Design of Feedback Control for Discrete-Time Systems Based on Iterative LMIs Subject to Stochastic Noise

Presented by:

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1. Problem statement
   - Control system subject to stochastic noise
   - Discrete-time observer-based state feedback

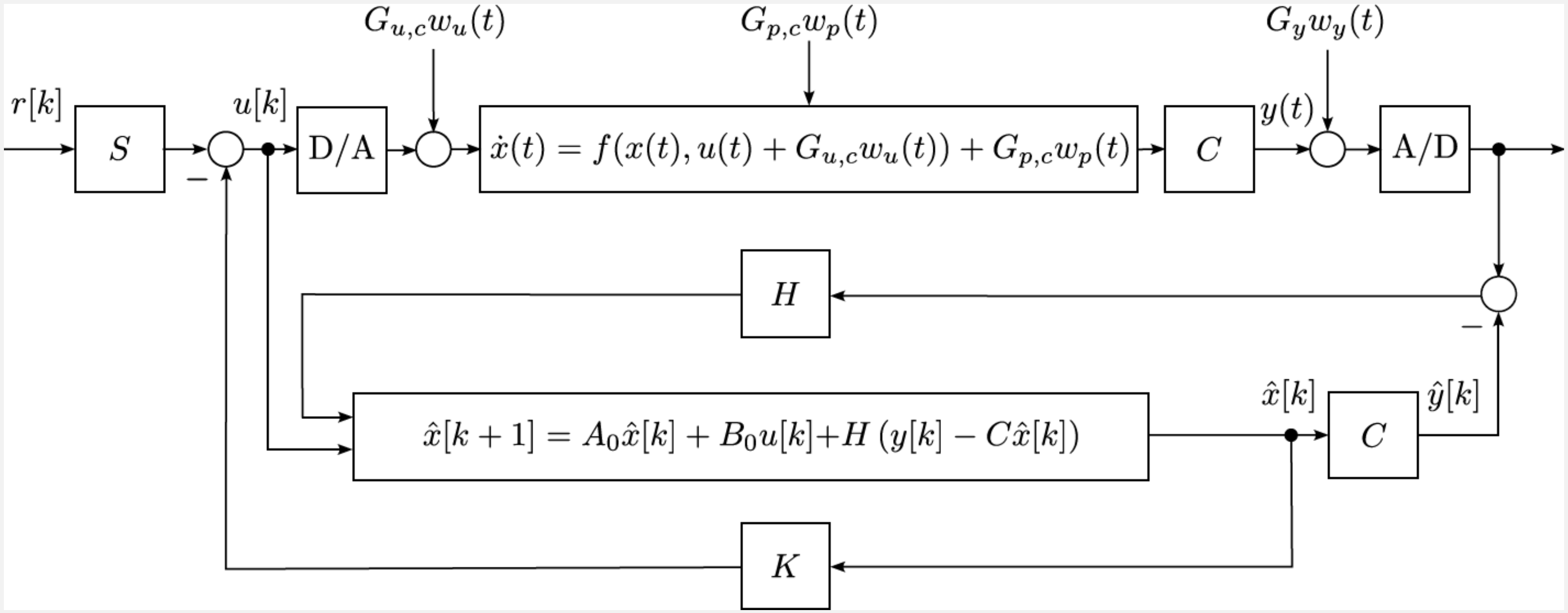
2. Fundamentals for the controller design
   - Robust Lyapunov stability and D_{R} regions
   - Generalization of the Lyapunov stability condition to stochastic noise

3. Developed LMI based algorithm
   - Superposed iteration rule
   - Optimization task

4. Example: Control of overhead traveling crane

5. Summary and Outlook

Control system subject to stochastic noise

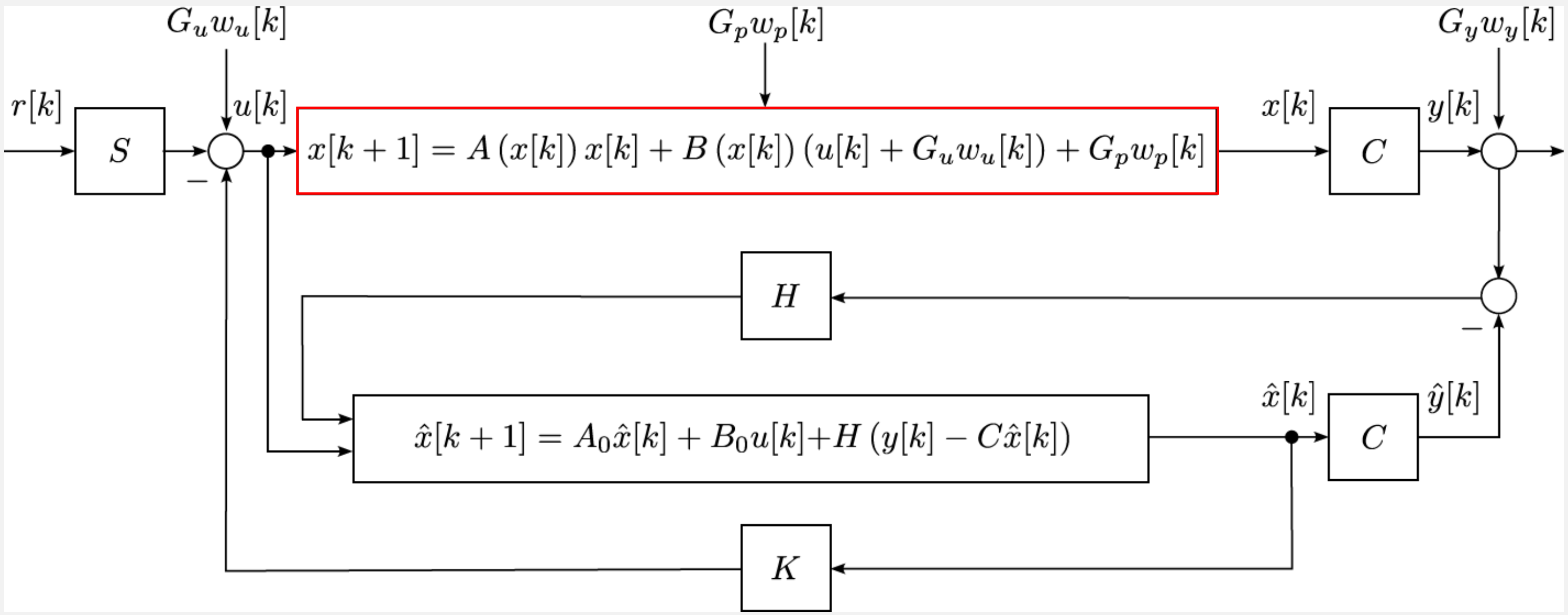


w_u, w_p, w_y: stochastically independent standard normally distributed actuator noise, process noise and sensor noise

$G_{u,c}, G_{p,c}, G_y$: disturbance input matrices contain standard deviations

Objective: Design observer-based state feedback controller

Design observer based state feedback controller



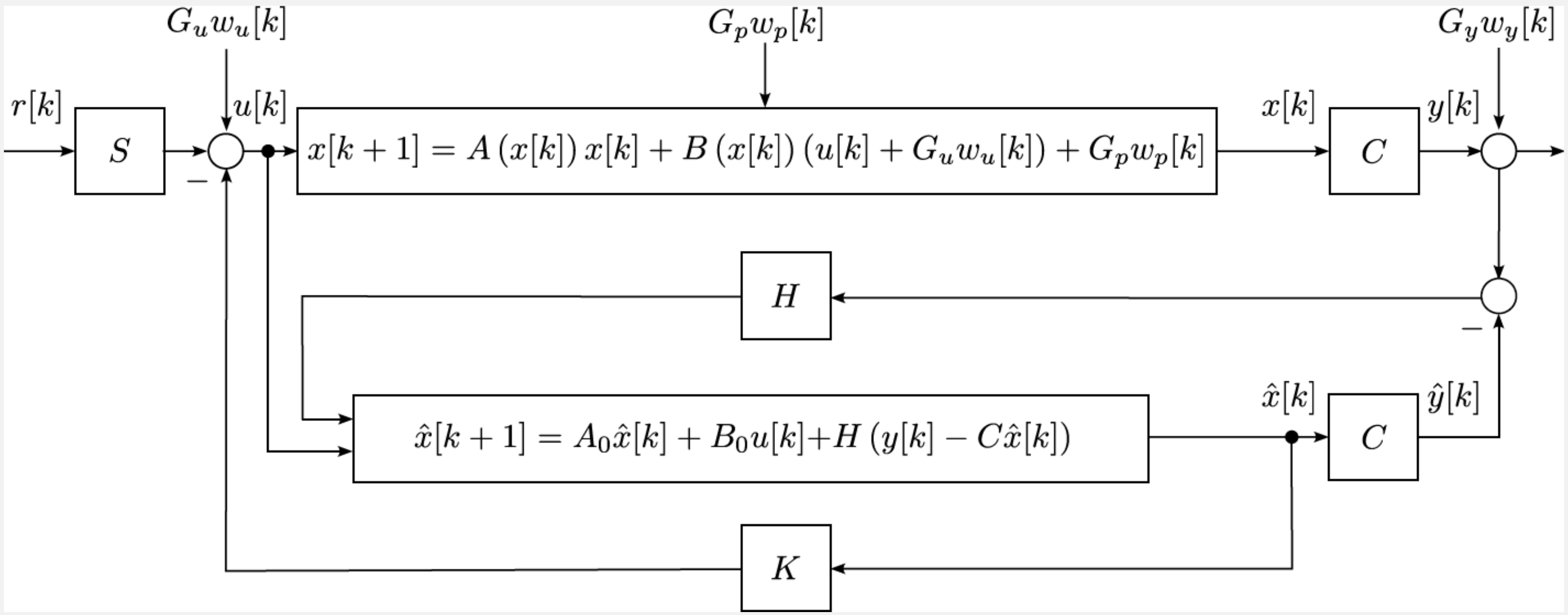
1. Convert the nonlinear system to a quasilinear form, with

$$\mathrm{x}[k] = (x_1[k],\ \dots,\ x_n[k])^T, \text{ with } x_i[k] = [\underline{x}_i,\ \overline{x}_i],\ i = 1, \dots, n$$

2. Discretization by first order explicit Euler approximation:

$$A(\mathrm{x}[k]) = A_c(\mathrm{x}[k])T_s + I, \qquad B(\mathrm{x}[k]) = B_c(\mathrm{x}[k])T_s,\ \dots$$

Design observer based state feedback controller

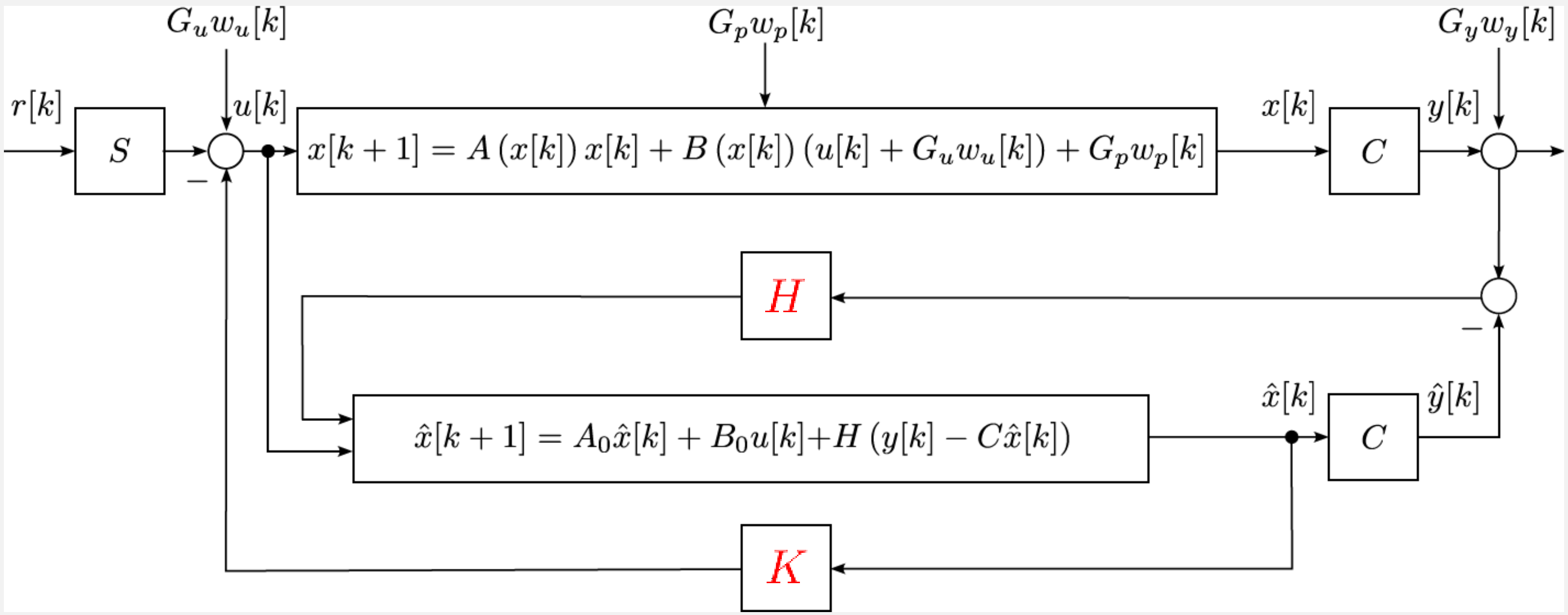


Augmented state space representation of the closed-loop ($r[k] = 0$):

$$z[k+1] = \underbrace{\begin{bmatrix} A(x[k]) - B(x[k])K & B(x[k])K \\ A(x[k]) - A_0 - (B(x[k]) - B_0)K & A_0 - HC - \big(B_0 - B(x[k])\big)K \end{bmatrix}}_{\mathcal{A}(x[k])} z[k] + \underbrace{\begin{bmatrix} B(x[k])G_u & G_p & 0 \\ B(x[k])G_u & G_p & -HG_y \end{bmatrix}}_{\mathcal{G}(x[k])} w[k]$$

with $z[k] = (x[k] \;\; e[k])^{\mathrm{T}}$ where $e[k] = x[k] - \hat{x}[k]$ and $w[k] = \big(w_u[k] \;\; w_p[k] \;\; w_y[k]\big)^{\mathrm{T}}$

Design observer based state feedback controller



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Objective: Determine observer gain H and controller gain K simultaneously

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Quasilinear system:

$$z[k+1] = \mathcal{A}(x[k])z[k] + \mathcal{G}(x[k])w[k] \quad \text{with} \quad x[k] = (x_1[k], \; \dots, \; x_n[k])^T,$$

$$y[k] = \mathcal{C}z[k] \qquad\qquad x_i[k] = \left[\underline{x}_i, \; \overline{x}_i\right], i = 1, \dots, n$$

Idea: Polytopic representation of $\mathcal{A}(x[k])$ and $\mathcal{G}(x[k])$:

$$[\mathcal{A}(x[k]), \mathcal{G}(x[k])] \in \left\{ [\mathcal{A}(\xi), \mathcal{G}(\xi)] = \sum_{v=1}^{n_v} \xi_v [\mathcal{A}_v, \mathcal{G}_v] \,\middle|\, \sum_{v=1}^{n_v} \xi_v = 1, \xi_v \geq 0 \right\}$$

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Example:

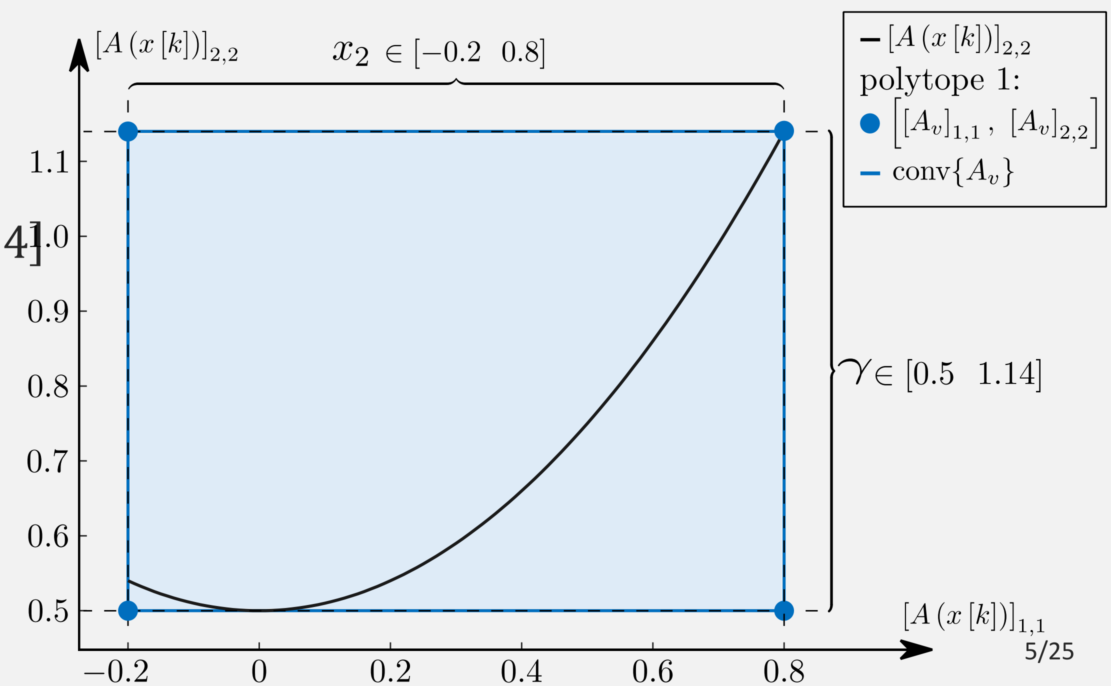
$$A(x[k]) = \begin{bmatrix} x_2 & 1 \\ 0 & 0.5 + x_2^2 \end{bmatrix} = \begin{bmatrix} x_2 & 1 \\ 0 & \gamma \end{bmatrix}$$

with $x_2 \in [-0.2 \ \ 0.8]$ and $\gamma \in [0.5 \ \ 1.14]$

$A(x[k]) \in$

$$\xi_1 \begin{bmatrix} -0.2 & 1 \\ 0 & 0.5 \end{bmatrix} + \xi_2 \begin{bmatrix} 0.8 & 1 \\ 0 & 0.5 \end{bmatrix} +$$

$$\xi_3 \begin{bmatrix} -0.2 & 1 \\ 0 & 1.14 \end{bmatrix} + \xi_4 \begin{bmatrix} 0.8 & 1 \\ 0 & 1.14 \end{bmatrix}$$



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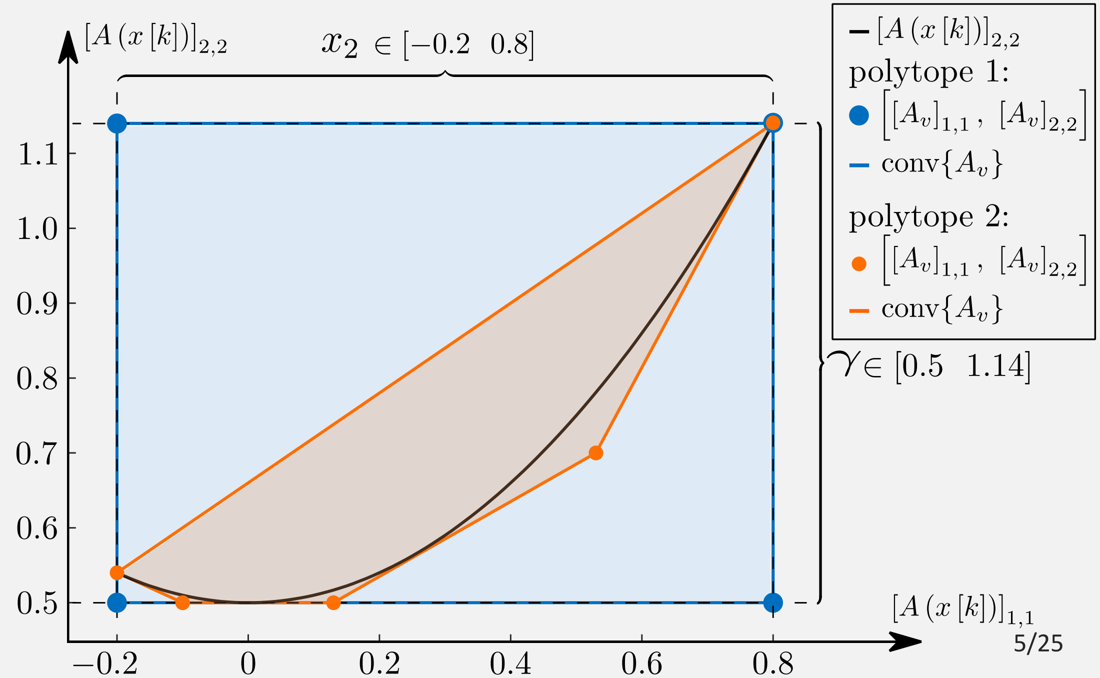
Example:

$$A(x[k]) = \begin{bmatrix} x_2 & 1 \\ 0 & 0.5 + x_2^2 \end{bmatrix}$$

$$A(x[k]) \in \xi_1 \begin{bmatrix} -0.2 & 1 \\ 0 & 0.54 \end{bmatrix} + \xi_2 \begin{bmatrix} -0.1 & 1 \\ 0 & 0.5 \end{bmatrix} + \xi_3 \begin{bmatrix} 0.13 & 1 \\ 0 & 0.5 \end{bmatrix} + \xi_4 \begin{bmatrix} 0.53 & 1 \\ 0 & 0.7 \end{bmatrix} + \xi_5 \begin{bmatrix} 0.8 & 1 \\ 0 & 1.14 \end{bmatrix}$$



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Robust Lyapunov stability and D_{R} regions (without noise $w[k] = 0$; deterministic system)

Quadratic Lyapunov function candidate

$$V(z\,[k]) = \frac{1}{2} z^{\mathrm{T}}[k]\, P\, z[k] \quad \text{with} \quad P = P^{T} > 0$$

Is a free LMI-decision variable

If the Lyapunov condition $\mathcal{A}(x[k])^{\mathrm{T}}\, P\, \mathcal{A}(x[k]) - P < 0$

with

$$\mathcal{A}(x[k]) = \begin{bmatrix} A(x[k]) - B(x[k])K & B(x[k])K \\ A(x[k]) - A_0 - (B(x[k]) - B_0)K & A_0 - HC - \left(B_0 - B(x[k])\right)K \end{bmatrix}$$

is fulfilled:

→ augmented closed-loop system quadratically stable for all $x_i[k] = [\underline{x}_i,\ \overline{x}_i]$

Polytopic representation:

→ This is true, if $\mathcal{A}_v{}^{\mathrm{T}} P \mathcal{A}_v - P < 0$ with $v = 1, \dots, n_v$ are satisfied.

Robust Lyapunov stability and D_R regions (without noise $w[k]=0$; deterministic system)

Extension to robust D_R regions:

All eigenvalues of all extremal realizations

$\mathcal{A}_v$, $v=1,\ldots,n_v$ are located within

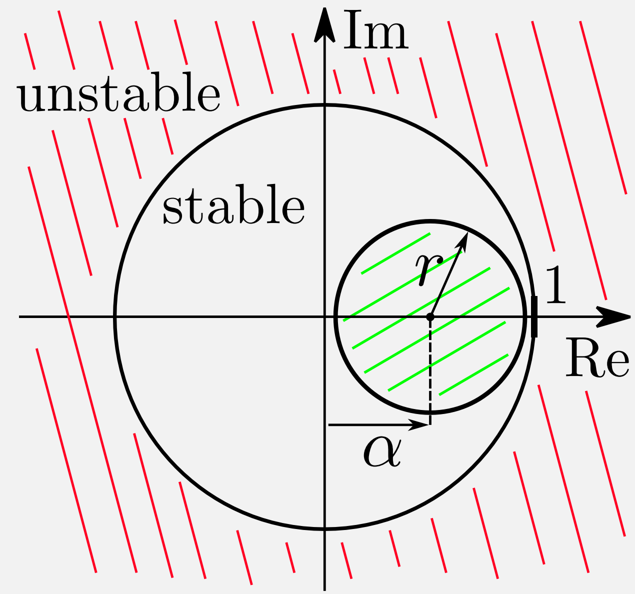
a circle with the midpoint α and radius r, if

$$(\mathcal{A}_v-\alpha I)^\mathrm{T} P(\mathcal{A}_v-\alpha I) - r^2 P < 0$$

or equivalent

$$\begin{bmatrix} P^{-1} & \mathcal{A}_v-\alpha I \\ (\mathcal{A}_v-\alpha I)^\mathrm{T} & r^2 P \end{bmatrix} > 0,$$

with $|\alpha|<1$ and $|\alpha|+r\leq 1$ are valid.



Generalization of the Lyapunov stability condition to stochastic noise (with noise $w[k] \neq 0$)

Stochastic noise affects $z[k+1] = \mathcal{A}_v z[k] + \mathcal{G}_v w[k]$, with

$$\mathcal{A}_v = \begin{bmatrix} A_v - B_v K & B_v K \\ A_v - A_0 - (B_v - B_0)K & A_0 - HC - (B_0 - B)K \end{bmatrix}, \mathcal{G}_v = \begin{bmatrix} B_v G_u & G_p & 0 \\ B_v G_u & G_p & -HG_y \end{bmatrix}$$

It follows the discrete-time version of the Itô differential operator:

$$L_D(V) = \frac{1}{2}\left(z^{\mathrm{T}}[k]\left(\mathcal{A}_v^{\mathrm{T}} P \mathcal{A}_v - P\right)z[k] + \mathrm{trace}\left(\mathcal{G}_v^{\mathrm{T}} P \mathcal{G}_v\right)\right)$$

Derivable from the expectation value:

$$E(\Delta V) = E\{V(z[k+1]) - V(z[k])\}$$

$$E(\Delta V) = E\left\{\frac{1}{2}\left(\left(z^{\mathrm{T}}[k]\mathcal{A}_v^{\mathrm{T}} + w^{\mathrm{T}}[k]\mathcal{G}_v^{\mathrm{T}}\right)P\left(\mathcal{A}_v z[k] + \mathcal{G}_v w[k]\right) - z^{\mathrm{T}}[k]Pz[k]\right)\right\}$$

Under assumptions: $w[k]$ and $z[k]$ stochastically independent; $w[k]$ is a zero mean process; variance of each noise process equals one

Robust Lyapunov stability and D_{R} regions (without noise $w[k]=0$; deterministic system)

Discrete- vs. continuous-time Lyapunov conditions

for state feedback controller: $u=-Kx$

Discrete-time (option 1):

$$(A_v-B_vK)^{\mathrm{T}}P(A_v-B_vK)-P<0$$

Schur-complement:

$$\begin{bmatrix} P^{-1} & A_v-B_vK \\ (A_v-B_vK)^{\mathrm{T}} & P \end{bmatrix}>0$$

Left/right multiplication with $\mathrm{diag}(I,P^{-1})$,

change of variables $\mathrm{Q}=P^{-1}, N=KP^{-1}$:

$$\begin{bmatrix} Q & A_vQ-B_vN \\ (A_vQ-B_vN)^{\mathrm{T}} & Q \end{bmatrix}>0$$

controller: $K=NQ^{-1}$

Continuous-time:

$$(A_v-B_vK)^{\mathrm{T}}P+P(A_v-B_vK)<0$$

Left/right multiplication with P^{-1} and

change of variables $\mathrm{Q}=P^{-1}, N=KP^{-1}$:

$$A_vQ+QA_v^{\mathrm{T}}-B_vN-N^{\mathrm{T}}B_v^{\mathrm{T}}<0$$

controller: $K=NQ^{-1}$

Robust Lyapunov stability and D_{R} regions (without noise $w[k] = 0$; deterministic system)

Discrete- vs. continuous-time Lyapunov conditions

for state feedback controller: $u = -Kx$

Discrete-time (option 1):

$$(A_v - B_v K)^{\mathrm{T}} P (A_v - B_v K) - P < 0$$

Schur-complement:

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$$\begin{bmatrix} Q & A_v Q - B_v N \\ (A_v Q - B_v N)^{\mathrm{T}} & Q \end{bmatrix} > 0$$

➡ controller: $K = NQ^{-1}$

Discrete-time (option 2):

$$\begin{bmatrix} P^{-1} & A_v - B_v K \\ (A_v - B_v K)^{\mathrm{T}} & P \end{bmatrix} > 0$$

Due to $P = P^{\mathrm{T}} > 0$ the quadratic form

$$(P - G)^{\mathrm{T}} P^{-1} (P - G) \geq 0$$

➡ $$G^{\mathrm{T}} P^{-1} G \geq G + G^{\mathrm{T}} - P$$

is always valid for any matrix G. This yields:

$$\begin{bmatrix} P & A_v G - B_v N \\ (A_v G - B_v N)^{\mathrm{T}} & G + G^{\mathrm{T}} - P \end{bmatrix} > 0$$

➡ controller: $K = NG^{-1}$

+ K independent of P

− requires change of variables

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Superposed iteration rule

$$\begin{bmatrix} P^{-1} & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0$$

Consider the quadratic form

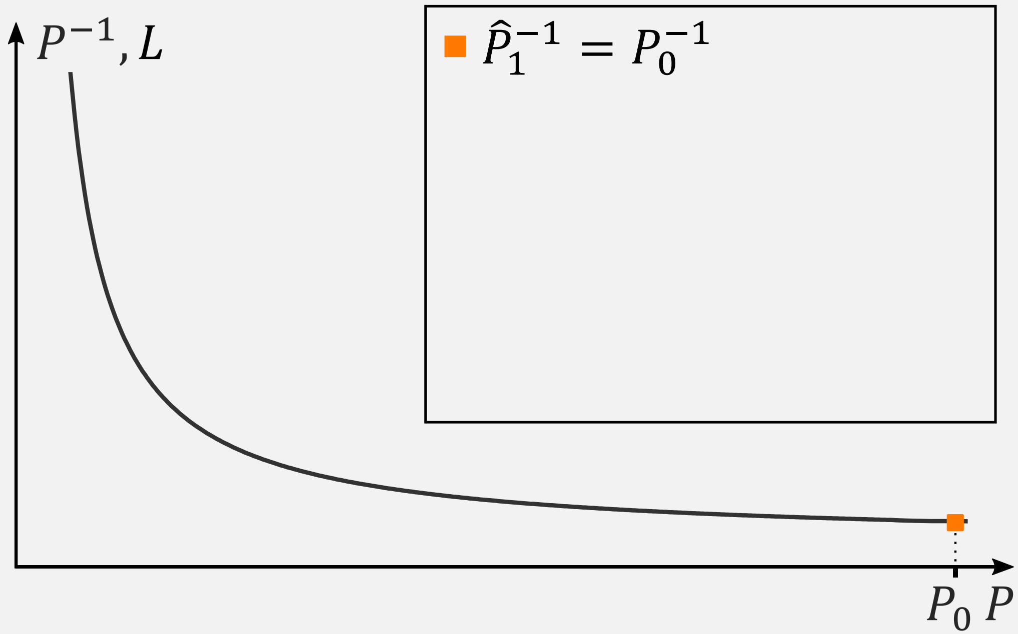
$$(P - G)^{\mathrm{T}} P^{-1} (P - G) \geq 0$$

Replace G by $\hat{P} = \hat{P}^{\mathrm{T}} > 0$ results in:

$$\hat{P} P^{-1} \hat{P} \geq 2\hat{P} - P$$

Left/right multiplication with $\hat{P}^{-1}$ yields:

$$P^{-1} \geq 2\hat{P}^{-1} - \hat{P}^{-1} P \hat{P}^{-1} = L$$



$$\Rightarrow \begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0$$

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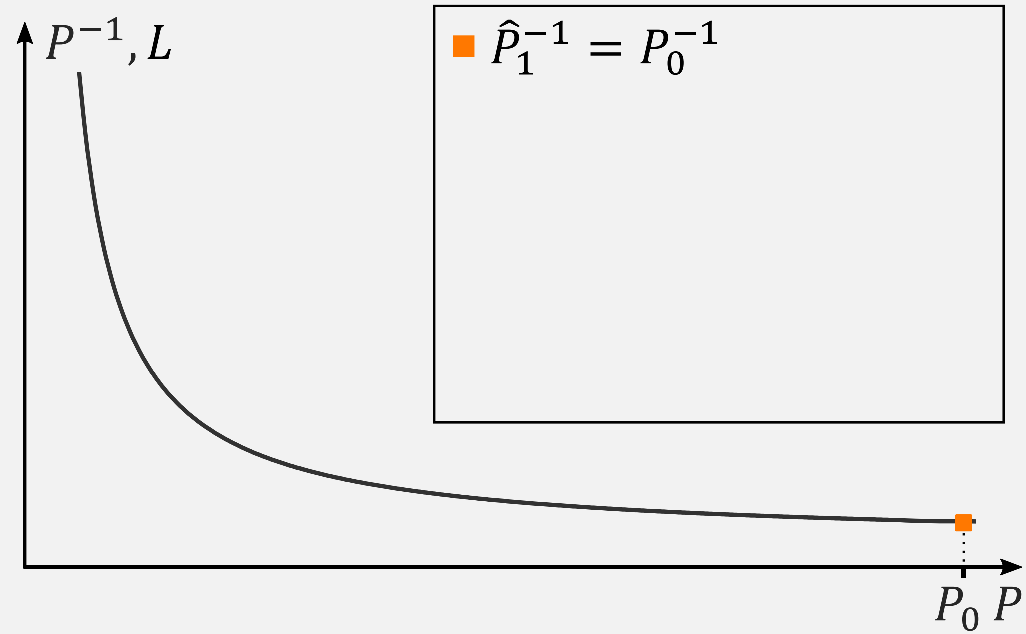

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$$(P - G)^{\mathrm{T}} P^{-1} (P - G) \geq 0$$

Replace G by $\hat{P} = \hat{P}^{\mathrm{T}} > 0$ results in:

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$$\Rightarrow \quad \begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0$$

Question: How to select $\hat{P}$ for a unknown decision variables P, such that $P - \hat{P}$ is small?

Solution: update rule

$$\hat{P}^{-1} = \left(P_{j-1}\right)^{-1}$$

$\Rightarrow$

Aim: Convergence of L vs. P^{-1}, such that

$$P^{-1} - L \approx 0$$

Superposed iteration rule

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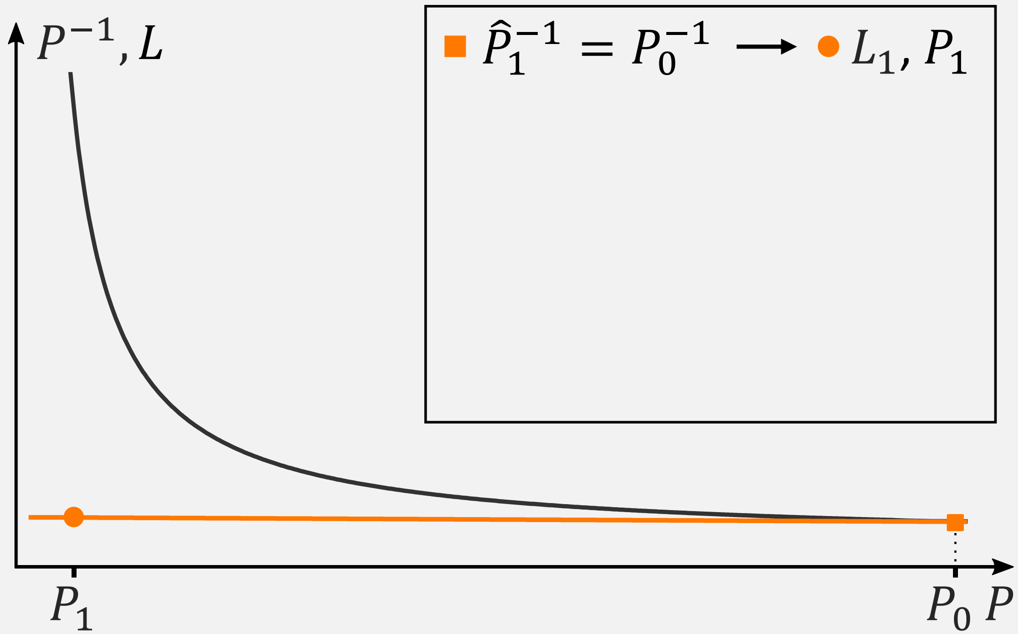

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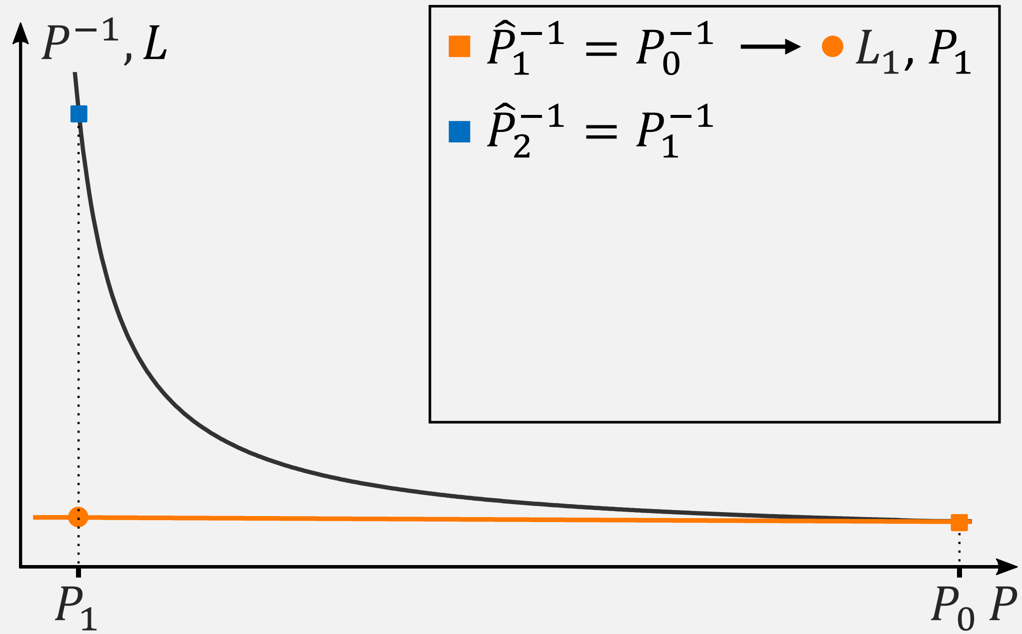

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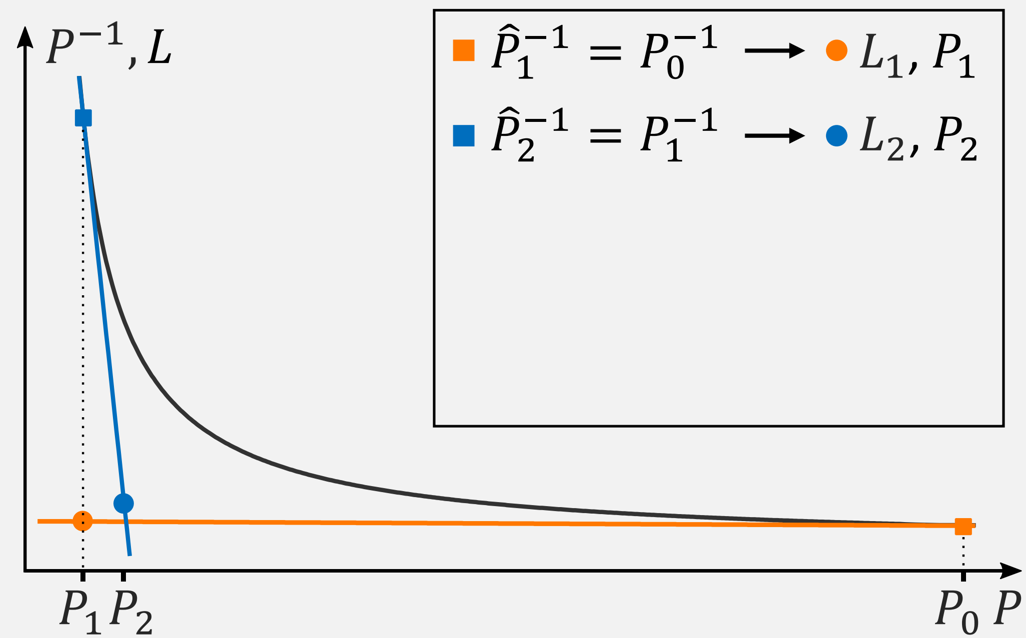

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Superposed iteration rule

$$\begin{bmatrix} P^{-1} & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^\mathrm{T} & r^2 P \end{bmatrix} > 0$$

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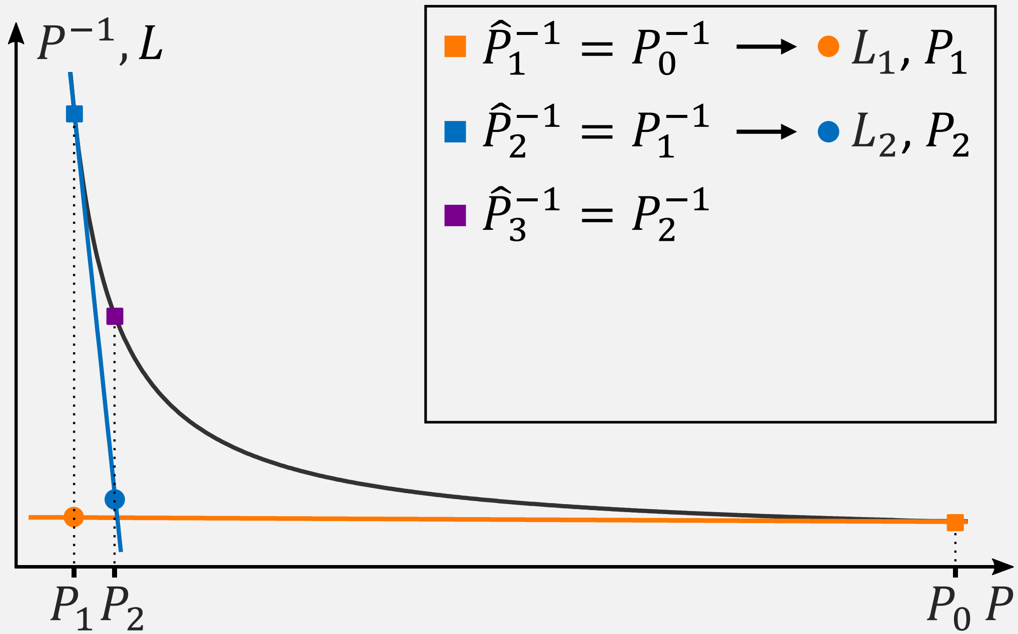

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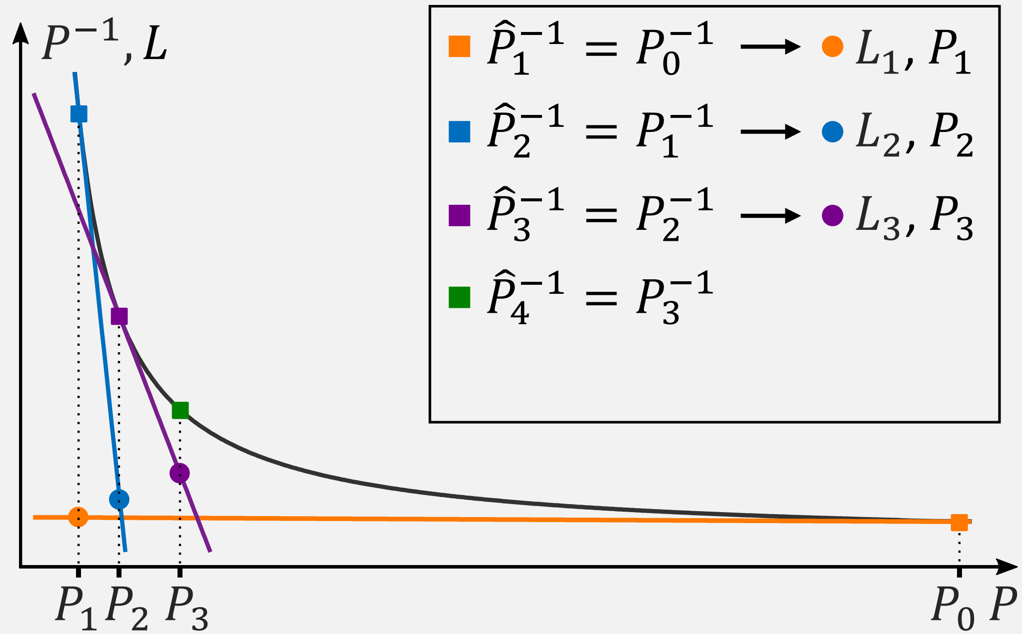

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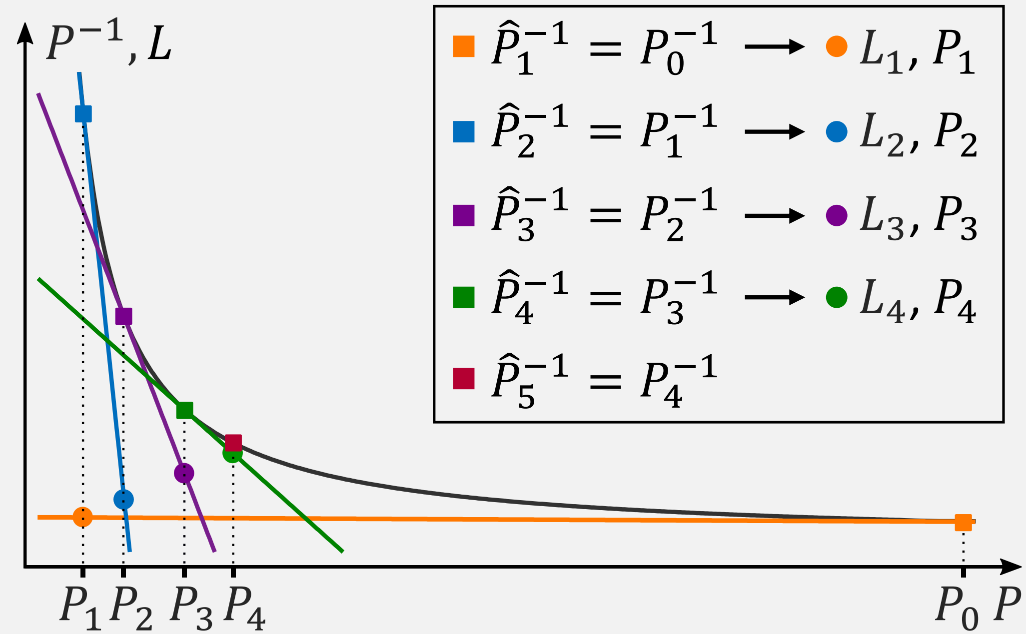


$$\begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0$$

Question: How to select $\hat{P}$ for a unknown decision variables P, such that $P - \hat{P}$ is small?

Solution: update rule

$$\hat{P}^{-1} = \left(P_{j-1}\right)^{-1}$$

Aim: Convergence of L vs. P^{-1}, such that

$$P^{-1} - L \approx 0$$

Superposed iteration rule

$$\begin{bmatrix} P^{-1} & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0$$

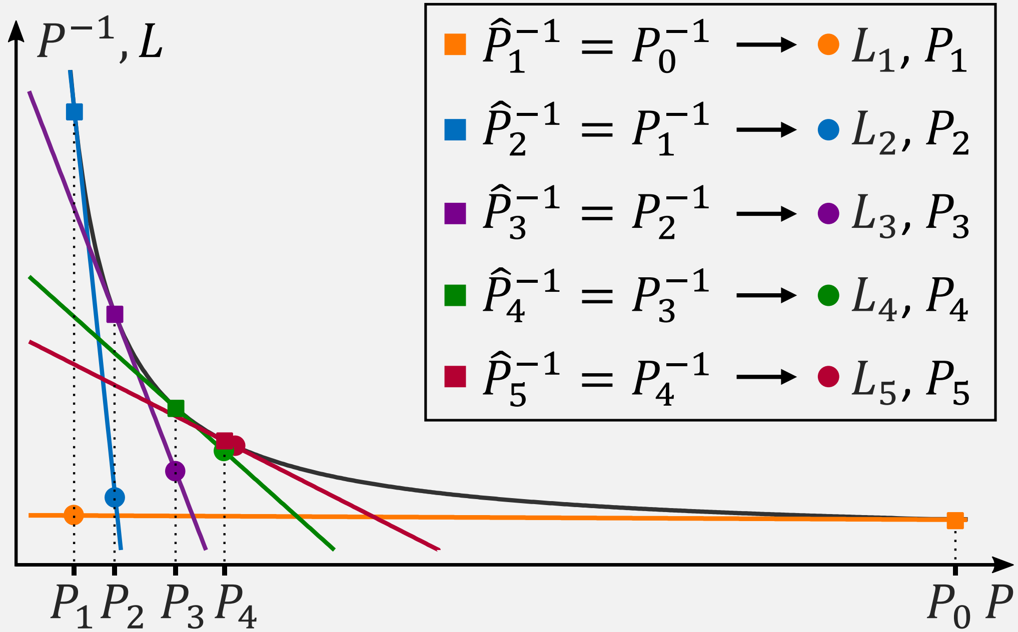
Consider the quadratic form

$$(P - G)^{\mathrm{T}} P^{-1} (P - G) \geq 0$$

Replace G by $\hat{P} = \hat{P}^{\mathrm{T}} > 0$ results in:

$$\hat{P} P^{-1} \hat{P} \geq 2\hat{P} - P$$

Left/right multiplication with $\hat{P}^{-1}$ yields:



$$P^{-1} \geq 2\hat{P}^{-1} - \hat{P}^{-1} P \hat{P}^{-1} = L \quad \Rightarrow \quad \begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0$$

Question: How to select $\hat{P}$ for a unknown decision variables P, such that $P - \hat{P}$ is small?

Solution: update rule

$$\hat{P}^{-1} = \left(P_{j-1}\right)^{-1}$$

Aim: Convergence of L vs. P^{-1}, such that

$$P^{-1} - L \approx 0$$

Superposed iteration rule (Stage 1)

Solution:

- Select constant α and $P_0 = I$
- Select $|\alpha| + r > 1$ in the first iteration
- Solve

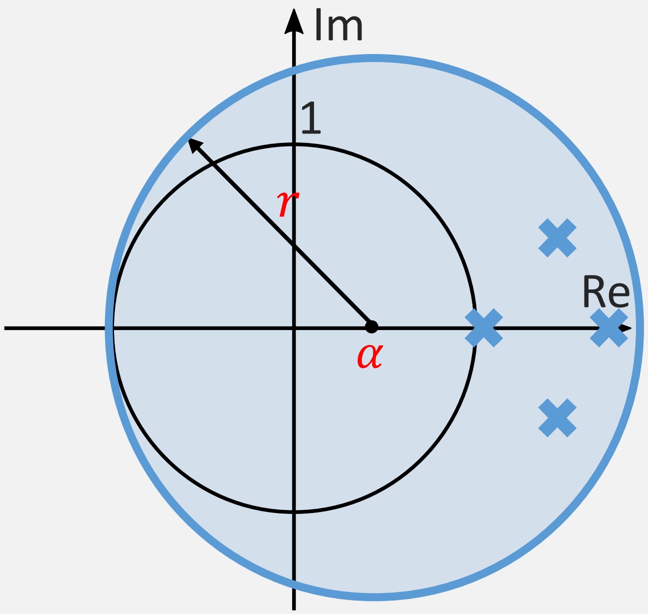
$$\begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0$$

with

$$L = 2\hat{P}^{-1} - \hat{P}^{-1} P \hat{P}^{-1}$$

$$\hat{P}^{-1} = \left(P_{j-1}\right)^{-1} \quad (j\text{: current iteration})$$

$|\alpha| + r > 1$: Closed-loop instable



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Superposed iteration rule (Stage 1)

Solution:

- Select constant α and $P_0 = I$
- Select $|\alpha| + r > 1$ in the first iteration
- Solve

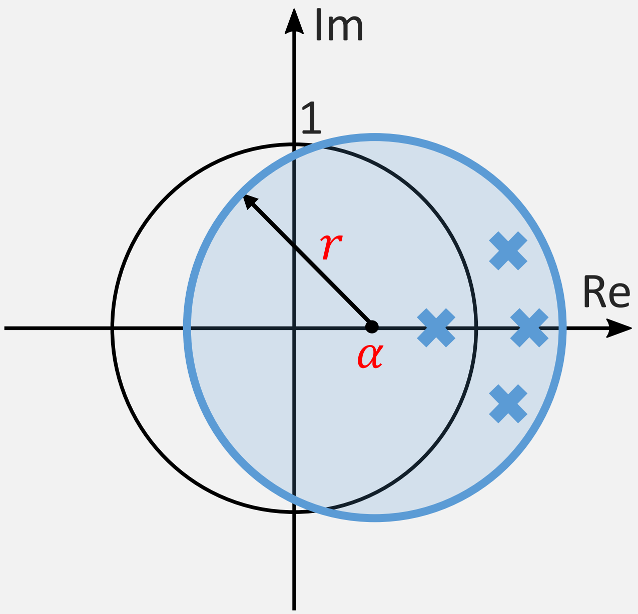
$$\begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0$$

with

$$L = 2\hat{P}^{-1} - \hat{P}^{-1} P \hat{P}^{-1}$$

$$\hat{P}^{-1} = \left(P_{j-1}\right)^{-1}$$ (j: current iteration)

$|\alpha| + r > 1$: Closed-loop instable



Superposed iteration rule (Stage 1)

Solution:

- Select constant α and $P_0 = I$
- Select $|\alpha| + r > 1$ in the first iteration
- Solve

$$\begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0$$

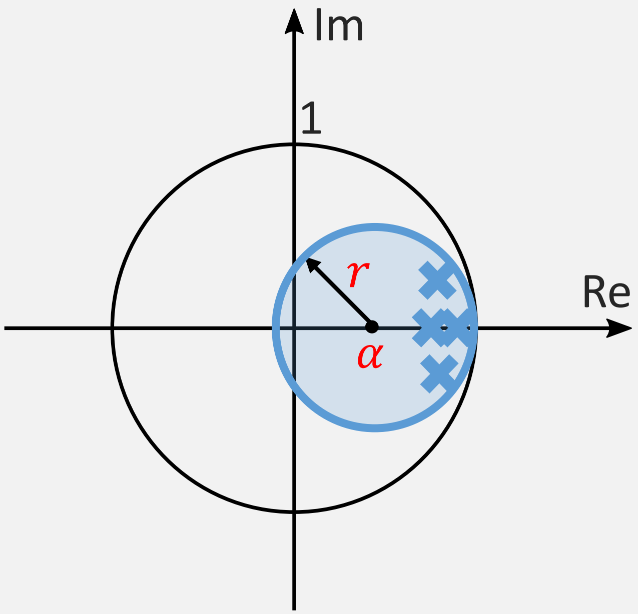
with

$$L = 2\hat{P}^{-1} - \hat{P}^{-1} P \hat{P}^{-1}$$

$$\hat{P}^{-1} = \left(P_{j-1}\right)^{-1} \quad (j\text{: current iteration})$$

$|\alpha| + r > 1$: Closed-loop instable

$|\alpha| + r = 1$: Closed-loop robust stable and all eigenvalues of $\mathcal{A}_v$ are located in the circular D_R region



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Superposed iteration rule (Stage 1)

Solution:

- Select constant α and $P_0 = I$
- Select $|\alpha| + r > 1$ in the first iteration
- Solve

$$\begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^\mathrm{T} & r^2 P \end{bmatrix} > 0$$

with

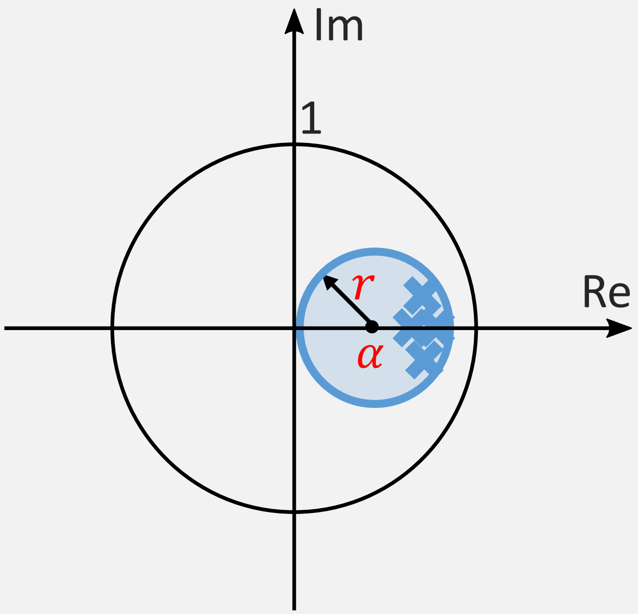
$$L = 2\hat{P}^{-1} - \hat{P}^{-1} P \hat{P}^{-1}$$

$$\hat{P}^{-1} = \left(P_{j-1}\right)^{-1} \quad (j\text{: current iteration})$$

$|\alpha| + r > 1$: Closed-loop instable

$|\alpha| + r = 1$: Closed-loop robust stable and all eigenvalues of $\mathcal{A}_v$ are located in the circular D_R region

$|\alpha| + r < 1$: increased distance to stability margin



Realizable objectives:
- Convergence of the linearization
- Stabilization of the closed loop
- Tuning control behavior

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Optimization task (Stage 2)

Discrete-time Itô differential operator:

$$L_D(V) = \frac{1}{2}\left(z^{\mathrm{T}}[k]\left(\mathcal{A}_v^{\mathrm{T}} P \mathcal{A}_v - P\right) z[k] + \mathrm{trace}\left(\mathcal{G}_v^{\mathrm{T}} P \mathcal{G}_v\right)\right)$$

If stochastic noise affects the closed-loop $z[k+1] = \mathcal{A}_v z[k] + \mathcal{G}_v w[k]$ ($\mathcal{G}_v$: non-zero)

- Maybe: $L_D(V) \geq 0$ in a neighborhood of $z[k] = 0$

- Non-provable stability region with boundary $L_D(V) = 0$

  is the interior of the ellipsoids:

$$z^{\mathrm{T}}[k]\left(\frac{-M_v}{\mathrm{trace}\left(\mathcal{G}_v^{\mathrm{T}} P \mathcal{G}_v\right)}\right) z[k] - 1 = 0 \quad \text{with} \quad M_v = \mathcal{A}_v^{\mathrm{T}} P \mathcal{A}_v - P$$

Decrease the non-provable stability region by minimizing the interior of the ellipsoids:

➡ Non-convex cost function $J = \sum_{v=1}^{n_v} \frac{\mathrm{trace}\left(\mathcal{G}_v^{\mathrm{T}} P \mathcal{G}_v\right)}{\det(-M_v)}$

Optimization task (Stage 2)

Integration into superposed iteration rule:

$$\min \quad J = \sum_{v=1}^{n_v} \frac{\mathrm{trace}\left(\mathcal{G}_v^{\mathrm{T}} P \mathcal{G}_v\right)}{\det(-M_v)}$$

subject to

$$P > 0,$$

$$\begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0,$$

with $L = 2\hat{P}^{-1} - \hat{P}^{-1} P \hat{P}^{-1}$

→

$$\min \quad J = \sum_{v=1}^{n_v} \frac{\mathrm{trace}(N)}{\det(-\hat{M}_v)}$$

subject to

$$P > 0,$$

$$N > 0,$$

$$N > \mathcal{G}_v^{\mathrm{T}} P \mathcal{G}_v,$$

$$\begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0,$$

with $L = 2\hat{P}^{-1} - \hat{P}^{-1} P \hat{P}^{-1}$,

$$\hat{M}_v = \hat{\mathcal{A}}_v^{\mathrm{T}} \hat{P} \hat{\mathcal{A}}_v - \hat{P}$$

Optimization task (Stage 2)

Integration into superposed iteration rule:

$$\min \quad J = \sum_{v=1}^{n_v} \frac{\mathrm{trace}\left(\mathcal{G}_v^{\mathrm{T}} P \mathcal{G}_v\right)}{\det(-M_v)}$$

subject to

$$P > 0,$$

$$\begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0,$$

with $L = 2\hat{P}^{-1} - \hat{P}^{-1} P \hat{P}^{-1}$

$$\min \quad J = \sum_{v=1}^{n_v} \frac{\mathrm{trace}(N)}{\det(-\hat{M}_v)}$$

subject to

$$P > 0,$$

$$N > 0,$$

$$\begin{bmatrix} P^{-1} & \mathcal{G}_v \\ \mathcal{G}_v^{\mathrm{T}} & P \end{bmatrix} \geq \begin{bmatrix} L & \mathcal{G}_v \\ \mathcal{G}_v^{\mathrm{T}} & P \end{bmatrix} > 0,$$

$$\begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0,$$

with $L = 2\hat{P}^{-1} - \hat{P}^{-1} P \hat{P}^{-1}$,

$$\hat{M}_v = \hat{\mathcal{A}}_v^{\mathrm{T}} \hat{P} \hat{\mathcal{A}}_v - \hat{P}$$

Optimization task (Stage 2)

Integration into superposed iteration rule:

$$\min \quad J = \sum_{v=1}^{n_v} \frac{\mathrm{trace}\left(\mathcal{G}_v^{\mathrm{T}} P \mathcal{G}_v\right)}{\det(-M_v)}$$

subject to

$$P > 0,$$

$$\begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0,$$

with $L = 2\hat{P}^{-1} - \hat{P}^{-1} P \hat{P}^{-1}$

convex optimization:

$$\min \quad J = \sum_{v=1}^{n_v} \frac{\mathrm{trace}(N)}{\det(-\hat{M}_v)}$$

subject to

$$P > 0,$$

$$N > 0,$$

$$\begin{bmatrix} L & \mathcal{G}_v \\ \mathcal{G}_v^{\mathrm{T}} & P \end{bmatrix} > 0,$$

$$\begin{bmatrix} L & \mathcal{A}_v - \alpha I \\ (\mathcal{A}_v - \alpha I)^{\mathrm{T}} & r^2 P \end{bmatrix} > 0,$$

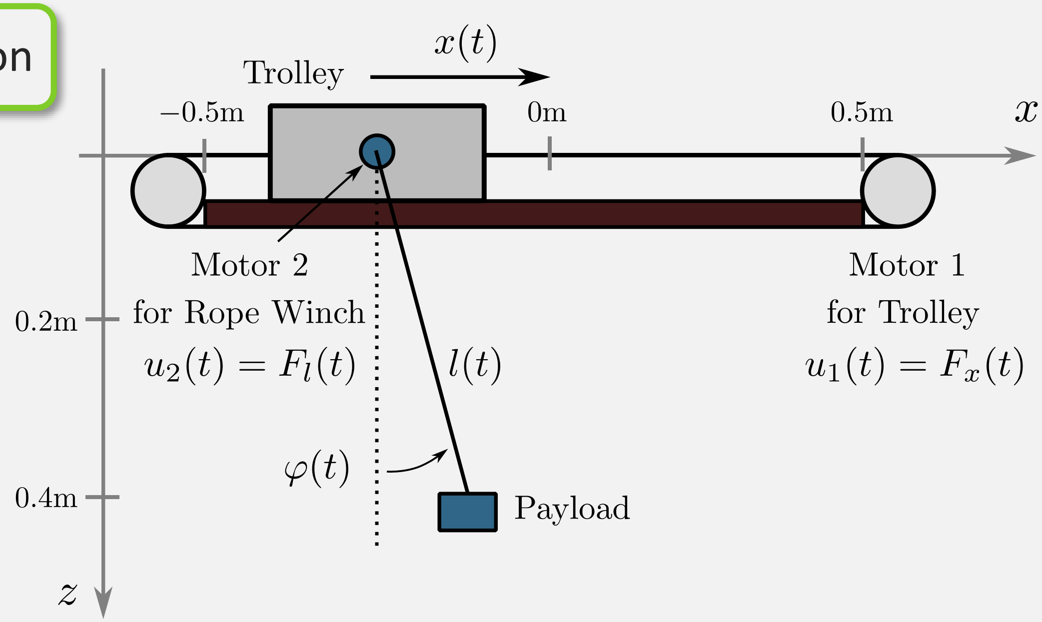
with $L = 2\hat{P}^{-1} - \hat{P}^{-1} P \hat{P}^{-1}$,

$$\hat{M}_v = \hat{\mathcal{A}}_v^{\mathrm{T}} \hat{P} \hat{\mathcal{A}}_v - \hat{P}$$

Realizable objectives:
- Optimality of the observer and controller gain
- Closed loop insensitive against noise

Modelling: Lagrange's equations of motion

Measurable outputs: $q = (x \quad \psi \quad l)^{\mathrm{T}}$



Lagrange function:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q_j$$

where

$$Q_j = \begin{pmatrix} F_x - F_{\mathrm{fc}}\dot{x} & -F_{\mathrm{fp}}\dot{\psi} & F_l - F_{\mathrm{fr}}\dot{l} \end{pmatrix}^{\mathrm{T}}$$

$$L = E_{\mathrm{kin}} - E_{\mathrm{pot}}$$

with

$$E_{\mathrm{kin}} = \frac{1}{2}\ \dot{x}^2\ (m_1 + m_2) + \frac{1}{2}\ \dot{l}^2\ m_2 + \frac{1}{2}\dot{\psi}^2\ l^2\ m_2 + \dot{x}\ \dot{l}\ m_2 \sin(\psi) + \dot{x}\ \psi\ l\ m_2 \cos(\psi) + \frac{1}{2}\ \theta\ \frac{\dot{l}^2}{R_{\mathrm{T}}^2}$$

$$E_{\mathrm{pot}} = -m_2\ \mathrm{g}\ l \cos(\psi)$$

System of differential equations

Idea: Transform differential equation to state space representation

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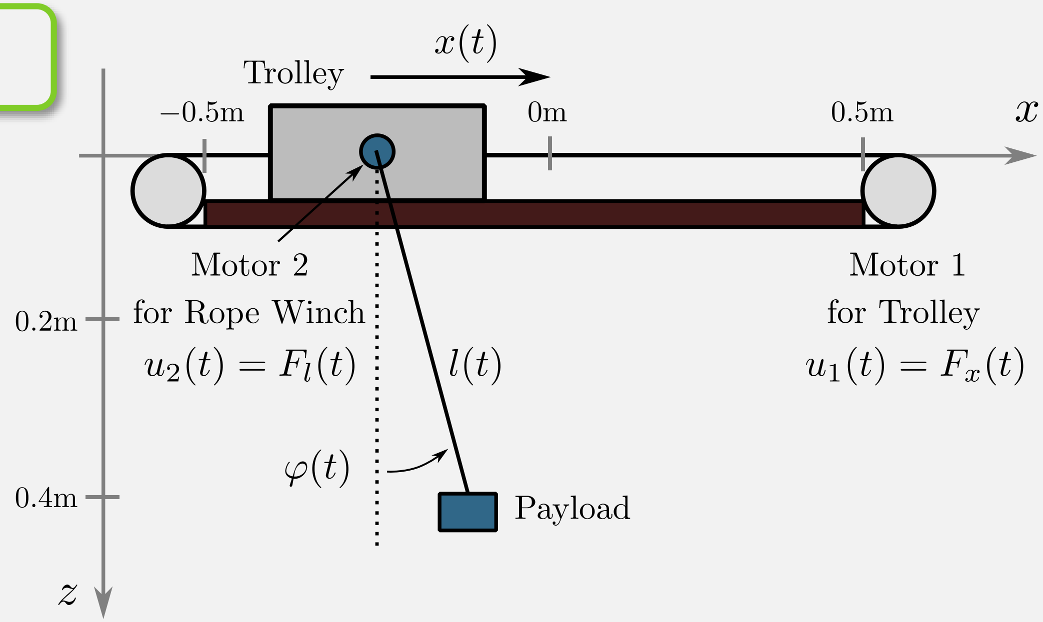
Modelling: Quasilinear representation



1. Simplifications for small angles:

$$\sin(\psi) = \psi, \quad \cos(\psi) = 1, \quad \dot{\psi}^2 = 0$$

2. Introduce the state vector:

$$x = \begin{pmatrix} x & \dot{x} & \psi & \dot{\psi} & l & \dot{l} \end{pmatrix}^{\mathrm{T}}$$

3. Quasilinear form:

$$\dot{x} = A_c(x)x + B_c(x)u$$

with

$$A_c(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -p_1 & p_2 & \frac{p_3}{x_5} & 0 & p_4 x_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{p_1}{x_5} & \frac{p_5}{x_5} & \frac{p_6}{x_5^2} & 0 & -p_4\frac{x_3}{x_5} - 2\frac{x_4}{x_5} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & p_7 x_3 & -p_8\frac{x_4}{x_5} & 0 & \frac{p_9}{x_5} & -p_{10} \end{pmatrix} \qquad B_c(x) = \begin{pmatrix} 0 & 0 \\ p_{11} & -p_{12}x_3 \\ 0 & 0 \\ -\frac{p_{11}}{x_5} & \frac{p_{12}x_3}{x_5} \\ 0 & 0 \\ -p_{12}x_3 & p_{13} \end{pmatrix}$$

Idea: Convex enclosure of nonlinearities

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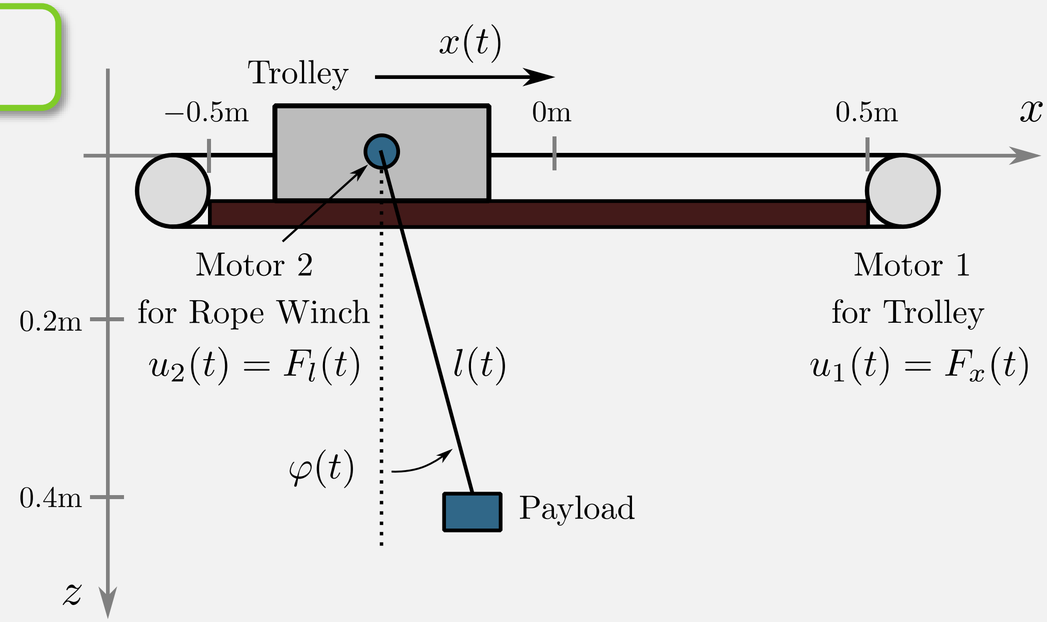
Modelling: Polytopic representation



4. States are constrained:

$$x_i = [\underline{x}_i, \overline{x}_i], i = 1, \dots, 6$$

5. Introduce independent parameters:

$$\delta_j = [\underline{\delta}_j, \quad \overline{\delta}_j], \quad j = 1, \dots, 5$$

for each nonlinear function yields the polytopic representation:

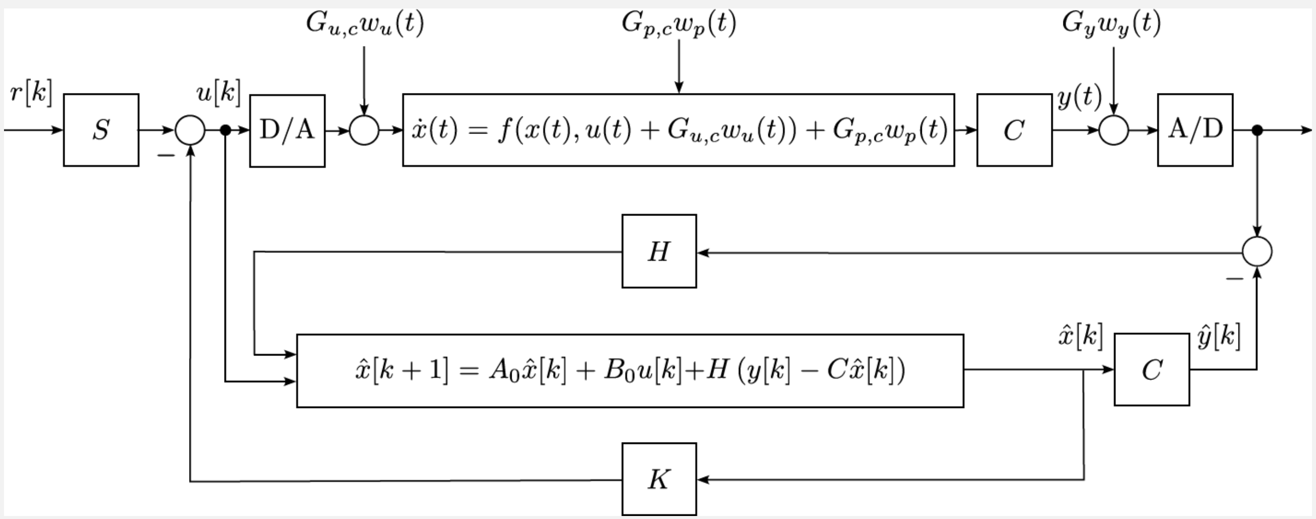
$$[A_c(\delta), B_c(\delta)] \in \left\{ [A_c(\zeta), B_c(\zeta)] = \sum_{v=1}^{n_v} \zeta_v \cdot [A_v, B_v]; \sum_{v=1}^{n_v} \zeta_v = 1; \zeta_v \geq 0 \right\}$$

with

$$A_c(\delta) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -p_1 & p_2 & p_3\delta_1 & 0 & p_4\delta_2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & p_1\delta_1 & p_5\delta_1 & p_6\delta_3 & 0 & -(p_4\delta_4 + 2\delta_5) \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & p_7\delta_2 & -p_8\delta_5 & 0 & p_9\delta_1 & -p_{10} \end{pmatrix} \quad B_c(\delta) = \begin{pmatrix} 0 & 0 \\ p_{11} & -p_{12}\delta_2 \\ 0 & 0 \\ -p_{11}\delta_1 & p_{12}\delta_4 \\ 0 & 0 \\ -p_{12}\delta_2 & p_{13} \end{pmatrix}$$

6. Discretization by first order Euler approximation: $A(\delta) = A_c(\delta)T_s + I, \; B(\delta) = B_c(\delta)T_s$

Simulation setup 1



disturbance input matrices:

$$G_u = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$G_p = \begin{pmatrix} 0 & 0 & 0 \\ 0.01 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}$$

$$G_y = \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}$$

Compared to LQG control: Filter and controller designed separately for the linearized system

Observer (filter) parameterization:

$$E\left[w_p(k)w_p^{\mathrm{T}}(k)\right] = Q_o$$
$$E\left[w_y(k)w_y^{\mathrm{T}}(k)\right] = R_o$$
$\Rightarrow H$

LQR applied to estimated states $\hat{x}[k]$:

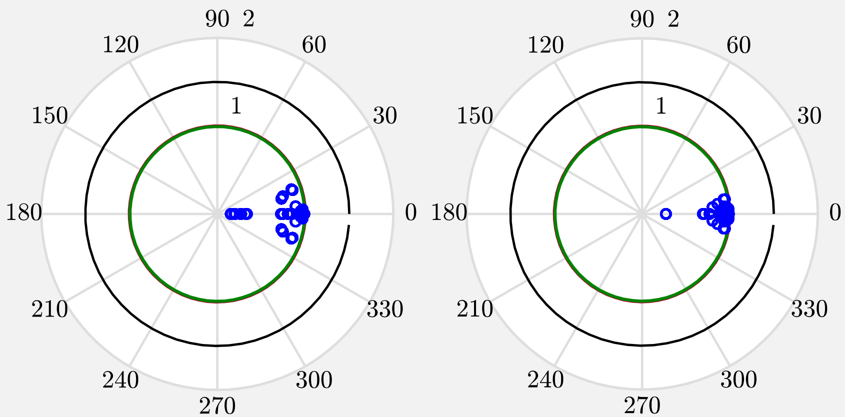
$$Q_c = \mathrm{diag}\left(\mu_{x,i}\frac{1}{x_{\max,i}^2}\right)$$
$$R_c = \mathrm{diag}\left(\mu_{u,j}\frac{1}{u_{\max,j}^2}\right)$$
$\Rightarrow K$

Disadvantages: Parameter tuning is semi-empirical; no proof of stability

Tuning the LMI controller: using robust D_{R} regions (circular sub-regions of the unit circle)

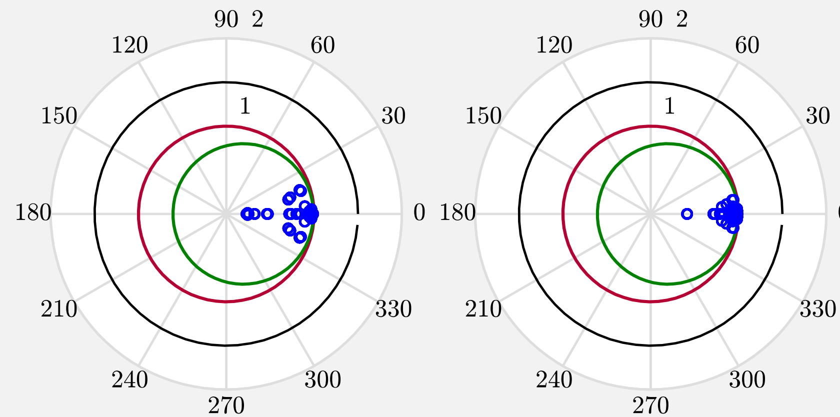
Case 1: $r = 0.993, \ \alpha = 0,$

before optimization | after optimization



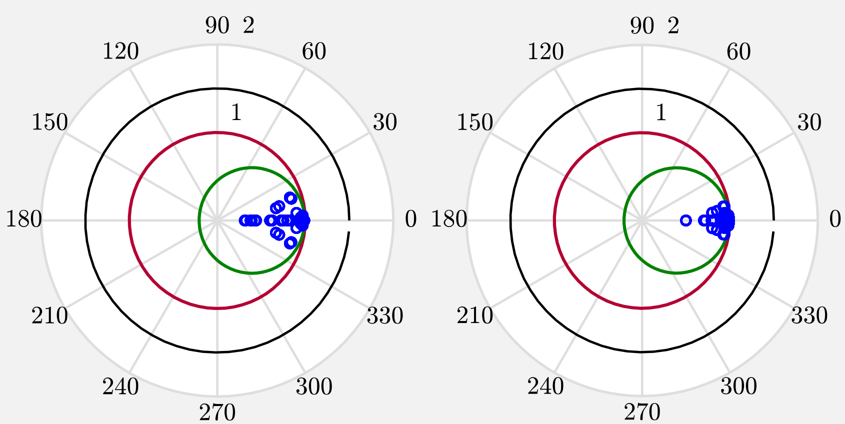
Case 2: $r = 0.8, \ \alpha = 0.193,$

before optimization | after optimization



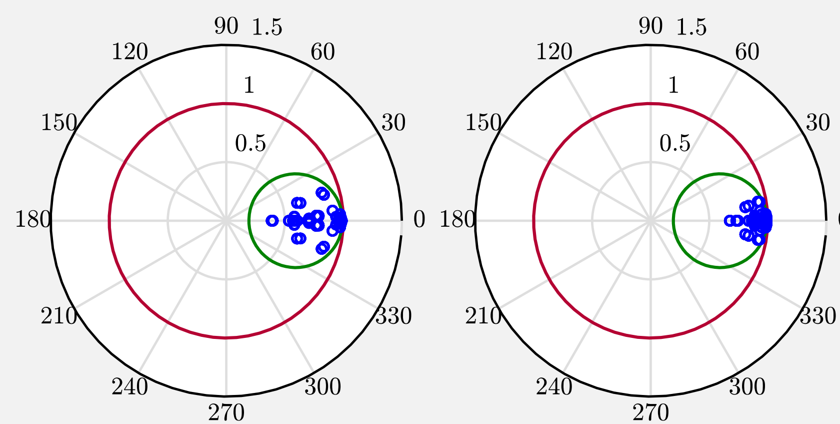
Case 3: $r = 0.6, \ \alpha = 0.393,$

before optimization | after optimization

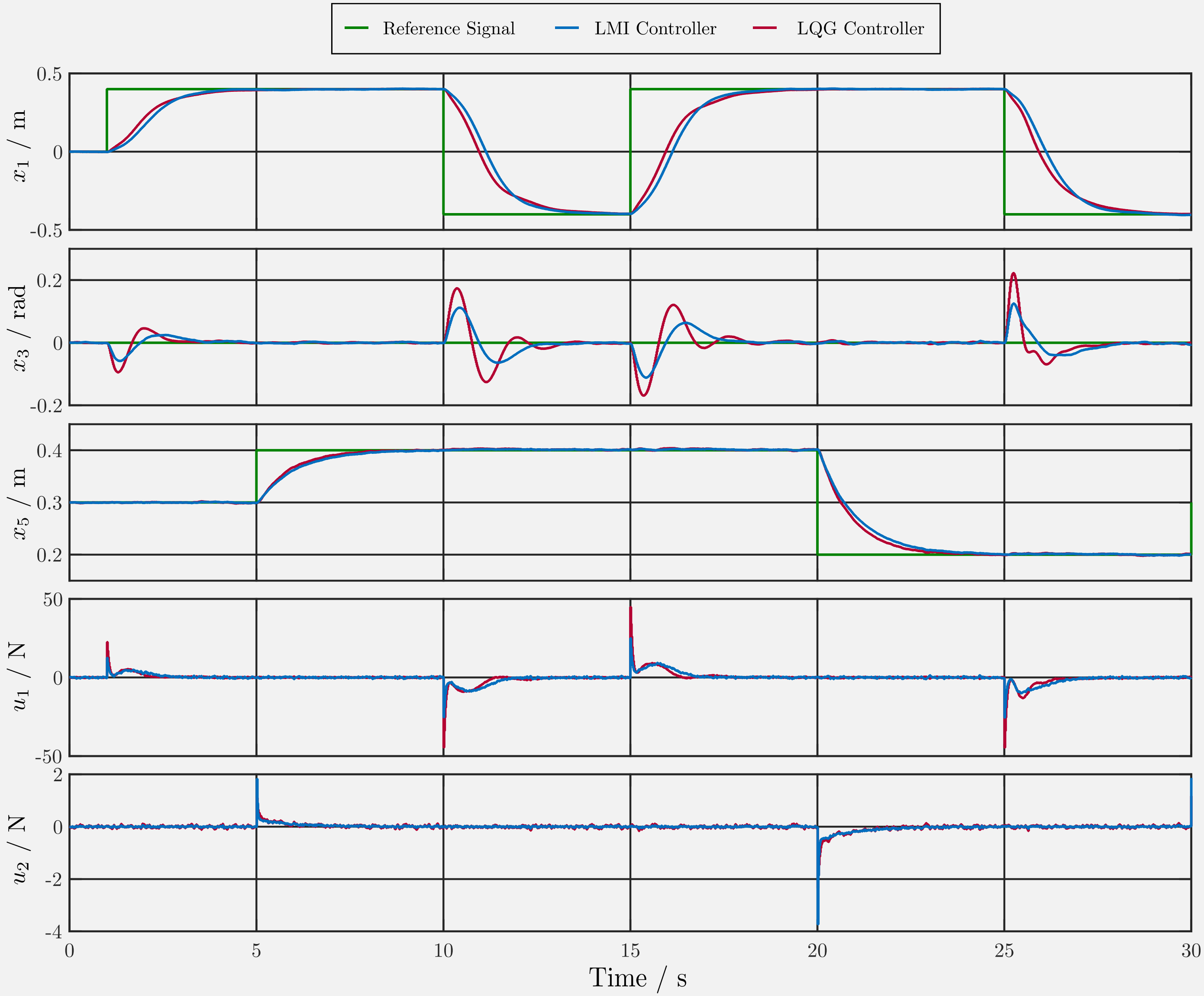


Case 4: $r = 0.4, \ \alpha = 0.593$

before optimization | after optimization



Idea: Select design settings with comparable control behaviors for both controller



Comparison 1

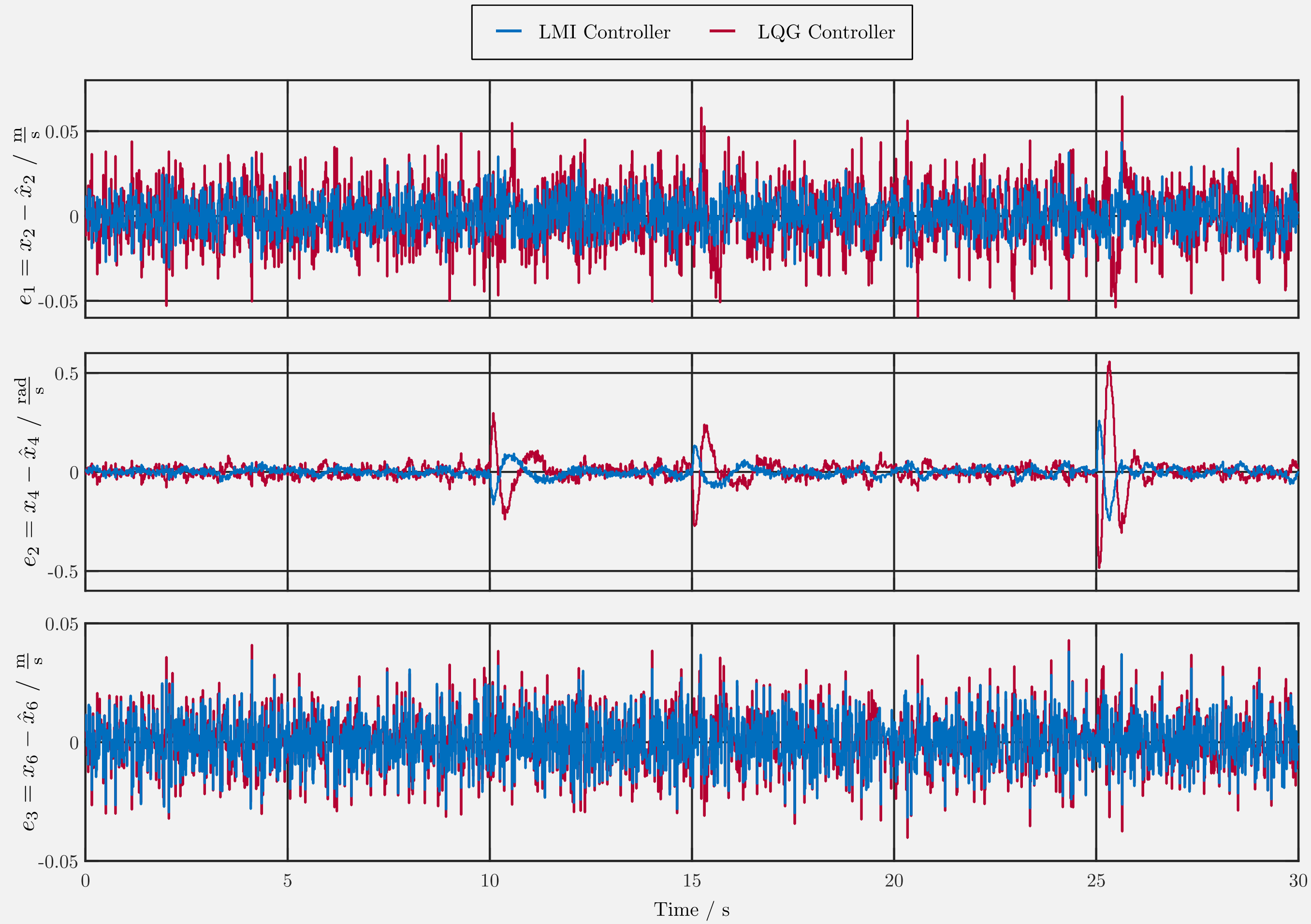
Cost function:

RMSE-Values

$$\Delta x_{j,O} = \sqrt{\frac{1}{N}\sum_{k=1}^{N}\left(x_j\left[k\right] - \hat{x}_j\left[k\right]\right)^2}$$

of the observer error

$N = 30s/T_S$=2000

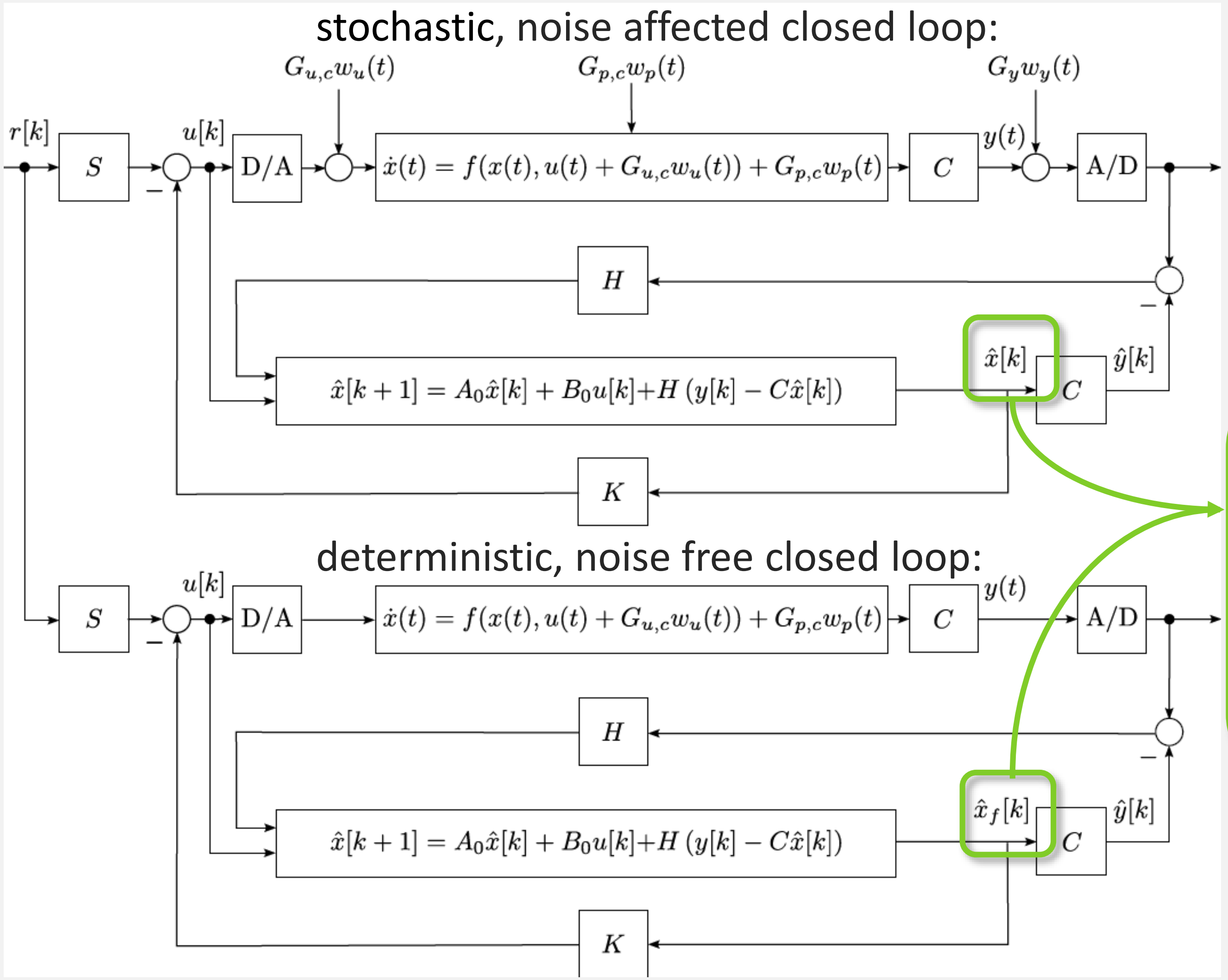


Evaluation:

Non-measurable states	$\Delta x_{2,O}$	$\Delta x_{4,O}$	$\Delta x_{6,O}$
LQG	0.0176	0.0679	0.0125
LMI	0.0107	0.0327	0.0106
Improvement	**39 %**	**52 %**	**15 %**

Improvement: Reduced observer error

Simulation setup 2



Cost function

RMSE-Values:

$$\Delta\hat{x}_{i,f} = \sqrt{\frac{1}{N}\sum_{k=1}^{N}\left(\hat{x}_i[k] - \hat{x}_{i,f}[k]\right)^2}$$

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Comparison 2: Measurable states

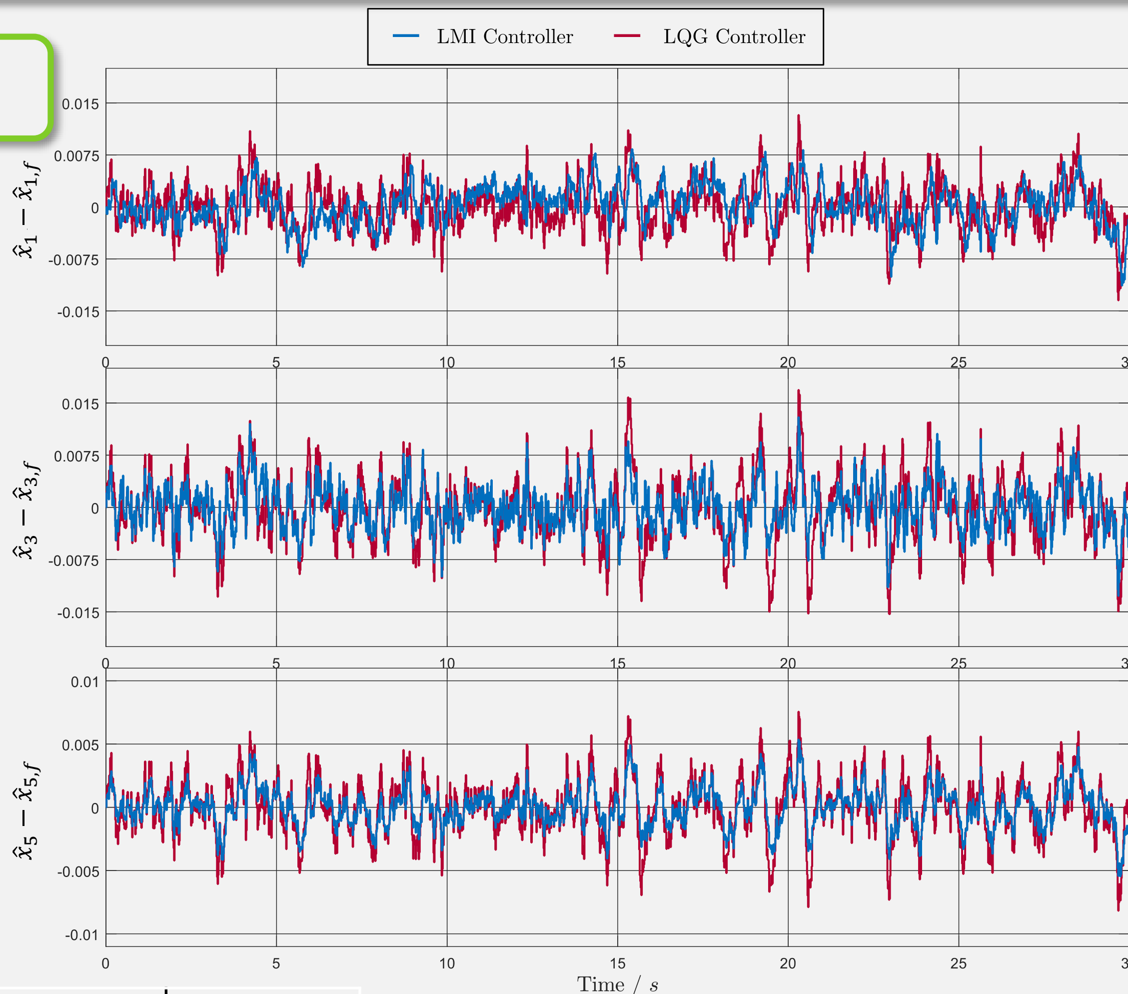
Cost function:

RMSE-Values

$$\Delta \hat{x}_{i,f} = \sqrt{\frac{1}{N}\sum_{k=1}^{N}\left(\hat{x}_i\left[k\right] - \hat{x}_{i,f}\left[k\right]\right)^2}$$

from all estimates $\hat{x}_i$

to ideal noise-free estimates $\hat{x}_{i,f}$



Evaluation:

Measurable states	$\Delta\hat{x}_{1,f}$	$\Delta\hat{x}_{3,f}$	$\Delta\hat{x}_{5,f}$
LQG	0.0035	0.0048	0.0023
LMI	0.0029	0.0034	0.0015
Improvement	**17 %**	**29 %**	**33 %**

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Comparison 2: Non-measurable states

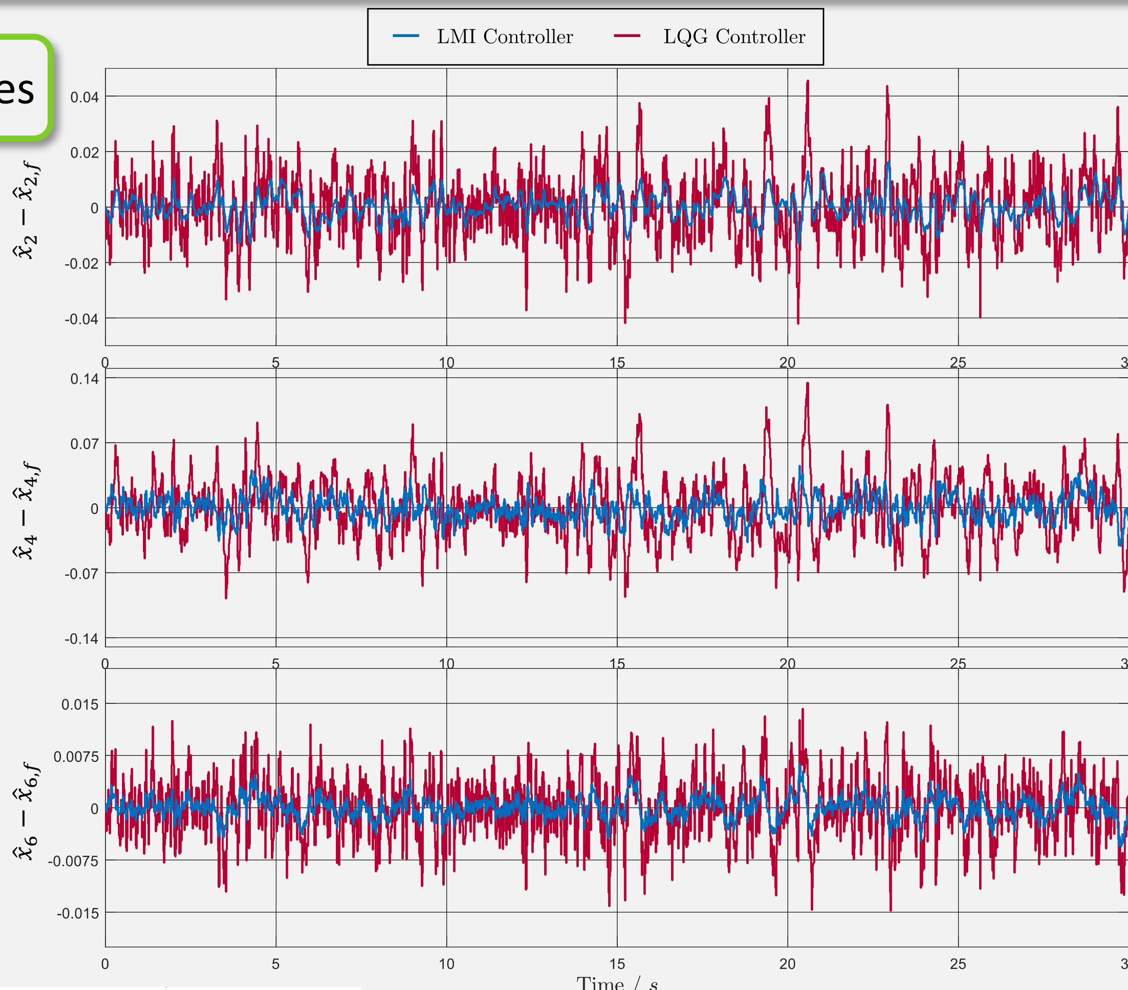
Cost function:

RMSE-Values

$$\Delta\hat{x}_{i,f} = \sqrt{\frac{1}{N}\sum_{k=1}^{N}\left(\hat{x}_i\left[k\right] - \hat{x}_{i,f}\left[k\right]\right)^2}$$

from all estimates $\hat{x}_i$

to ideal noise-free estimates $\hat{x}_{i,f}$



Evaluation:

Non-measurable states	$\Delta\hat{x}_{2,f}$	$\Delta\hat{x}_{4,f}$	$\Delta\hat{x}_{6,f}$
LQG	0.0124	0.0333	0.0046
LMI	0.0046	0.0133	0.0016
Improvement	**63 %**	**60 %**	**65 %**

Significant improvements in noise reduction compared to the LQG controller

Summary and Outlook

- Iterative LMI design method for observer-based state feedback controller subject to stochastic noise

- Advantages of the method:

  - Closed-loop less sensitive to noise compared to LQG

  - Provides control parameters and a proof of stability (for deterministic part)

  - Consideration of uncertainties and non-linearities by polytopic representations

  - Various control structures with identical LMI conditions possible

- Further work:

  - Controller design for real mechatronic systems

  - Dealing with the non-unique nature of the quasilinear form and the polytopic representation

Thank you for your attention

Results published in:

1 R. Dehnert, M. Damaszek, S. Lerch, A. Rauh and B. Tibken, ”Robust Feedback Control for Discrete-Time Systems Based on Iterative LMIs with Polytopic Uncertainty Representations subject to Stochastic Noise”, in Frontiers in Control Engineering, Vol. 2, 2022.

Selection of preliminary work:

2 Rauh, A.; Romig S.; Aschemann H. When Is Naive Low-Pass Filtering of Noisy Measurements Counter-Productive for the Dynamics of Controlled Systems?, in 2018 23rd International Conference on Methods Models in Automation Robotics (MMAR), Miedzyzdroje, Poland, August 27–30, 2018, 809–814.

3 Rauh, A.; Romig, S. Linear Matrix Inequalities for an Iterative Solution for the Robust Output Feedback Control of Systems with Bounded and Stochastic Uncertainty. Sensors 2021, 21, 3285. https://doi.org/10.3390/s21093285

4 Rauh, A.; Dehnert, R.; Romig, S.; Lerch, S.; Tibken, B. Iterative Solution of Linear Matrix Inequalities for the Combined Control and Observer Design of Systems with Polytopic Parameter Uncertainty and Stochastic Noise. Algorithms 2021, 14, 205. https:// doi.org/10.3390/a14070205

5 R. Dehnert, S. Lerch, T. Grunert, M. Damaszek and B. Tibken, ”A Less Conservative Iterative LMI approach for Output Feedback Controller Synthesis for Saturated Discrete-Time Linear Systems”, Proceedings of the 25th International Conference on System Theory, Control and Computing, Iasi, Romania, October 20-23, 2021