Event-Triggered Interval Observers Design for Continuous-Time Linear Systems: L_1 - Gain Approach

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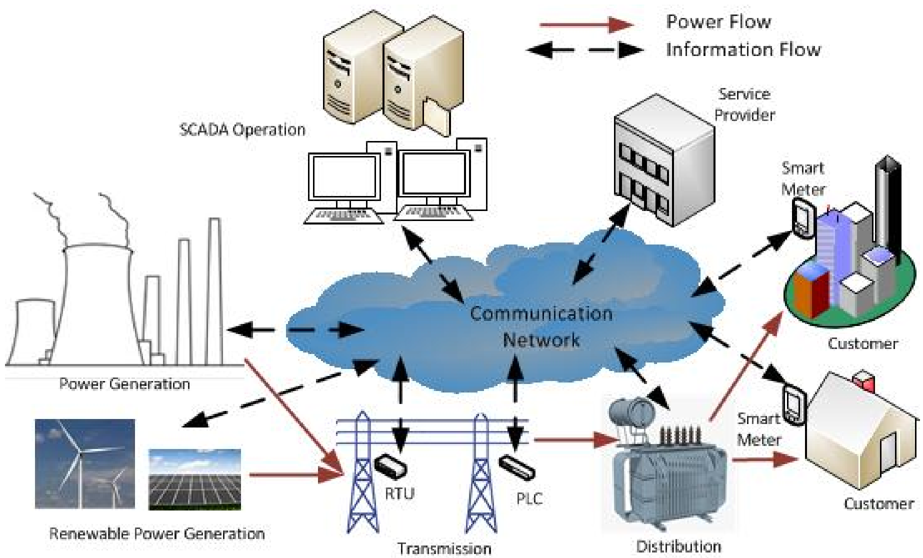
Introduction

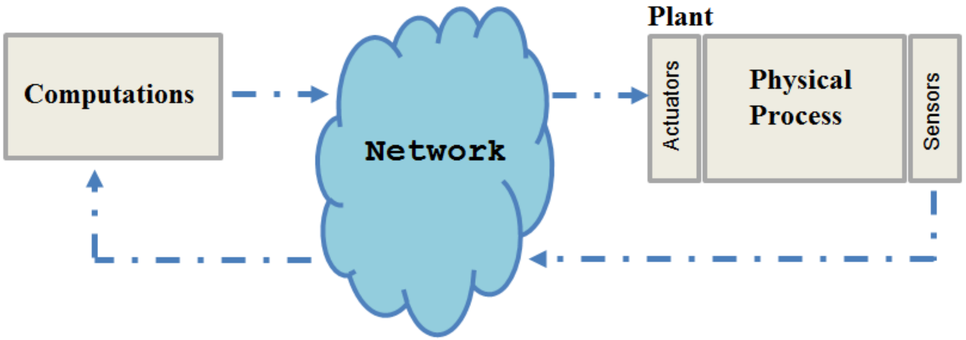
Cyber-physical systems



- Physical processes – **continuous models**
- Cyber components : computers, communication networks – **discrete models**

Introduction

Irregularity of time-sampling



- Network : finite bandwidth,
- Real-time computing : microprocessor latency

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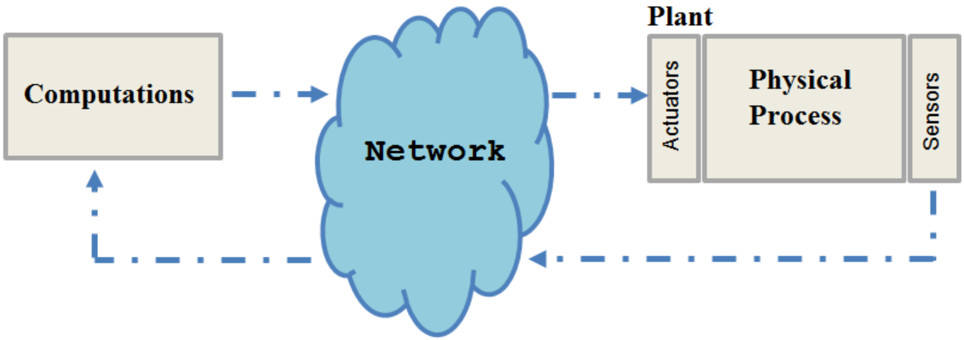

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$\Longrightarrow$ New challenges for Control Theory

$\rightsquigarrow$ *Sampling is not necessarily periodic* Hetel *et al.*, 17

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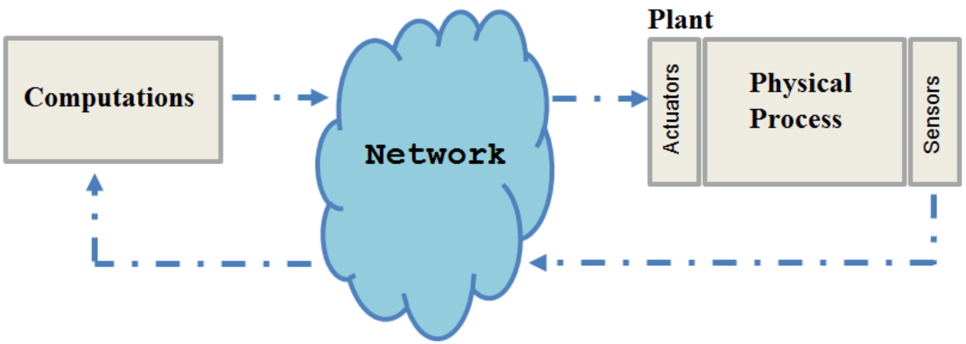

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$\rightsquigarrow$ *Event-based scheduling* Tabuada *et al.*, 07

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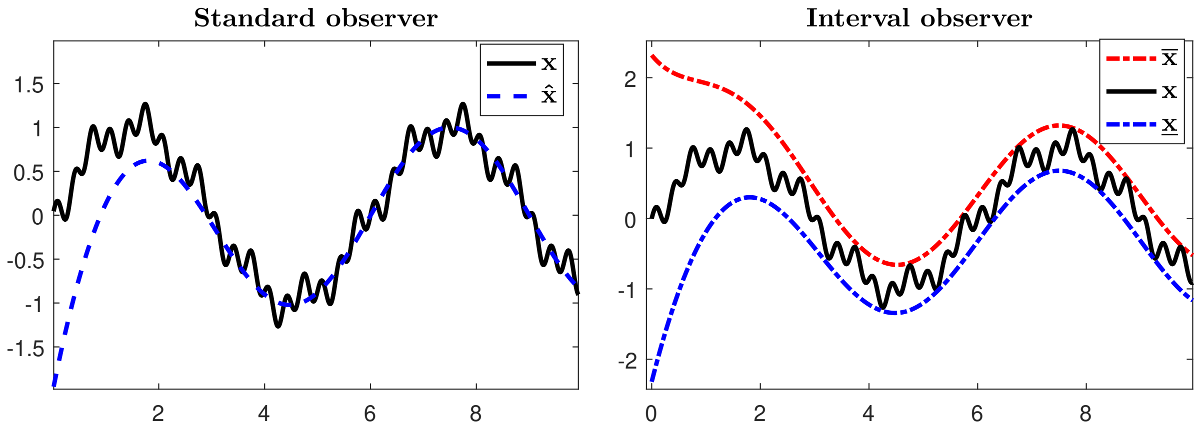
Interval observers

The interval observer provides upper and lower bounds estimates (Gouzé *et al.*,00)



Examples of applications of interval observer

- *Estimation of biological systems*: Gouzé *et al.*,00; Moisan *et al.*,09 ...
- *Fault detection and diagnosis*: Raïssi *et al.*,10; Efimov et al.,11; ...

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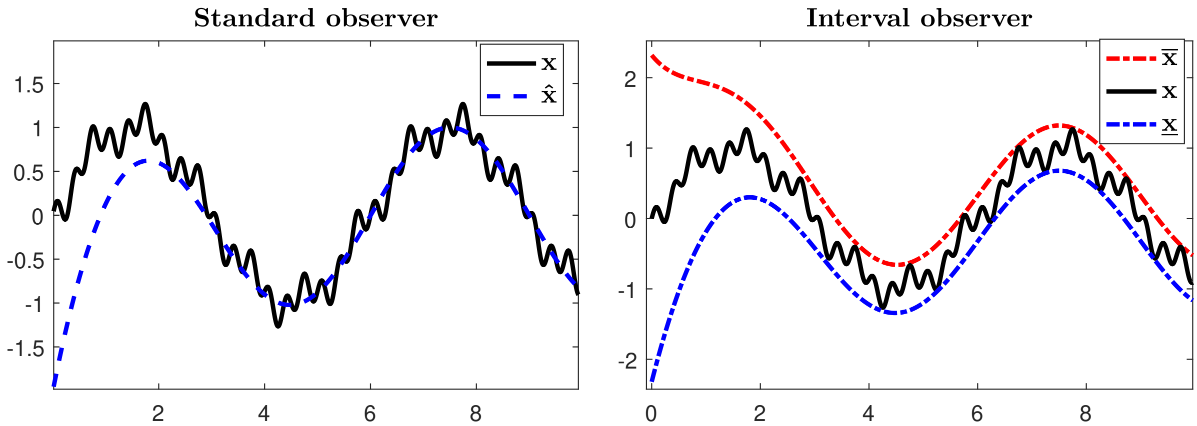

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Interval observer: previous works related to CPS

- *Switched systems*: Rabehi *et al.*,17; Briat *et al.*,17; Ethabet *et al.*,18 ...
- *Impulsive systems*: Dugue *et al.*,16; Briat *et al.*,18 ...
- *Sampled-data system*: Mazenc *et al.*,15; Efimov et al.,16

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Objectives

- *Interval observers with event-triggered sampling for linear continuous-time systems*
- *Co-design of the interval observer gain and the event-triggered conditions for linear systems*

Outline

1. Positive and hybrid systems
2. Finite L_1-gain Event-Triggered Interval observer
3. Co-design of the event-triggered mechanism and the interval observer gain
4. Conclusions

Problem statement

The system

$$\mathcal{S}_2 : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t), \\ y(t_k) = Cx(t_k) + Fd(t_k), \quad k \in \mathbb{N} \end{cases}$$

$d(t) \in \mathbb{R}^n$: the disturbance.

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$d(t) \in \mathbb{R}^n$: the disturbance.

Working assumption

- Assumption 1: Let $\underline{d}, \overline{d} \in \mathbb{R}^n$, be given s.t.

$$\forall t \geq 0 \quad \underline{d}(t) \leq d(t) \leq \overline{d}(t),$$

Problem statement

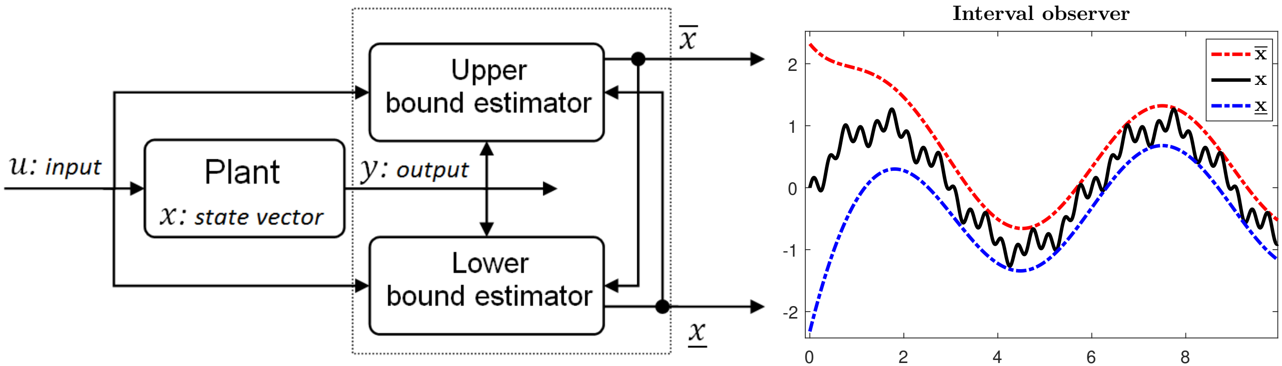

Figure 1: Interval observer principle - scheme

The interval observer should satisfy

- Inclusion: $\underline{x}(t) \leq x(t) \leq \overline{x}(t), \forall t \geq 0$ provided that $\underline{x}(0) \leq x(0) \leq \overline{x}(0) \implies$ **Nonnegativity (Positive dynamics)**
- Convergence of $\underline{e} = x - \underline{x}$ and $\overline{e} = \overline{x} - x \implies$ **Stability (Hybrid systems)**

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Positive (Nonnegative) dynamics

Definition: Monotonicity (Smith, 08)

$\dot{x} = f(x),\ \forall x \in D$ is monotone iif its solution $\phi(x(t_0), t)$ verifies

$$\forall x_1, x_2 \in D : x_1(t_0) \leq x_2(t_0) \implies \phi(x_1(t_0), t) \leq \phi(x_2(t_0), t),\ \forall t \geq t_0,$$

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Definitions (Farina et al ,00)

- Metzler matrix $\mathcal{M} = \{A \in \mathbb{R}^{n\times n} \mid A_{i,j} \geq 0, \ \forall i \neq j\}$
- Nonnegative matrix $\mathcal{N} = \{A \in \mathbb{R}^{n\times n} \mid A_{i,j} \geq 0, \ \forall i, j\}$;

e.g.;

$$\begin{bmatrix} * & + & + \\ + & * & + \\ + & + & * \end{bmatrix} \in \mathcal{M}; \quad \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix} \in \mathcal{N};$$

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Definition: Cooperativity (Hirsch'04)

- A continuous-time linear system $\dot{x}(t) = Ax(t)$ is cooperative if $A \in \mathcal{M}$

Definition: Nonnegativity

- A discrete-time linear system $x(k+1) = Ax(k)$ is nonnegative if $A \in \mathcal{N}$

Positive (Nonnegative) dynamics

Change of coordinates: linear continuous-time system

If $A \notin \mathcal{M} \;\rightarrow\; \exists \mathcal{P}$ for $z = \mathcal{P}x$ s.t. $\mathcal{P}A\mathcal{P}^{-1} \in \mathcal{M}$.
$\rightarrow \dot{z} = \mathcal{P}A\mathcal{P}^{-1}z$ is cooperative

Change of coordinates: linear discrete-time system

If $A \notin \mathcal{N} \;\rightarrow\; \exists \mathcal{T}$ for $z = \mathcal{T}x$ s.t. $\mathcal{T}A\mathcal{T}^{-1} \in \mathcal{N}$.
$\rightarrow z(k+1) = \mathcal{T}A\mathcal{T}^{-1}z(k)$ is nonnegative

Examples:
Diagonalization,
Jordan form,
Time-varying transformation (Mazenc et al., 11),
Time-invariant transformation (Raïssi et al., 12) .

Positive (Nonnegative) dynamics

Internal nonnegativity

- Continuous case:[a]
  If $A \notin \mathcal{M} \rightarrow \exists$ nonnegative realisation $A^M = d_A + (A - d_A)^+$ and $A^N = A^M - A$ with d_A contains only the diagonal elements of A.
- Discrete case:
  If $A \notin \mathcal{N} \rightarrow \exists$ nonnegative realisation A^+ and $A^- = A^+ - A$ s.t. $A^+ - A^- = A$

[a] $A^+ = \max\{A, 0\}$

Example#1: $A = \begin{bmatrix} 1 & -2 \\ 0.8 & -0.2 \end{bmatrix} \rightsquigarrow A^+ = \begin{bmatrix} 1 & 0 \\ 0.8 & 0 \end{bmatrix}; A^- = \begin{bmatrix} 0 & 2 \\ 0 & 0.2 \end{bmatrix}$

Example#2: $A = \begin{bmatrix} -1 & -2 \\ 0.8 & -0.2 \end{bmatrix} \rightsquigarrow A^M = \begin{bmatrix} -1 & 0 \\ 0.8 & -0.2 \end{bmatrix}; A^N = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

Hybrid systems

$$\mathcal{H}: \quad \begin{cases} \dot{x}(t) = f(x(t)), & x \in \mathcal{C}, \\ x(t_k^+) = g(x(t_k)), & x \in \mathcal{D}. \end{cases}$$

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Exp: Bouncing ball

$$f(x) = \begin{bmatrix} x_2 \\ -a \end{bmatrix}, \quad \mathcal{C} = \{x : x_1 \geq 0\},$$

$$g(x) = \begin{bmatrix} 0 \\ -\lambda x_2 \end{bmatrix}, \quad \mathcal{D} = \{x : x_1 = 0,\ x_2 < 0\}.$$

Hybrid systems: finite $\mathcal{L}_p$-gain stability

Impulsive system

$$\mathcal{H}: \quad \begin{cases} \dot{x} = f(x,d), & \forall (x,d) \in \mathcal{C}, \\ x^+ = g(x,d), & \forall (x,d) \in \mathcal{D}, \\ y = h(x,d). \end{cases}$$

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Definition

Given $p \in [1, +\infty)$, system $\mathcal{H}$ is finite-gain $\mathcal{L}_p$ stable from d to y with gain (upper bounded by) $\gamma_p \geq 0$ if $\exists \beta \in \mathbb{R}_{\geq 0}$ s.t., any solution to $\mathcal{H}$ satisfies

$$||y||_p \leq \beta |x(0,0)| + \gamma_p ||d||_p \tag{1}$$

for all $d \in \mathcal{L}_p^{n_d}$.

HS : $\mathcal{L}_p$-gain stability by using Storage function

Definition (Nešić et al., 13)

Given $p \in [1, +\infty)$, a positive semi-definite continuously differentiable $V : \mathbb{R}^{n_V} \to \mathbb{R}_+$ is:

- A finite-gain $\mathcal{L}_p$ storage function for $\mathcal{H}$ if $\exists$ $c_2,\ \gamma_{yf},\ \gamma_{yg} \in \mathbb{R}_{>0}$, and $\gamma_{dg},\ \gamma_{df} \in \mathbb{R}_{\geq 0}$, s.t.,

$$0 \leq V(x) \leq c_2 |x|^p, \qquad \forall (x,d) \in \mathcal{C} \cup \mathcal{D}, \quad (2a)$$

$$\langle \nabla V(x), f(x,d) \rangle \leq -\gamma_{yf} |h(x,d)|^p + \gamma_{df} |d|^p, \qquad \forall (x,d) \in \mathcal{C}, \quad (2b)$$

$$V(g(x,d)) - V(x) \leq -\gamma_{yg} |h(x,d)|^p + \gamma_{dg} |d|^p, \qquad \forall (x,d) \in \mathcal{D}. \quad (2c)$$

- Moreover, $\mathcal{H}$ is finite-gain $\mathcal{L}_p$ stable, and the gain of the operator $d \to y$ is upper bounded by

$$\gamma_p = \sqrt[p]{\max\{\gamma_{df}, \gamma_{dg}\} / \min\{\gamma_{yf}, \gamma_{yg}\}}.$$

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Finite $\mathcal{L}_1$-gain Event-Triggered interval observer

Interval observer structure

$$\mathcal{S}_2 : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t), \\ y(t_k) = Cx(t_k) + Fd(t_k), \quad k \in \mathbb{N} \end{cases}$$

Open-loop estimation:

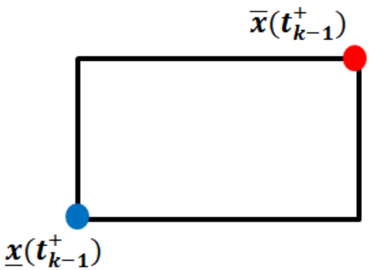
$$\mathcal{IO}_{2.a} : \begin{cases} \dot{\underline{x}}(t) = A^M \underline{x}(t) - A^N \overline{x}(t) + Bu(t) + E^+ \underline{d}(t) - E^- \overline{d}(t) \\ \dot{\overline{x}}(t) = A^M \overline{x}(t) - A^N \underline{x}(t) + Bu(t) + E^+ \overline{d}(t) - E^- \underline{d}(t) \end{cases} \quad \forall t \in [t_{k-1},\ t_k]$$

where

$$x(0) \in [\underline{x}(0), \overline{x}(0)] \tag{3}$$

Finite $\mathcal{L}_1$-gain Event-Triggered interval observer

Interval observer structure



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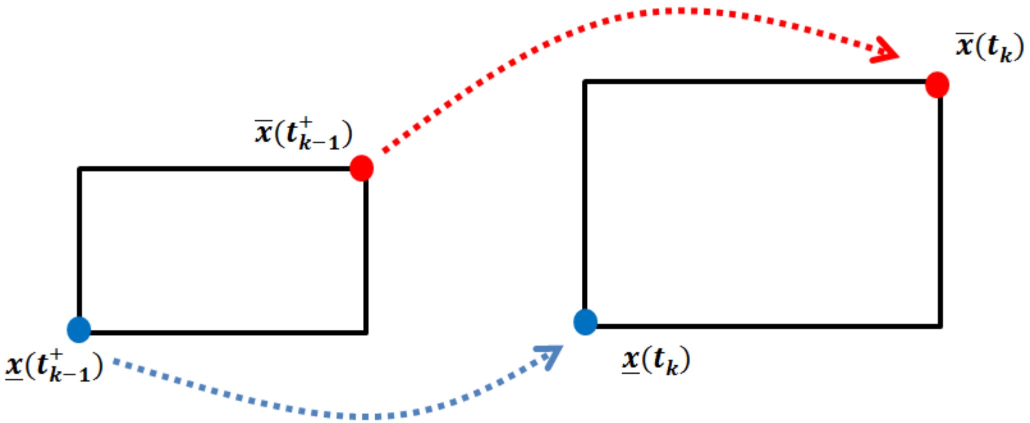

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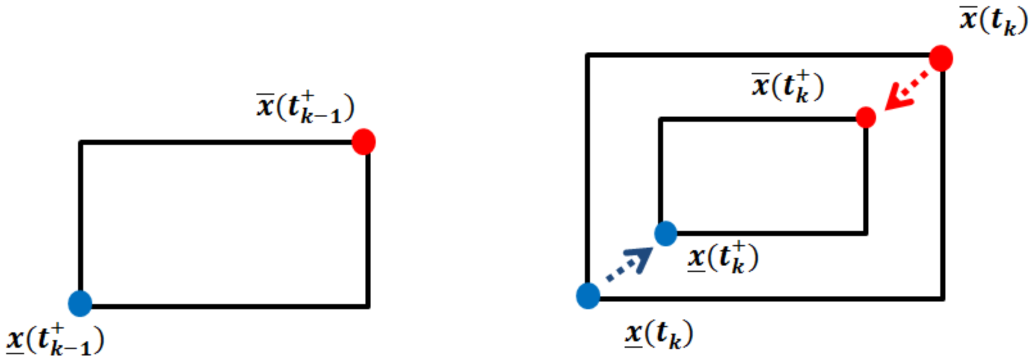

At correction:

$$\mathcal{IO}_{2.b}: \begin{cases} \underline{x}(t_k^+) = (I_n + LC)^+ \underline{x}(t_k) - (I_n + LC)^- \overline{x}(t_k) \\ \qquad + (LF)^+ \underline{d}(t_k) - (LF)^- \overline{d}(t_k) - Ly(t_k) \\ \overline{x}(t_k^+) = (I_n + LC)^+ \overline{x}(t_k) - (I_n + LC)^- \underline{x}(t_k) \\ \qquad + (LF)^+ \overline{d}(t_k) - (LF)^- \underline{d}(t_k) - Ly(t_k) \end{cases} \quad \forall k \in \mathbb{N}$$

Finite $\mathcal{L}_1$-gain Event-Triggered interval observer

Interval observer errors ($\underline{e} = x - \underline{x}, \quad \overline{e} = \overline{x} - x$)

$$\begin{cases} \begin{bmatrix} \dot{\underline{e}}(t) \\ \dot{\overline{e}}(t) \end{bmatrix} = \mathcal{M}(A) \begin{bmatrix} \underline{e}(t) \\ \overline{e}(t) \end{bmatrix} + \tilde{E}\psi(t), & \forall t \in [t_{k-1}, \ t_k], \ \forall k \in \mathbb{N} \\ \begin{bmatrix} \underline{e}(t_k^+) \\ \overline{e}(t_k^+) \end{bmatrix} = \Gamma(L) \begin{bmatrix} \underline{e}(t_k) \\ \overline{e}(t_k) \end{bmatrix} + \tilde{F}(L)\psi(t_k), & \forall k \in \mathbb{N} \end{cases} \tag{4}$$

where

$$\mathcal{M}(A) = \begin{bmatrix} A^M & A^N \\ A^N & A^M \end{bmatrix}; \tilde{E} = \begin{bmatrix} E^+ & E^- \\ E^- & E^+ \end{bmatrix}; \quad \psi(t) = \begin{bmatrix} d(t) - \underline{d}(t) \\ \overline{d}(t) - d(t) \end{bmatrix}$$

$$\Gamma(L) = \begin{bmatrix} (I_n + LC)^+ & (I_n + LC)^- \\ (I_n + LC)^- & (I_n + LC)^+ \end{bmatrix}; \tilde{F}(L) = \begin{bmatrix} (LF)^+ & (LF)^- \\ (LF)^- & (LF)^+ \end{bmatrix}.$$

Finite $\mathcal{L}_1$-gain Event-Triggered interval observer

The augmented error $\xi = (\underline{e}, \overline{e})$.

$$\mathcal{H}_o : \begin{cases} \dot{\xi}(t) = \mathcal{M}(A)\xi(t) + \tilde{E}\psi(t) & \forall \xi \in \mathcal{C}_\xi \\ \xi(t_k^+) = \Gamma(L)\xi(t_k) + \tilde{F}(L)\psi(t_k) & \forall \xi \in \mathcal{D}_\xi \end{cases}$$

With the flow and jump sets

$$\mathcal{C}_\xi = \{(\xi, \psi) \in \mathbb{R}^n \times \mathbb{R}^{n_d} \ : \ |\xi|_1 \leq \ \beta|\psi|_1\}$$
$$\mathcal{D}_\xi = \{(\xi, \psi) \in \mathbb{R}^n \times \mathbb{R}^{n_d} \ : \ |\xi|_1 \geq \ \beta|\psi|_1\}$$

where $\beta \in \mathbb{R}_{>0}$.
$\omega(t) = \overline{x}(t) - \underline{x}(t) = \overline{e}(t) + \underline{e}(t)$: the width of the estimate,
$\delta(t) = \overline{d}(t) - \underline{d}(t)$: the width of the feasible domain of uncertainties.

$$|\omega(t)|_1 = |\xi(t)|_1$$
$$|\delta(t)|_1 = |\psi(t)|_1$$

Equivalence

The $\mathcal{L}_1$-gain of the operator $\psi \rightarrow \xi$, is equivalent to the $\mathcal{L}_1$-gain of operator $\delta \rightarrow \omega$.

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Finite $\mathcal{L}_1$-gain Event-Triggered interval observer

Theorem: Event-Triggered Mechanism (ETM) design

Let Assumptions hold. For a given matrix $L \in \mathbb{R}^{n\times n_y}$,
if $\exists\lambda \in \mathbb{R}^{2n}_{\geq 0}$, and $\zeta_{\mathcal{C}}, \zeta_{\mathcal{D}}, \gamma_{\omega f}, \gamma_{\omega g} \in \mathbb{R}_{>0}$, $\gamma_{\delta f}, \gamma_{\delta g} \in \mathbb{R}_{\geq 0}$ and β satisfying

$$\mathcal{M}^\top(A)\lambda + (\gamma_{\omega f} - \zeta_{\mathcal{C}})\mathbb{1}_{2n} \leq 0 \tag{6a}$$

$$\tilde{E}^\top\lambda - (\gamma_{\delta f} - \zeta_{\mathcal{C}}\beta)\mathbb{1}_{2n_d} \leq 0 \tag{6b}$$

$$\Gamma^\top(L)\lambda - \lambda + (\gamma_{\omega g} + \zeta_{\mathcal{D}})\mathbb{1}_{2n} \leq 0 \tag{6c}$$

$$\tilde{F}^\top(L)\lambda - (\gamma_{\delta g} + \zeta_{\mathcal{D}}\beta)\mathbb{1}_{2n_d} \leq 0 \tag{6d}$$

then,

- the system $\mathcal{IO}_2$ is a finite $\mathcal{L}_1$-gain interval observer for the system $\mathcal{S}_2$.
- the $\mathcal{L}_1$-gain from δ to ω is upper bounded by $\gamma_{\mathcal{L}_1} = \frac{\max\{\gamma_{\delta f}, \gamma_{\delta g}\}}{\min\{\gamma_{\omega f}, \gamma_{\omega g}\}}$

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Proof: Using Linear Copositive Lyapunov function $V(\xi) = \xi^\top \lambda$.

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Let Assumptions hold. For a given matrix $L \in \mathbb{R}^{n\times n_y}$,
if $\exists \lambda \in \mathbb{R}^{2n}_{\geq 0}$, and $\zeta_{\mathcal{C}}, \zeta_{\mathcal{D}}, \gamma_{\omega f}, \gamma_{\omega g} \in \mathbb{R}_{>0}, \gamma_{\delta f}, \gamma_{\delta g} \in \mathbb{R}_{\geq 0}$ and β satisfying

$$\mathcal{M}^\top(A)\lambda + (\gamma_{\omega f} - \zeta_{\mathcal{C}})\mathbb{1}_{2n} \leq 0 \tag{6a}$$

$$\tilde{E}^\top \lambda - (\gamma_{\delta f} - \zeta_{\mathcal{C}}\beta)\mathbb{1}_{2n_d} \leq 0 \tag{6b}$$

$$\Gamma^\top(L)\lambda - \lambda + (\gamma_{\omega g} + \zeta_{\mathcal{D}})\mathbb{1}_{2n} \leq 0 \tag{6c}$$

$$\tilde{F}^\top(L)\lambda - (\gamma_{\delta g} + \zeta_{\mathcal{D}}\beta)\mathbb{1}_{2n_d} \leq 0 \tag{6d}$$

then,

- the system $\mathcal{IO}_2$ is a finite $\mathcal{L}_1$-gain interval observer for the system $\mathcal{S}_2$.
- the $\mathcal{L}_1$-gain from δ to ω is upper bounded by $\gamma_{\mathcal{L}_1} = \frac{\max\{\gamma_{\delta f}, \gamma_{\delta g}\}}{\min\{\gamma_{\omega f}, \gamma_{\omega g}\}}$

Remark

This ETM cannot guarantee the existence of Minimum Inter-Event Time.

Finite $\mathcal{L}_1$-gain Event-Triggered interval observer

The new dynamic ETM that we propose is similar to Girard, 2015

$$\begin{aligned} \mathcal{C}_\eta &= \left\{(\omega, \delta, \eta) \in \mathbb{R}^n \times \mathbb{R}^{n_d} \;:\; |\omega(t)|_1 \leq \beta|\delta(t)|_1 + \frac{\eta(t)}{\theta}\right\} \\ \mathcal{D}_\eta &= \left\{(\omega, \delta, \eta) \in \mathbb{R}^n \times \mathbb{R}^{n_d} \;:\; |\omega(t)|_1 \geq \beta|\delta(t)|_1 + \frac{\eta(t)}{\theta}\right\} \end{aligned} \tag{7}$$

η : the state of the following auxiliary scalar dynamical system

$$\begin{aligned} \dot{\eta}(t) &= -\alpha\eta(t) + \beta|\delta(t)|_1 - |\omega(t)|_1 \\ \eta(0) &\geq |\omega(0)|_1 - \beta|\delta(0)|_1 \end{aligned} \tag{8}$$

The initial condition of the auxiliary system is chosen in a way to initialize the observer in the flow set $\mathcal{C}_\eta$.

Finite $\mathcal{L}_1$-gain Event-Triggered interval observer

Corollary: ETM design guarantees the existence of MIET

Let Assumptions hold. For a given $L \in \mathbb{R}^{n \times n_y}$ and $\theta \in \mathbb{R}_{>0}$,
if $\exists \lambda \in \mathbb{R}^{2n}_{\geq 0}$, $\zeta_{\mathcal{C}}$, $\zeta_{\mathcal{D}}$, $\gamma_{\omega f}, \gamma_{\omega g}, \alpha$, $\beta \in \mathbb{R}_{>0}$, $\gamma_{\delta f}$, $\gamma_{\delta g} \in \mathbb{R}_{\geq 0}$, satisfying

$$\left.\begin{aligned} \mathcal{M}^\top(A)\lambda + (-1 + \gamma_{\omega f} - \zeta_{\mathcal{C}})\mathbb{1}_{2n} &\leq 0 \\ \tilde{E}^\top \lambda + (\beta - \gamma_{\delta f} + \zeta_{\mathcal{C}}\beta)\mathbb{1}_{2n_d} &\leq 0 \\ -\alpha + \zeta_{\mathcal{C}}\tfrac{1}{\theta} &\leq 0 \end{aligned}\right\} \tag{9a}$$

$$\left.\begin{aligned} \Gamma^\top(L)\lambda - \lambda + (\gamma_{\omega g} + \zeta_{\mathcal{D}})\mathbb{1}_{2n} &\leq 0 \\ \tilde{F}^\top(L)\lambda - (\gamma_{\delta g} + \zeta_{\mathcal{D}}\beta)\mathbb{1}_{2n_d} &\leq 0 \end{aligned}\right\} \tag{9b}$$

$$\gamma_{\delta g} - \beta\gamma_{\omega g} \leq 0 \tag{9c}$$

then, the system $\mathcal{IO}_2$ with the ETM (7)-(8) is a finite $\mathcal{L}_1$-gain interval observer for the system $\mathcal{S}_2$ guaranteeing the existence of positive MIET.

Proof: Using the Lyapunov function $W(\xi, \eta) = V(\xi) + \eta = \xi^\top \lambda + \eta$. Then analyzing the variation of the ratio $\kappa(t) = \frac{|\omega(t)|_1}{\beta|\delta(t)|_1 + \frac{\eta(t)}{\theta}}$ in between sampling.

Outline

1. Positive and hybrid systems
2. Finite L_1-gain Event-Triggered Interval observer
3. **Co-design of the event-triggered mechanism and the interval observer gain**
4. Conclusions

ETM and Interval observer gain Co-design

Estimation error dynamics

$$\mathcal{H}_o : \begin{cases} \dot{\xi}(t) = \mathcal{M}(A)\xi(t) + \tilde{E}\psi(t) & \forall \xi \in \mathcal{C}_\xi \\ \xi(t_k^+) = \Gamma(L)\xi(t_k) + \tilde{F}(L)\psi(t_k) & \forall \xi \in \mathcal{D}_\xi \end{cases}$$

$$\Gamma(L) = \begin{bmatrix} (I_n + LC)^+ & (I_n + LC)^- \\ (I_n + LC)^- & (I_n + LC)^+ \end{bmatrix} ; \tilde{F}(L) = \begin{bmatrix} (LF)^+ & (LF)^- \\ (LF)^- & (LF)^+ \end{bmatrix} .$$

ETM and Interval observer gain Co-design

Nonnegative realization based synthesis

Let $G = [I_n + LC]$,
$\forall G_p \geq 0$, $G_n \geq 0$ $\exists \Delta \geq 0$ s.t.

$$\begin{aligned} G &= G_p - G_n \\ &= (G^+ + \Delta) - (G^- + \Delta). \end{aligned}$$

Example:

$$\underbrace{\begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix}}_{G} = \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}}_{G_p} - \underbrace{\begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}}_{G_n}$$

$$= \left(\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{G^+} + \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}}_{\Delta} \right) - \left(\underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}}_{G^-} + \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}}_{\Delta} \right)$$

ETM and Interval observer gain Co-design

Result

$\Gamma(G_p, G_n) = \Gamma(L) + \mathbb{1}_{2\times 2} \otimes \Delta$ where $\Gamma(G_p, G_n) = \begin{bmatrix} G_p & G_n \\ G_n & G_p \end{bmatrix}$

ETM and Interval observer gain Co-design

Result

$\Gamma(G_p, G_n) = \Gamma(L) + \mathbb{1}_{2\times 2} \otimes \Delta$ where $\Gamma(G_p, G_n) = \begin{bmatrix} G_p & G_n \\ G_n & G_p \end{bmatrix}$

$$\mathcal{H}_o : \begin{cases} \dot{\xi}(t) = \mathcal{M}(A)\xi(t) + \tilde{E}\psi(t) & \forall \xi \in \mathcal{C}_\xi \\ \xi(t_k^+) = \Gamma(L)\xi(t_k) + \tilde{F}(L)\psi(t_k) & \forall \xi \in \mathcal{D}_\xi \end{cases}$$

Proposition

$$\chi_D(k+1) = A_D \chi_D(k) \tag{10}$$

$A_D \in \mathcal{N}$ and $\chi_D \in \mathbb{R}^n$.
Assume A_D is Schur stable. If $\exists\ A_d, E \in \mathcal{N}$ s.t. $A_D = A_d + E$, then the system

$$\chi_d(k+1) = A_d \chi_d(k)$$

satisfies $\chi_d(k) \leq \chi_D(k)\ \forall k \in \mathbb{N}$ provided that $0 \leq \chi_d(0) \leq \chi_D(0)$.

ETM and Interval observer gain Co-design

Theorem: Co-design

Let Assumption 1 hold,
if $\exists L \in \mathbb{R}^{n\times n_y}$, $G_p, G_n \in \mathbb{R}^{n\times n}_{\geq 0}$, $R_p, R_n \in \mathbb{R}^{n\times n_d}_{\geq 0}$, $\lambda \in \mathbb{R}^{2n}_{\geq 0}$, $\zeta_{\mathcal{C}}$, $\zeta_{\mathcal{D}}$, $\gamma_{\omega f}$, $\gamma_{\omega g}$, α, $\beta \in \mathbb{R}_{>0}$ $\gamma_{\delta f}$, $\gamma_{\delta g} \in \mathbb{R}_{\geq 0}$, satisfying inequalities (9a), (9c), and the following

$$\begin{bmatrix} \Gamma^\top(G_p, G_n)\lambda - \lambda + (\gamma_{\omega g} + \zeta_{\mathcal{D}})\mathbb{1}_{2n} \\ \tilde{F}^\top(R_p, R_n)\lambda - (\gamma_{\delta g} + \zeta_{\mathcal{D}}\beta)\mathbb{1}_{2n_d} \end{bmatrix} \leq 0\,, \tag{11a}$$

$$G_p - G_n = I_n + LC\,, \tag{11b}$$

$$R_p - R_n = LF\,, \tag{11c}$$

where $\Gamma(G_p, G_n) = \begin{bmatrix} G_p & G_n \\ G_n & G_p \end{bmatrix}$, $\quad \tilde{F}(R_p, R_n) = \begin{bmatrix} R_p & R_n \\ R_n & R_p \end{bmatrix}$
then, the system $\mathcal{IO}_2$ with the ETM (7) is a finite $\mathcal{L}_1$-gain interval observer for the system $\mathcal{S}_2$ guaranteeing the existence of positive MIET.

Double spring-mass-damper system

The LTI system

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1+k2)}{m_1} & -\frac{(c_1+c_2)}{m_1} & \frac{k_2}{m_1} & \frac{c2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{c_2}{m_2} & -\frac{(k_2+k_3)}{m_2} & -\frac{(c_2+c_3)}{m_2} \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix},$$

$$E = \begin{bmatrix} 0.1 & -0.2 \\ -0.7 & 0.6 \\ 0.2 & -0.2 \\ -0.5 & 0.6 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0.6 & -0.8 \\ -0.4 & 0.5 \end{bmatrix}$$

Based on the FMINCON solver. For $\theta = 2$:
The designed parameters of the ETM (7) are $\alpha = 1.3081$ and $\beta = 3.9244$,
The observation gain matrix is

$$L = \begin{bmatrix} 0.4535 & 0.0558 \\ 0 & 0 \\ 0.0528 & 0.5166 \\ 0 & 0 \end{bmatrix}.$$

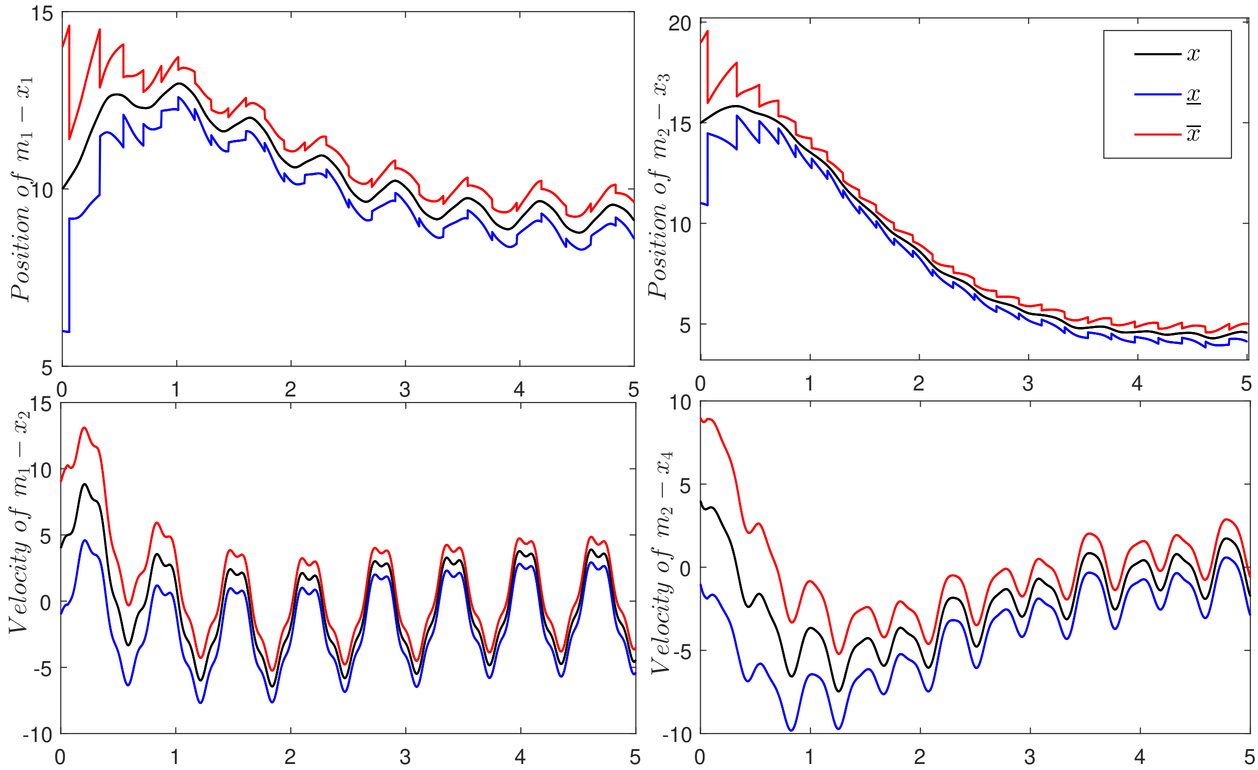

Figure 2: Simulation results: the upper and the lower estimate bounds for the masses position (x_1, x_3), and masses velocity (x_2, x_4).

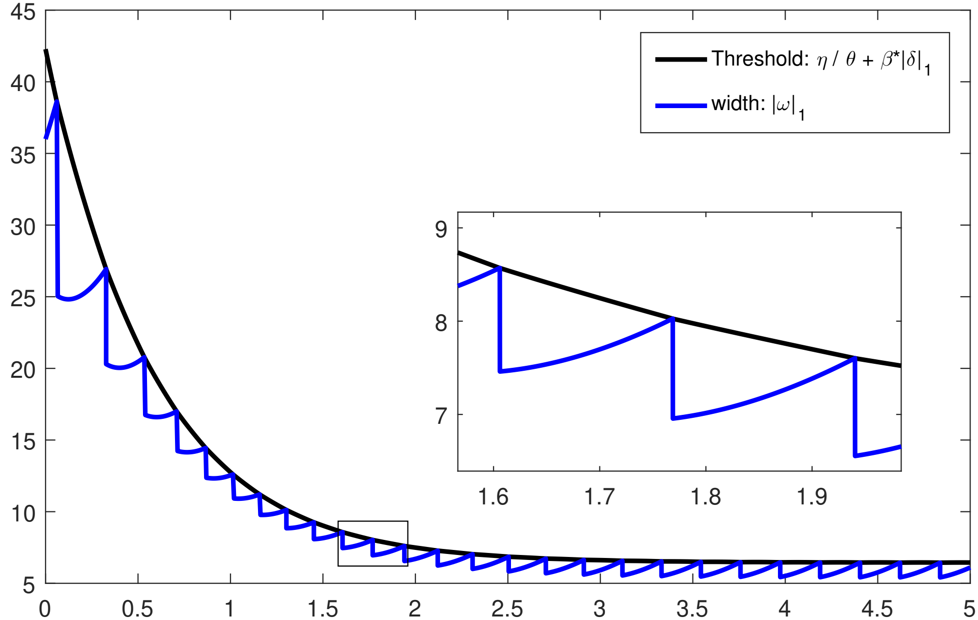

Figure 3: The evolution of the triggering mechanism : (blue) the width of the state estimate, (black) the threshold for the width.

From the simulation: $\forall k \quad t_{k+1} - t_k \geq \tau_{min} = 0.0609$.

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The following contributions have been proposed

- Design of event-triggered observer that can reduce the occupation of network in CPS estimation.
- New co-design methodology for the interval observer gain and the dynamic event-triggered mechanism for LTI systems.

Perspectives

- Extend this approaches to nonlinear system, e.g., systems that can be over-approximated by linear system
- Design of an event-triggered output feedback stabilizing law using interval observer approaches.

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Thank you !