Observer design for a class of uncertain nonlinear systems with sampled outputs

Application to the estimation of reaction heat in chemical reactors

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SHGO design

Consider the class of nonlinear systems that are diffeomorphic to the following triangular form:

$$\begin{cases} \dot{x}(t) = Ax(t) + \varphi(u(t), x(t)) + B\varepsilon(t) \\ y(t_k) = Cx(t_k) + \omega(t_k) = x_1(t_k) + \omega(t_k) \end{cases} \tag{1}$$

with $x = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^T \in \mathbb{R}^n$, $\varphi(u,x) = \begin{pmatrix} \varphi_1(u, x_1) \\ \varphi_2(u, x_1, x_2) \\ \vdots \\ \varphi_{n-1}(u, x_1, \dots, x_{n-1}) \\ \varphi_n(u, x) \end{pmatrix}$

$A = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^n$ for $n \geq 2$, $B = \begin{pmatrix} 0 & 0 & \dots & 1 \end{pmatrix}^T$, $C = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$ where x is the state vector of the system, u is the input signal, and $y(t_k)$ is the system output which is available only at the sampling instants t_k. $\varepsilon(t)$ and $\omega(t)$ are system uncertainties and output noise. $T_s = t_{k+1} - t_k$ is defined as the sampling partition

SHGO design

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Assume that:

- The state x is bounded;
- The functions φ_i for $i \in [1, n]$ are Lipschitz with respect to x uniformly in u, i.e. $\|\varphi_i(u, x) - \varphi_i(u, \bar{x})\| \leq L\|x - \bar{x}\|$;
- The unknown function ε is essentially bounded, i.e. $\exists \delta_\varepsilon, \underset{t \geq 0}{supEss}\|\varepsilon(t)\| \leq \delta_\varepsilon$;
- The noise signal ω is essentially bounded, i.e. $\exists \delta_\omega, \underset{t \geq 0}{supEss}\|\omega(t)\| \leq \delta_\omega$

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Objective

Design a continuous-discrete time observer providing a continuous time estimation of the full state of system (1) by using the measurements that are available only at the sampling instants.

SHGO design

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Based on the continuous-time high gain observer in (Farza, M'Saad, *et al.* 2004), the discrete-continuous time observer is designed as (Bouraoui *et al.* 2015):

Standard discrete-continuous observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta\Delta_\theta^{-1} K e^{-\theta k_1 (t - t_k)} (C\hat{x}(t_k) - y(t_k)), \quad t \in [t_k, t_{k+1}] \quad (2)$$

where $\hat{x} = \begin{pmatrix} \hat{x}_1 & \dots & \hat{x}_n \end{pmatrix}^T$ is the estimated state vector, $K = \begin{pmatrix} k_1 & \dots & k_n \end{pmatrix}^T$ is the gain matrix where $k_i, i = 1, \dots, n$ are chosen such that the matrix $\bar{A} = A - KC$ is Hurwitz, and $\Delta_\theta = diag \begin{pmatrix} 1 & \frac{1}{\theta} & \dots & \frac{1}{\theta^{n-1}} \end{pmatrix}$ is the diagonal matrix with $\theta \geq 1$.

SHGO design

Since the matrix $\bar{A} = A - KC$ is Hurwitz, there exist a $n \times n$ symmetric positive definite (SPD) matrix P and a positive real μ such that $P\bar{A} + \bar{A}^T P \leq -2\mu I_n$. Then, for the standard discrete-continuous observer, we have:

Theorem

$\forall \rho > 0, \exists \theta_0 > 0, \forall \theta > \theta_0, \forall u \; s.t. \|u\|_\infty \leq \rho$, there exist positive constants $\chi_\theta > 0$ and $\eta_\theta(T_s) > 0$ such that if the sampling interval T_s is chosen such that $T_s < \chi_\theta$, then for every $\hat{x}(0) \in \mathbb{R}$, we have:

$$\begin{aligned}\|\hat{x}(t) - x(t)\| \leq & \sigma\theta^{q-1} e^{-\eta_\theta t} \|\hat{x}(0) - x(0)\| \\ & + N_\theta \left(\frac{\delta_\varepsilon}{\theta} + \theta^{n-1} \|K\| (2 - e^{-k_1 \theta T_s}) \delta_\omega \right)\end{aligned}$$

where$N_\theta = \sigma\theta T_s \frac{2 - e^{-\eta_\theta(T_s)T_s}}{1 - e^{-\eta_\theta(T_s)T_s}}$, $\theta_0 = max(1, \frac{2L\sqrt{n}\lambda_M}{\mu})$, $\sigma = \sqrt{\frac{\lambda_M}{\lambda_m}}$ is the conditioning number of the matrix P, λ_M and λ_m are the maximum and minimum eigenvalue of the matrix P.
$\eta_\theta = a_\theta(1 - \frac{T_s}{\chi_\theta})e^{-a_\theta T_s}$, $\chi_\theta = \frac{\mu}{2(L+\theta)\sigma\|K\|\lambda_M}$, $a_\theta = \frac{\mu\theta}{2\lambda_M}$.

Proof of Theorem

Lemma

Consider a differentialable function $v(t) \in \mathbb{R}^+$ satisfying the following inequality:

$$\dot{v}(t) \leq -av(t) + b\int_{t_k}^{t} v(s)ds + p(t), \quad t \in [t_k, t_{k+1}]$$

with $p(t)$ is an essentially bounded function with $c = \underset{t \geq 0}{supEss}\ p(t)$, a and b are positive reals satisfying $\frac{bT_s}{a} < 1$.
Then, the function $v(t)$ satisfies:

$$v(t) \leq e^{-\eta(t-t_0)}v(t_0) + cT_s\frac{2 - e^{-\eta T_s}}{1 - e^{-\eta T_s}}$$

with $\eta = (a + bT_s)e^{-aT_s} > 0$.

The proof of this Lemma can be found in (Bouraoui *et al.* 2015).

Proof of Theorem

Let $\widetilde{x}(t) = \hat{x}(t) - x(t)$ be the observation error, it can be easily checked that:

$$\dot{\widetilde{x}} = A\widetilde{x} + \phi(u, \hat{x}, x) - \theta\Delta_\theta^{-1} K e^{-\theta k_1 (t-t_k)} C\widetilde{x}(t_k) - B\varepsilon(t) + \theta\Delta_\theta^{-1} K e^{-\theta k_1 (t-t_k)} \omega(t_k)$$

where $\phi(u, \hat{x}, x) = \varphi(u, \hat{x}) - \varphi(u, x)$.

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where $\phi(u, \hat{x}, x) = \varphi(u, \hat{x}) - \varphi(u, x)$.

- Setting $\bar{x} = \Delta_\theta \widetilde{x}$, and taking into account the following identities:

  $$\Delta_\theta A \Delta_\theta^{-1} = \theta A, \; C\Delta_\theta^{-1} = C$$

  one gets:

  $$\dot{\bar{x}} = \theta A\bar{x} + \Delta_\theta\phi(u, \hat{x}, x) - \theta Ke^{-\theta k_1(t-t_k)}C\bar{x}(t_k) - \frac{1}{\theta^{n-1}}B\varepsilon(t) + \theta Ke^{-\theta k_1(t-t_k)}\omega(t_k) \quad (1)$$

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one gets:

$$\dot{\bar{x}} = \theta A\bar{x} + \Delta_\theta \phi(u, \hat{x}, x) - \theta K e^{-\theta k_1(t-t_k)} C\bar{x}(t_k) - \frac{1}{\theta^{n-1}} B\varepsilon(t) + \theta K e^{-\theta k_1(t-t_k)} \omega(t_k) \quad (1)$$

- Adding and subtracting the term $\theta KC\bar{x}$ yields:

$$\dot{\bar{x}} = \theta \bar{A}\bar{x} + \theta kz - \frac{1}{\theta^{n-1}} B\varepsilon(t) + \theta K e^{-\theta k_1(t-t_k)} \omega(t_k) \quad (2)$$

where $\bar{A} = A - KC$, $z = C\bar{x} - e^{-\theta k_1(t-t_k)} C\bar{x}(t_k) = \bar{x}_1 - e^{-\theta k_1(t-t_k)} \bar{x}_1(t_k)$

Proof of Theorem

Considering the following Lyapunov function: $V(\bar{x}) = \bar{x}^T P \bar{x}$ where P is the SPD matrix defined before, one can show that:

$$\begin{aligned}\dot{V}(\bar{x}) =& \theta \bar{x}^T (\bar{A}^T P + P\bar{A})\bar{x} + 2\bar{x}^T P(\Delta_\theta \phi(u, \hat{x}, x) + \theta K z) \\ & + 2\bar{x}^T P \frac{1}{\theta^{n-1}} B\varepsilon(t) + 2\bar{x}^T P\theta K e^{-\theta k_1 (t-t_k)} \omega(t_k)\end{aligned} \quad (3)$$

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- According to the Lipschitz assumption and the triangular structure of φ, for $\theta > 1$, one has (Farza, M'Saad, *et al.* 2004):

$$2\bar{x}^T P \Delta_\theta \phi(u, \hat{x}, x) \leq 2\sqrt{n}\lambda_M L \|\bar{x}\|^2$$

$$2\bar{x}^T P \frac{1}{\theta^{n-1}} B\varepsilon(t) \leq 2\frac{\|\varepsilon(t)\|\sqrt{\lambda_M}}{\theta^{n-1}} \sqrt{V(\bar{x})}$$

$$2\bar{x}^T P\theta K e^{-\theta k_1 (t-t_k)} \omega(t_k) \leq 2\theta \|\omega(t_k)\| \sqrt{\lambda_M} \|K\| \sqrt{V(\bar{x})}$$

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- One gets:

$$\begin{aligned}\dot{V}(\bar{x}) \leq & -2(\mu\theta - \sqrt{n}\lambda_M L)\|\bar{x}\|^2 + 2\theta\sqrt{\lambda_M}\|K\|\|z\|\sqrt{V(\bar{x})} \\ &+ 2\sqrt{\lambda_M}\left(\frac{\|\varepsilon(t)\|}{\theta^{n-1}} + \theta\|K\|\|\omega(t_k)\|\right)\sqrt{V(\bar{x})}\end{aligned} \tag{4}$$

Proof of Theorem

Considering the following Lyapunov function: $V(\bar{x}) = \bar{x}^T P \bar{x}$ where P is the SPD matrix defined before, one can show that:

$$\begin{aligned} \dot{V}(\bar{x}) =& \theta \bar{x}^T (\bar{A}^T P + P\bar{A})\bar{x} + 2\bar{x}^T P(\Delta_\theta \phi(u, \hat{x}, x) + \theta K z) \\ &+ 2\bar{x}^T P \frac{1}{\theta^{n-1}} B\varepsilon(t) + 2\bar{x}^T P\theta K e^{-\theta k_1 (t-t_k)} \omega(t_k) \end{aligned} \tag{3}$$

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- Choosing θ such that $2(\mu\theta - \sqrt{n}\lambda_M L) > \mu\theta$, i.e. $\theta > \theta_0 = 2\sqrt{n}\lambda_M L/\mu$, the last inequality becomes:

$$\begin{aligned} \dot{V}(\bar{x}) \leq & -\frac{\mu\theta}{\lambda_M} V(\bar{x}) + 2\theta\sqrt{\lambda_M}\|K\|\|z\|\sqrt{V(\bar{x})} \\ &+ 2\sqrt{\lambda_M}\left(\frac{\|\varepsilon(t)\|}{\theta^{n-1}} + \theta\|K\|\|\omega(t_k)\|\right)\sqrt{V(\bar{x})} \end{aligned} \tag{5}$$

Proof of Theorem

Calculation of $\|z\|$

As mentioned before, $z = C\bar{x} - e^{-\theta k_1(t-t_k)}C\bar{x}(t_k) = \bar{x}_1 - e^{-\theta k_1(t-t_k)}\bar{x}_1(t_k)$, the time derivative of z can be written as follows:

$$\dot{z} = \dot{\bar{x}}_1 + k_1\theta e^{-\theta k_1(t-t_k)}\bar{x}_1(t_k) = \theta\bar{x}_2 + \phi_1(u, \hat{x}_1, x_1) + k_1\theta e^{-\theta k_1(t-t_k)}\omega(t_k) \quad (6)$$

where $\phi_1(u, \hat{x}_1, x_1) = \varphi_1(u, \hat{x}_1) - \varphi_1(u, x_1)$.

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where $\phi_1(u, \hat{x}_1, x_1) = \varphi_1(u, \hat{x}_1) - \varphi_1(u, x_1)$.

- Integrating (6) from t_k to t while using the fact that $z(t_k) = 0$ yields:

$$z = \int_{t_k}^{t} (\theta x_2(s) + \phi_1(u(s), \hat{x}_1(s), x_1(s)))\, ds + (1 - e^{-k_1\theta(t-t_k)})\omega(t_k) \qquad (7)$$

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- Integrating (6) from t_k to t while using the fact that $z(t_k) = 0$ yields:

$$z = \int_{t_k}^{t} \left(\theta x_2(s) + \phi_1(u(s),\hat{x}_1(s),x_1(s))\right) ds + (1 - e^{-k_1\theta(t-t_k)})\omega(t_k) \quad (7)$$

- Bearing in mind that $\phi(u,\hat{x}_1,x_1) = \varphi_1(u,\hat{x}_1) - \varphi_1(u,x_1)$, one gets:

$$\begin{aligned} \|z\| &\leq (\theta + L)\int_{t_k}^{t} \|\bar{x}(s)\| ds + (1 - e^{-k_1\theta(t-t_k)})\delta_\omega \\ &\leq ((\theta + L)/\sqrt{\lambda_m})\int_{t_k}^{t} \|\sqrt{V(\bar{x}(s))}\| ds + (1 - e^{-k_1\theta T_s})\delta_\omega \end{aligned} \quad (8)$$

Proof of Theorem

Combing the above inequalities, one obtains:

$$\dot{V}(\bar{x}) \leq -\frac{\mu\theta}{\lambda_M}V(\bar{x}) + 2\theta\sqrt{\frac{\lambda_M}{\lambda_m}}\|K\|(\theta+L)\sqrt{V(\bar{x})}\int_{t_k}^{t}\|\sqrt{V(\bar{x}(s))}\|ds + 2\sqrt{\lambda_M}\left(\frac{\|\varepsilon\|}{\theta^{n-1}} + \theta\|K\|(2-e^{-k_1\theta T_s})\delta_\omega\right)\sqrt{V(\bar{x})} \quad (9)$$

Equivalently, one has:

$$\frac{d}{dt}\sqrt{V(\bar{x})} \leq -\frac{\mu\theta}{2\lambda_M}\sqrt{V(\bar{x})} + \theta\sigma\|K\|(\theta+L)\int_{t_k}^{t}\|\sqrt{V(\bar{x}(s))}\|ds + \sqrt{\lambda_M}\left(\frac{\|\varepsilon\|}{\theta^{n-1}} + \theta\|K\|(2-e^{-k_1\theta T_s})\delta_\omega\right) \quad (10)$$

where $\sigma = \sqrt{\lambda_M/\lambda_m}$ is the conditioning number of P.

Finally, let $a_\theta = \frac{\mu\theta}{2\lambda_M}$, $b_\theta = \theta\sigma\|K\|(\theta + L)$, $p_\theta = \sqrt{\lambda_M}\left(\frac{\|\varepsilon\|}{\theta^{n-1}} + \theta\|K\|(2 - e^{-k_1\theta T_s})\delta_\omega\right)$, and assume that the upper diameter of the sampling partition T_s satisfies the following condition: $T_s \leq \frac{a_\theta}{b_\theta} = \frac{\mu}{2(\theta+L)\sigma\|K\|\lambda_M} = \chi_\theta$.
Then, according to Lemma, one has:

$$\sqrt{V(\bar{x})} \leq e^{-\eta_\theta t}\sqrt{V(\bar{x}(0))} + c_\theta T_s \frac{2 - e^{-\eta_\theta T_s}}{1 - e^{-\eta_\theta T_s}} \qquad (11)$$

where $c_\theta = \underset{t\geq 0}{supEss}\ p_\theta = \sqrt{\lambda_M}\left(\frac{\delta_\varepsilon}{\theta^{n-1}} + \theta\|K\|(2 - e^{-k_1\theta T_s})\delta_\omega\right)$, and $\eta_\theta = (a_\theta - b_\theta T_s)e^{-a_\theta T_s}$.

Proof of Theorem

Finally, let $a_\theta = \frac{\mu\theta}{2\lambda_M}$, $b_\theta = \theta\sigma\|K\|(\theta+L)$, $p_\theta = \sqrt{\lambda_M}\left(\frac{\|\varepsilon\|}{\theta^{n-1}} + \theta\|K\|(2-e^{-k_1\theta T_s})\delta_\omega\right)$, and assume that the upper diameter of the sampling partition T_s satisfies the following condition: $T_s \leq \frac{a_\theta}{b_\theta} = \frac{\mu}{2(\theta+L)\sigma\|K\|\lambda_M} = \chi_\theta$.
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$\eta_\theta = (a_\theta - b_\theta T_s)e^{-a_\theta T_s}$.

- Thus, one obtains:

$$\|\bar{x}\| \leq \sigma e^{-\eta_\theta t}\|\bar{x}(0)\| + \frac{c_\theta T_s}{\sqrt{\lambda_m}} \frac{2-e^{-\eta_\theta T_s}}{1-e^{-\eta_\theta T_s}} \tag{12}$$

Proof of Theorem

- Thus, one obtains:

$$\|\bar{x}\| \leq \sigma e^{-\eta_\theta t}\|\bar{x}(0)\| + \frac{c_\theta T_s}{\sqrt{\lambda_m}} \frac{2 - e^{-\eta_\theta T_s}}{1 - e^{-\eta_\theta T_s}} \tag{11}$$

- Recall that $\bar{x} = \Delta_\theta \widetilde{x}$ with $\Delta_\theta = diag\begin{pmatrix} 1 & \frac{1}{\theta} & \dots & \frac{1}{\theta^{n-1}} \end{pmatrix}$.
- Then, coming back to the original coordinates of the observation error $\widetilde{x}$, for $\theta > 1$, one has:

$$\frac{1}{\theta^{n-1}}\|\widetilde{x}\| \leq \|\Delta_\theta \widetilde{x}\| = \|\bar{x}\| \leq \sigma e^{-\eta_\theta t}\|\Delta_\theta\|\|\widetilde{x}(0)\| + \frac{c_\theta T_s}{\sqrt{\lambda_m}} \frac{2 - e^{-\eta_\theta T_s}}{1 - e^{-\eta_\theta T_s}} \tag{12}$$

Proof of Theorem

- Recall that $\bar{x} = \Delta_\theta \widetilde{x}$ with $\Delta_\theta = diag\begin{pmatrix} 1 & \frac{1}{\theta} & \dots & \frac{1}{\theta^{n-1}} \end{pmatrix}$.
- Then, coming back to the original coordinates of the observation error $\widetilde{x}$, for $\theta > 1$, one has:

$$\frac{1}{\theta^{n-1}}\|\widetilde{x}\| \leq \|\Delta_\theta \widetilde{x}\| = \|\bar{x}\| \leq \sigma e^{-\eta_\theta t}\|\Delta_\theta\|\|\widetilde{x}(0)\| + \frac{c_\theta T_s}{\sqrt{\lambda_m}} \frac{2 - e^{-\eta_\theta T_s}}{1 - e^{-\eta_\theta T_s}} \tag{11}$$

- Substituting in c_θ defined before, one gets:

$$\begin{aligned} \|\widetilde{x}\| \leq & \sigma\theta^{q-1} e^{-\eta_\theta t}\|\widetilde{x}(0)\| \\ & + N_\theta \left(\frac{\delta_\varepsilon}{\theta} + \theta^{n-1}\|K\|(2 - e^{-k_1 \theta T_s})\delta_\omega \right) \end{aligned}$$

where $N_\theta = \sigma\theta T_s \frac{2 - e^{-\eta_\theta(T_s)T_s}}{1 - e^{-\eta_\theta(T_s)T_s}}$. Theorem has been proved.

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Improved NPHGO design

In order to eliminate the peaking phenomenon, an improved NP observer has been proposed for the continuous-time system in (Farza, Ragoubi, *et al.* 2021). Therefore, for the nonlinear system with sampled measurements (1), an improved NP discrete-continuous observer can be written as follows:

Improved NP discrete-continuous observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta^{1+\eta}H(t), \quad t \in [t_k, t_{k+1}] \tag{12}$$

Improved NP discrete-continuous observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta^{1+\eta} H(t), \quad t \in [t_k, t_{k+1}] \tag{12}$$

where $H(t) = \begin{pmatrix} H_1(t) & \dots & H_n(t) \end{pmatrix}^T$ is defined as follows:

$$\begin{cases} H_1(t) = sat_{\nu,1}\left(\theta^{-\eta} k_1 e^{-\theta k_1 (t-t_k)} (C\hat{x}(t_k) - y(t_k))\right) \\ H_i(t) = sat_{\nu,i}\left(\theta \dfrac{k_i}{k_{i-1}} H_{i-1}(t_k)\right), \quad i = 2, \dots, n \end{cases} \tag{13}$$

with $k_i \in K = \begin{pmatrix} k_1 & \dots & k_n \end{pmatrix}^T$ are the gain parameters which are chosen such that the matrix $\bar{A} = A - KC$ is Hurwitz. η is a small positive constant. $sat_\nu(\cdot)$ is a saturation function defined as:

$$\forall z \in \mathbb{R}, \quad sat_\nu(z) = \begin{cases} z, & if |z| \le \nu \\ \nu \cdot sign(z), & if |z| > \nu \end{cases} \tag{14}$$

where $\nu > 0$ is a positive constant; and $sign(\cdot)$ is the usual signum function.

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Mathematical model of the reactor

A main reaction (perhydrolysis of formic acid to peroxyformic acid by hydrogen peroxide) with three decomposition reactions take place in the considered batch reactor (Vernières-Hassimi *et al.* 2017; Leveneur *et al.* 2012; Zheng *et al.* 2016).

Mathematical model

$$\begin{cases} \dot{C}_{HCOOH} = -r_{perh} + r_{decomp,2} \\ \dot{C}_{H_2O_2} = -r_{perh} - r_{decomp,3} \\ \dot{C}_{HCOOOH} = r_{perh} - r_{decomp,1} - r_{decomp,2} \\ \dot{C}_{H_2O} = r_{perh} + r_{decomp,1} + r_{decomp,3} \\ \dot{T}_R = \dfrac{q_R}{\rho V C_P} + \dfrac{UA}{\rho V C_P}(T_J - T_R) - \dfrac{q_{loss}}{\rho V C_P} \end{cases} \tag{15}$$

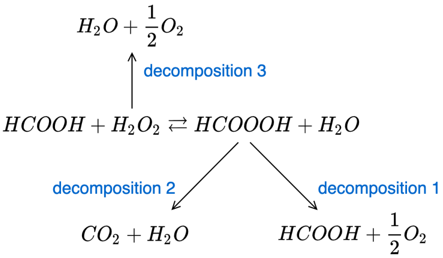


Figure: Chemical reactions occur in the batch reactor

Mathematical model of the reactor

A main reaction (perhydrolysis of formic acid to peroxyformic acid by hydrogen peroxide) with three decomposition reactions take place in the considered batch reactor (Vernières-Hassimi *et al.* 2017; Leveneur *et al.* 2012; Zheng *et al.* 2016).

Mathematical model

$$\begin{cases} \dot{C}_{HCOOH} = -r_{perh} + r_{decomp,2} \\ \dot{C}_{H_2O_2} = -r_{perh} - r_{decomp,3} \\ \dot{C}_{HCOOOH} = r_{perh} - r_{decomp,1} - r_{decomp,2} \\ \dot{C}_{H_2O} = r_{perh} + r_{decomp,1} + r_{decomp,3} \\ \dot{T}_R = \frac{q_R}{\rho V C_P} + \frac{UA}{\rho V C_P}(T_J - T_R) - \frac{q_{loss}}{\rho V C_P} \end{cases} \quad (15)$$

where

- T: temperature (K)
- C: concentration $(\mathrm{mol \cdot L^{-1}})$
- r: reaction rate $(\mathrm{mol \cdot L^{-1} {\cdot} s^{-1}})$
- $q_R = -\sum_i r_i \Delta H_{R,i} V$ is the reaction heat, with ΔH_R is the reaction enthalpy
- $q_{loss} = 0$ is the heat loss
- subscript R, J are the reaction, the jacket of the reactor, respectively
- Others are constants

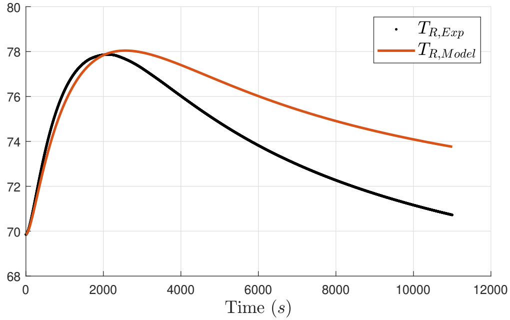


Figure: Temperature comparison between the experimental data and the simulated data

Initial conditions:

- $C_{HCOOH} = 2.5 \text{ mol} \cdot \text{L}^{-1}$
- $C_{H_2O_2} = 2.8 \text{ mol} \cdot \text{L}^{-1}$
- $C_{HCOOOH} = 0 \text{ mol} \cdot \text{L}^{-1}$
- $C_{H_2O} = 50.27 \text{ mol} \cdot \text{L}^{-1}$
- $T_R = 69.8511\ ^\circ\text{C}$

Temperature comparison

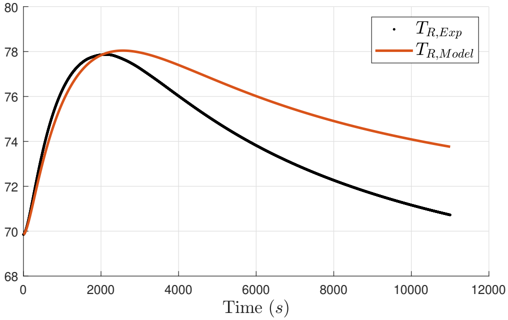


Figure: Temperature comparison between the experimental data and the simulated data

Initial conditions:

- $C_{HCOOH} = 2.5 \text{ mol} \cdot \text{L}^{-1}$
- $C_{H_2O_2} = 2.8 \text{ mol} \cdot \text{L}^{-1}$
- $C_{HCOOOH} = 0 \text{ mol} \cdot \text{L}^{-1}$
- $C_{H_2O} = 50.27 \text{ mol} \cdot \text{L}^{-1}$
- $T_R = 69.8511\ ^\circ\text{C}$

We focus on $t \in [0, 3600]$

Reaction heat estimation by sampled measurements

- Rewrite the reactor model into the observer-construction form
  Define $x_1(t) = T_R(t)$, $x_2(t) = \frac{q_R(t)}{\rho V C_P}$, one gets:

$$\begin{cases} \dot{x}_1(t) = \dot{T}_R(t) = x_2(t) + \varphi_1(t) \\ \dot{x}_2(t) = \dfrac{\dot{q}_R(t)}{\rho V C_P} = \varepsilon(t) \end{cases} \tag{16}$$

  with $\varphi_1(t) = \frac{UA}{\rho V C_P}(T_J - T_R(t)) - \frac{q_{loss}}{\rho V C_P}$
- SHGO construction

Standard discrete-continuous observer

$$\begin{cases} \dot{\hat{T}}_R(t) = \dfrac{\hat{q}_R(t)}{\rho V C_P} + \dfrac{UA}{\rho V C_P}(T_J - \hat{T}_R(t)) - \dfrac{q_{loss}}{\rho V C_P} - \theta k_1 e^{-\theta k_1 (t-t_k)}(\hat{T}_R(t_k) - T_R(t_k)) \\ \dot{\hat{q}}_R(t) = -\rho V C_P \theta^2 e^{-\theta k_1 (t-t_k)}(\hat{T}_R(t_k) - T_R(t_k)) \end{cases} \tag{17}$$

Reaction heat estimation by sampled measurements

Improved NP discrete-continuous observer

$$\begin{cases} \dot{\hat{T}}_R(t) = \frac{\hat{q}_R(t)}{\rho V C_P} + \frac{UA}{\rho V C_P}(T_J - sat_{\nu,T_R}(\hat{T}_R(t))) - \frac{q_{loss}}{\rho V C_P} - \theta^{1+\eta} H_1(t) \\ \dot{\hat{q}}_R(t) = -\theta^{1+\eta} H_2(t) \end{cases} \quad (18)$$

where

$$\begin{cases} H_1(t) = sat_{\nu,1}\left(\theta^{-\eta} k_1 e^{-\theta k_1 (t-t_k)} (\hat{T}_R(t_k) - T_R(t_k))\right) \\ H_2(t) = sat_{\nu,2}\left(\rho V C_P \theta \frac{k_2}{k_1} H_1(t)\right) \end{cases} \quad (19)$$

Initial conditions

In the following simulations:

- $\hat{C}_{HCOOH} = C_{HCOOH}$
- $\hat{C}_{H_2O_2} = C_{H_2O_2}$
- $\hat{C}_{HCOOOH} = C_{HCOOOH}$
- $\hat{C}_{H_2O} = C_{H_2O}$
- $\hat{T}_R = 65\ ^{\circ}\mathrm{C}$
- $\hat{q}_R = 0$

- $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$
- $\theta = 1$
- $\nu, T_R = 85\ ^{\circ}\mathrm{C}$
- $\nu, 1 = 10$
- $\nu, 2 = 1$
- $\eta = 0.01$
- $t_0 = 0$

Model based estimation. Case 1: $T_s = 2s$

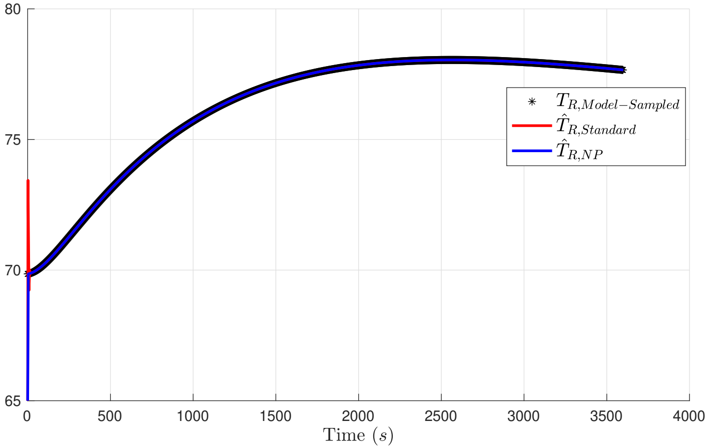


Figure: Temperature T_R $(^\circ\mathrm{C})$ and its estimation $\hat{T}_R$ based on the reactor model

Model based estimation. Case 1: $T_s = 2s$

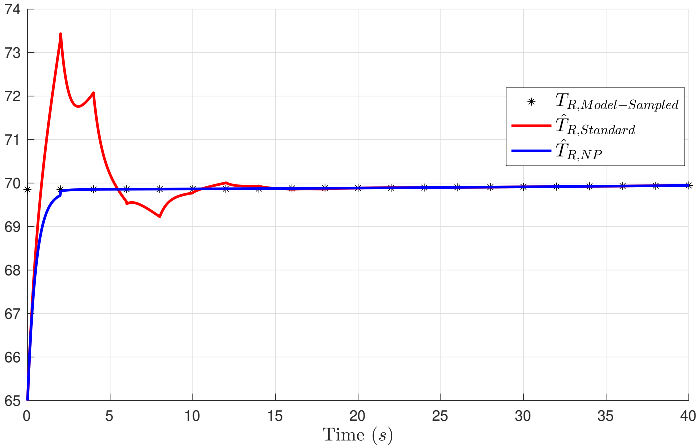


Figure: Zoom in

Reaction heat estimation by sampled measurements

Model based estimation. Case 1: $T_s = 2s$

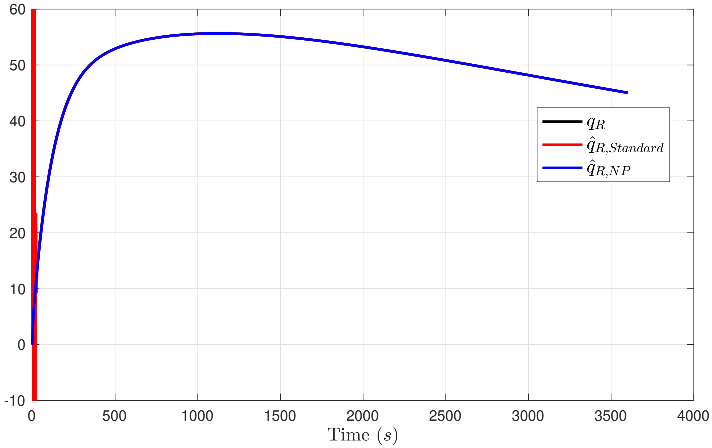


Figure: Reaction heat q_R $(\mathrm{J \cdot s^{-1}})$ and its estimation $\hat{q}_R$ based on the reactor model

Model based estimation. Case 1: $T_s = 2s$

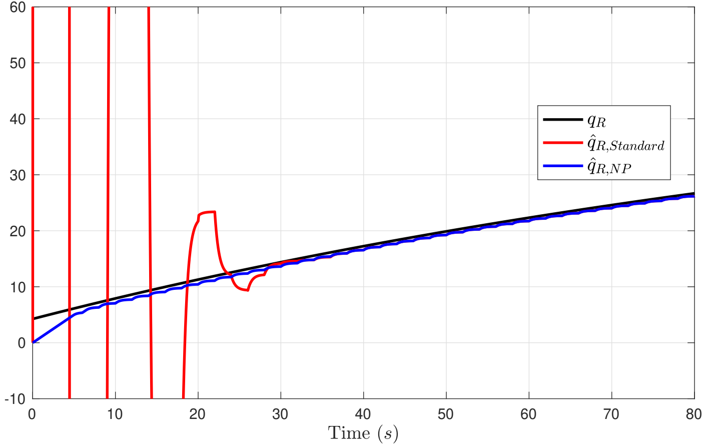


Figure: Zoom in

Reaction heat estimation by sampled measurements

Model based estimation. Case 2: $T_s = 4s$

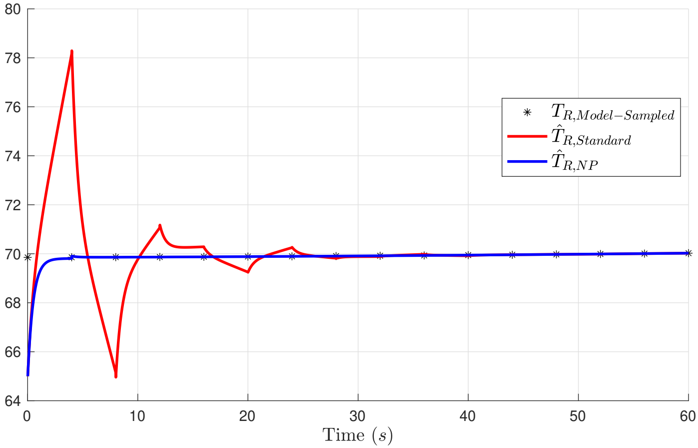


Figure: Temperature T_R $(^\circ\mathrm{C})$ and its estimation $\hat{T}_R$ based on the reactor model

Model based estimation. Case 2: $T_s = 4s$

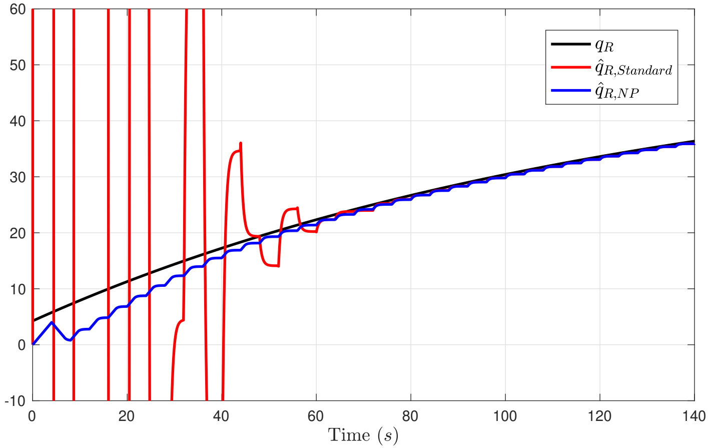


Figure: Reaction heat q_R $(\mathrm{J} \cdot \mathrm{s}^{-1})$ and its estimation $\hat{q}_R$ based on the reactor model

Experimental data based estimation. Case 1: $T_s = 2s$

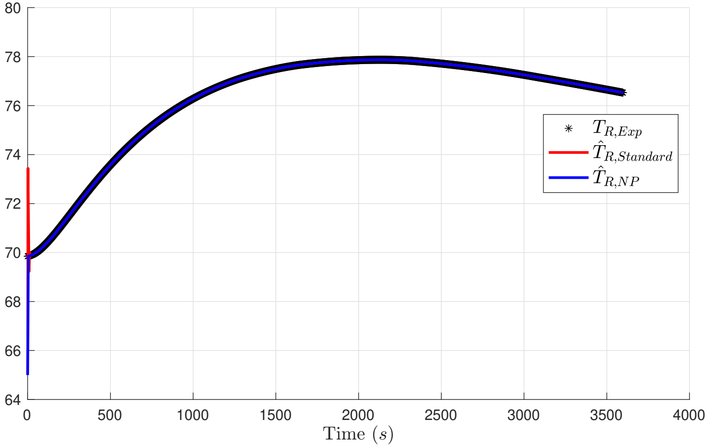


Figure: Temperature T_R $(^{\circ}\mathrm{C})$ and its estimation $\hat{T}_R$ based on experimental data

Experimental data based estimation. Case 1: $T_s = 2s$

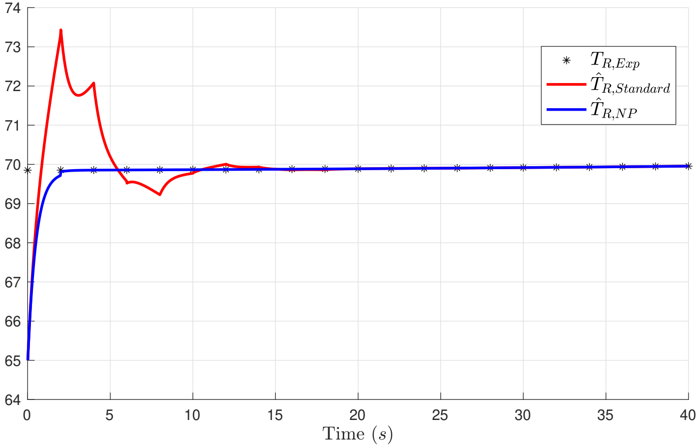


Figure: Zoom in

Experimental data based estimation. Case 1: $T_s = 2s$

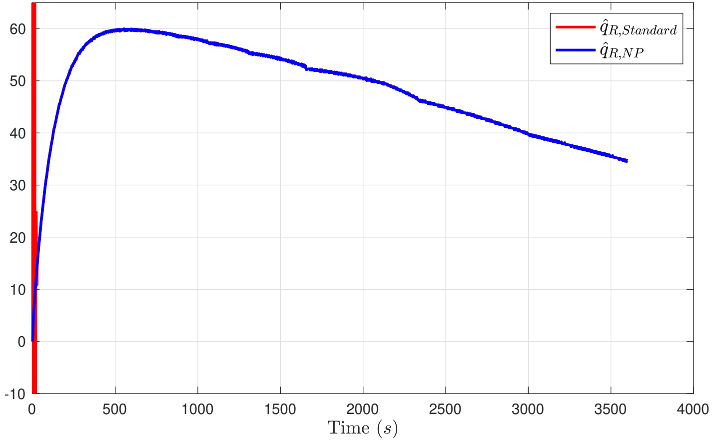


Figure: Reaction heat q_R $(\mathrm{J} \cdot \mathrm{s}^{-1})$ and its estimation $\hat{q}_R$ based on experimental data

Experimental data based estimation. Case 1: $T_s = 2s$

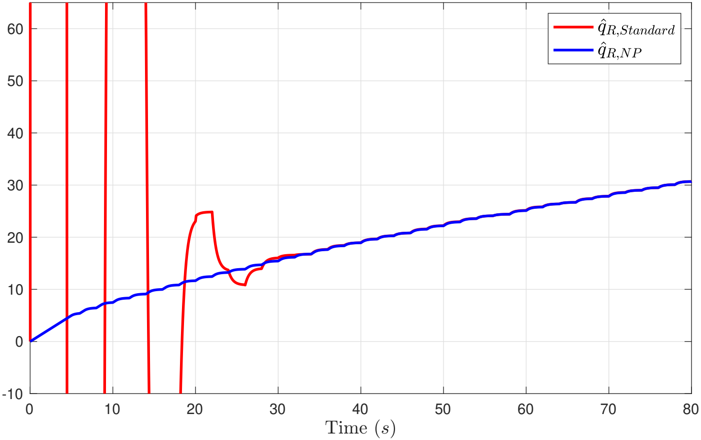
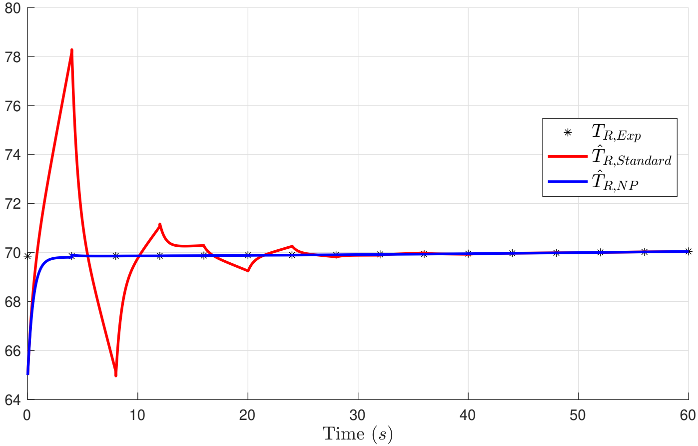


Figure: Zoom in

Experimental data based estimation. Case 2: $T_s = 4s$



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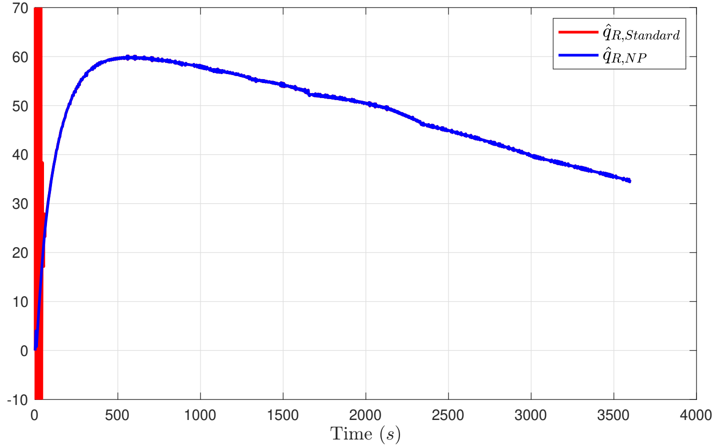

Figure: Reaction heat q_R $(\mathrm{J \cdot s^{-1}})$ and its estimation $\hat{q}_R$ based on experimental data

Experimental data based estimation. Case 2: $T_s = 4s$

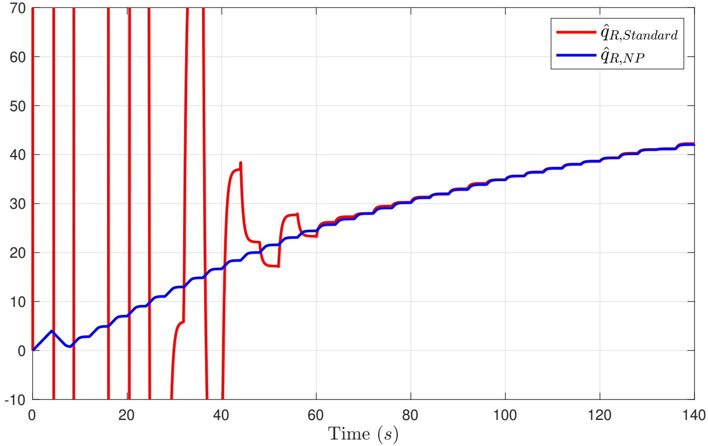

Figure: Zoom in

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Conclusion

- A standard high gain observer (SHGO) has been presented for the system with sampled measurements.
- An improved non-peaking observer (NPHGO) has been proposed for the mentioned system, and the peaking phenomenon has been eliminated compared to the SHGO.
- The reaction heat of the batch reactor has been well estimated by the proposed observes.
- Both of the observers have been proved to be efficient with an enlarged sampling interval.

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Thank You!