{b}}$

Then:

$$\{\mathbf{x} \in \bar{X}_k : \mathbf{g}(\mathbf{x}, \mathbf{u}_k, \mathbf{v}) = \mathbf{y},\ \mathbf{v} \in V\} \subseteq \hat{X}_k = \bar{X}_k \cap_{\mathbf{C}} Y, \quad \mathbf{C} \triangleq \nabla_x^T \mathbf{g}(\gamma_x, \mathbf{u}_k, \gamma_v)$$

$$\begin{aligned} Y = {} & (\mathbf{y} - \mathbf{g}(\gamma_x, \mathbf{u}_k, \gamma_v) + \nabla_x^T\mathbf{g}(\gamma_x, \mathbf{u}_k, \gamma_v)\gamma_x + \nabla_v^T\mathbf{g}(\gamma_x, \mathbf{u}_k, \gamma_v)\gamma_v) \\ & \oplus (-\nabla_v^T\mathbf{g}(\gamma_x, \mathbf{u}_k, \gamma_v)V) \oplus (-R) \end{aligned}$$

$$R = \tilde{\mathbf{c}} \oplus [\tilde{\mathbf{G}}\ \ \tilde{\mathbf{G}}_\mathbf{v}]B_\infty(\tilde{\mathbf{A}}, \tilde{\mathbf{b}}) \oplus \triangleleft(\mathbf{L}, (\mathbf{c}_z - \gamma_z) \oplus 2\mathbf{G}_z B_\infty(\mathbf{A}_z, \mathbf{b}_z))$$

- Generalized intersection: less conservative

Consistency step

Consistency step (mean value extension, CZ)

- Nonlinear mapping $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^{n_h}$ is of class $\mathcal{C}^1$
- Constrained zonotope $\hat{X}_k$, choose one $\gamma_x \in \square\hat{X}_k$
- Real matrix $\tilde{\mathbf{J}} \in \mathbb{R}^{n_h \times n}$

Then: $\{\mathbf{x} \in \hat{X}_k : \mathbf{h}(\mathbf{x}) = \mathbf{0}\} \subseteq \tilde{X}_k = \hat{X}_k \cap_{\mathbf{D}} H, \quad \mathbf{D} \triangleq \tilde{\mathbf{J}}$

$$H = (\tilde{\mathbf{J}}\gamma_x - \mathbf{h}(\gamma_x)) \oplus \triangleleft(\tilde{\mathbf{J}} - \mathbf{J}, \hat{X} - \gamma_x)$$

Consistency step (first-order Taylor extension, CZ)

- Nonlinear mapping $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^{n_h}$ is of class $\mathcal{C}^2$
- Constrained zonotope $\hat{X}_k$, choose one $\gamma_x \in \square\hat{X}_k$
- Interval matrices $\mathbf{Q}^{[q]} \supseteq \mathbf{H}_x h_q(\square\bar{X}_k)$, $\tilde{\mathbf{Q}}^{[q]} \supseteq \mathbf{G}_x^T \mathbf{Q}^{[q]} \mathbf{G}_x$, $\mathbf{L}_{q,:} \supseteq (\mathbf{c}_x - \gamma_x)^T \mathbf{Q}^{[q]}$
- $\tilde{\mathbf{c}}$, $\tilde{\mathbf{G}}$, $\tilde{\mathbf{G}}_{\mathbf{v}}$, $\tilde{\mathbf{A}}$, $\tilde{\mathbf{b}}$

Then: $\{\mathbf{x} \in \hat{X}_k : \mathbf{h}(\mathbf{x}) = \mathbf{0}\} \subseteq \tilde{X}_k = \hat{X}_k \cap_{\mathbf{D}} H, \quad \mathbf{D} = \nabla_x^T \mathbf{h}(\gamma_x)$

$$H = (\nabla_x^T \mathbf{h}(\gamma_x)\gamma_x - \mathbf{h}(\gamma_x)) \oplus (-R)$$

$$R = \tilde{\mathbf{c}} \oplus [\tilde{\mathbf{G}} \;\; \tilde{\mathbf{G}}_{\mathbf{v}}] B_\infty(\tilde{\mathbf{A}}, \tilde{\mathbf{b}}) \oplus \triangleleft(\mathbf{L}, (\mathbf{c}_x - \gamma_x) \oplus 2\mathbf{G}_x B_\infty(\mathbf{A}_x, \mathbf{b}_x))$$

Example 1: nonlinear measurement

System equations: nonlinear dynamics and measurement

$$x_{1,k} = 3x_{1,k-1} - \frac{x_{1,k-1}^2}{7} - \frac{4x_{1,k-1}x_{2,k-1}}{4 + x_{1,k-1}} + w_{1,k-1}$$
$$x_{2,k} = -2x_{2,k-1} + \frac{3x_{1,k-1}x_{2,k-1}}{4 + x_{1,k-1}} + w_{2,k-1}$$
$$y_{1,k} = x_{1,k} - \sin\left(\frac{x_{2,k}}{2}\right) + v_{1,k}$$
$$y_{2,k} = -x_{1,k}x_{2,k} + x_{2,k} + v_{2,k}$$

Known bounds

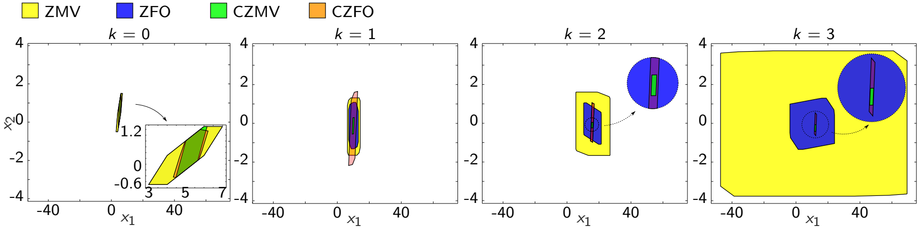
$$\bar{X}_0 = \left\{ \begin{bmatrix} 0.5 & 1 & -0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0.5 \end{bmatrix} \right\}$$

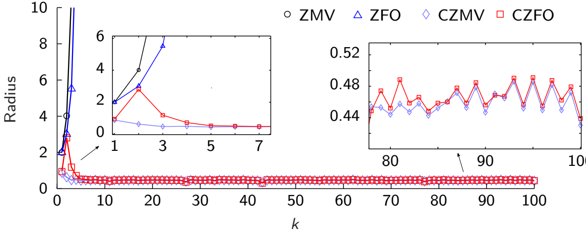
$$\|\boldsymbol{w}_k\|_\infty \leq 0.4, \quad \|\boldsymbol{v}_k\|_\infty \leq 0.4$$

Set complexity

- 8 generators
- 3 constraints

Example 1: nonlinear measurement





ZMV: 30.5 ms
ZFO: 47.4 ms
CZMV: 65.3 ms
CZFO: 72.2 ms

Example 2: nonlinear measurement and invariants

Attitude estimation using quaternions [Teixeira et al., 2009, Lefferts et al., 1982]

$$\mathbf{x}_k = \left(\cos(p(\mathbf{u}_k, \mathbf{w}_k))\mathbf{I}_4 - \frac{T_s}{2}\frac{\sin(p(\mathbf{u}_k, \mathbf{w}_k))}{p(\mathbf{u}_k, \mathbf{w}_k)}\boldsymbol{\Omega}(\mathbf{u}_k, \mathbf{w}_k)\right)\mathbf{x}_{k-1}, \qquad \|\mathbf{x}_k\|_2 = 1$$

$$\mathbf{y}_k = (\mathbf{C}(\mathbf{x}_k)\mathbf{r}^{[1]}, \mathbf{C}(\mathbf{x}_k)\mathbf{r}^{[2]}) + \mathbf{v}_k$$

$$p(\mathbf{u}_k, \mathbf{w}_k) \triangleq \frac{T_s}{2}\|\check{\mathbf{u}}_k\|_2, \;\; \boldsymbol{\Omega}(\mathbf{u}_k, \mathbf{w}_k) \triangleq \begin{bmatrix} 0 & \check{u}_{3,k} & -\check{u}_{2,k} & \check{u}_{1,k} \\ -\check{u}_{3,k} & 0 & \check{u}_{1,k} & \check{u}_{2,k} \\ \check{u}_{2,k} & -\check{u}_{1,k} & 0 & \check{u}_{3,k} \\ -\check{u}_{1,k} & -\check{u}_{2,k} & -\check{u}_{3,k} & 0 \end{bmatrix}, \check{\mathbf{u}}_k \triangleq \mathbf{u}_k - \mathbf{w}_k$$

$$\mathbf{C}(\mathbf{x}_k) = \begin{bmatrix} x_{1,k}^2 - x_{2,k}^2 - x_{3,k}^2 + x_{4,k}^2 & 2(x_{1,k}x_{2,k} + x_{3,k}x_{4,k}) & 2(x_{1,k}x_{3,k} - x_{2,k}x_{4,k}) \\ 2(x_{1,k}x_{2,k} - x_{3,k}x_{4,k}) & -x_{1,k}^2 + x_{2,k}^2 - x_{3,k}^2 + x_{4,k}^2 & 2(x_{1,k}x_{4,k} + x_{2,k}x_{3,k}) \\ 2(x_{1,k}x_{3,k} + x_{2,k}x_{4,k}) & 2(-x_{1,k}x_{4,k} + x_{2,k}x_{3,k}) & -x_{1,k}^2 - x_{2,k}^2 + x_{3,k}^2 + x_{4,k}^2 \end{bmatrix},$$

$\mathbf{r}^{[1]} = [1\ 0\ 0]^T$, $\mathbf{r}^{[2]} = [0\ 1\ 0]^T$

Known bounds

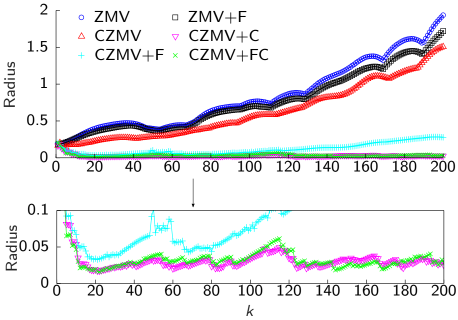
$$\bar{X}_0 = \left\{0.18\mathbf{I}_4, [0\ 1\ 0\ 0]^T\right\}, \quad X_{\mathrm{F}} = \{\mathbf{I}_4, \mathbf{0}\}$$

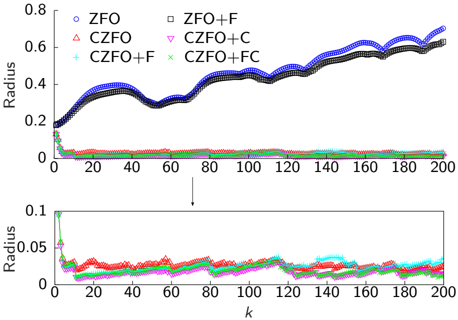
$$\|\mathbf{w}_k\|_\infty \leq 3{\cdot}10^{-3}, \quad \|\mathbf{v}_k\|_\infty \leq 0.15$$

Set complexity

- 12 generators
- 5 constraints

Example 2: nonlinear measurement and invariants





	Average radius ratio	Average execution time
CZMV+C/ZMV+F	6.71%	122.2%
CZFO+C/ZFO+F	5.12%	110.5%

Outline

Overview and future work

Overview

- State estimation of systems with nonlinear dynamics
  - Nonlinear prediction step usinc CZs
    - Mean value extension (CZMV)
    - First-order Taylor extension (CZFO)
- State estimation of systems with nonlinear measurement and invariants
  - Nonlinear update step using CZs
  - Consistency step using CZs
    - Mean value extension (CZMV+C)
    - First-order Taylor extension (CZFO+C)
- More accurate bounds in comparison to zonotopes

Future work

- Reduce computational times
- Alternatives to mean value theorem and first-order Taylor expansion

Set-based State Estimation of Nonlinear Discrete-time Systems Using Constrained Zonotopes

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