Intl. Online Seminar on Interval Methods in Control Engineering
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Fixed-Time Interval Observer Design Using an Artificial Delay: A Study of Discrete-Time Case

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Outline

Introduction

Remarks

- In applied mathematics, using time may help to overcome obstacles
- Using an artificial delay: intentionally inserting a delay in the observer design of a system, without the system having to rely on past inputs or states
- We will review two families of systems where using an artificial delay can be useful for fixed-time observer design

Context and Goal

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No appropriate knowledge of the initial conditions is known + no monotonicity property

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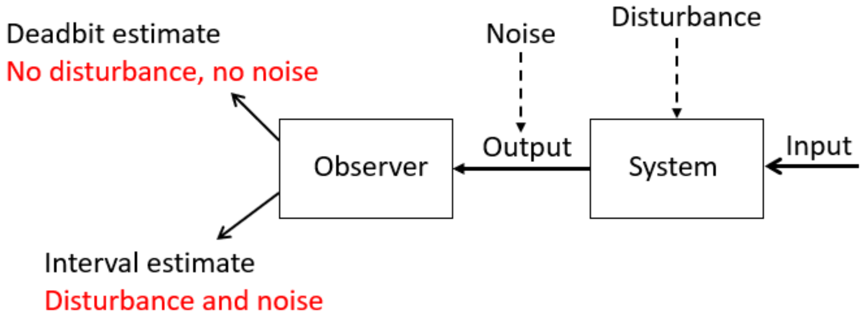
Goal:

- If known disturbances (≡no disturbances) $\cdots$ exact estimate after h
- If unknown disturbances $\cdots$ interval estimate $\cdots$ $\forall k >$ fixed-time h

  cf. Classical observers = Asymptotic estimate $\cdots$ $k \to \infty$

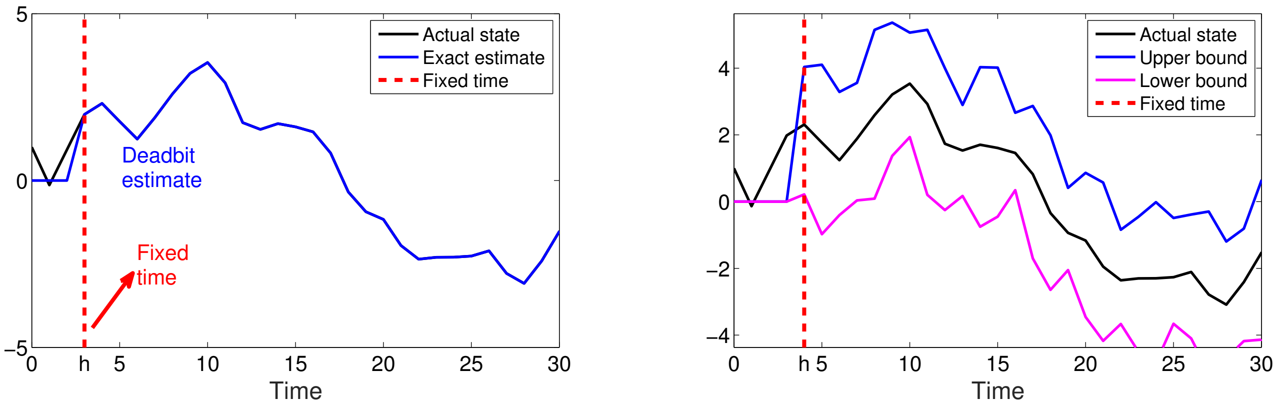
  cf. Interval observers = Two bounds $\forall k$ $\cdots$
  $\cdots$ known bound of initial condition + nonnegativity properties



What are deadbit estimate & fixed-time interval estimate?

- Exact estimation after a fixed time which can be tuned in the absence of disturbances

- Interval after a fixed time which can be tuned in the presence of disturbances under assumptions of knowing bounds on uncertain terms



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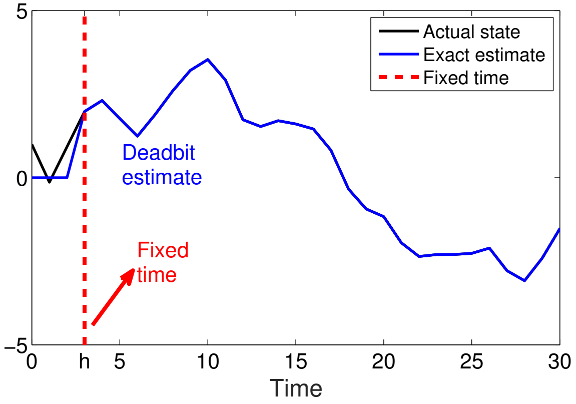

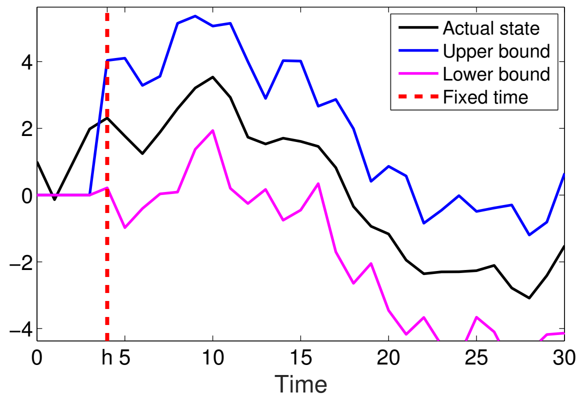

Use of past values of the input and the output of the studied system (+ Frames of disturbances) = deadbit($\rightarrow$ interval) estimate

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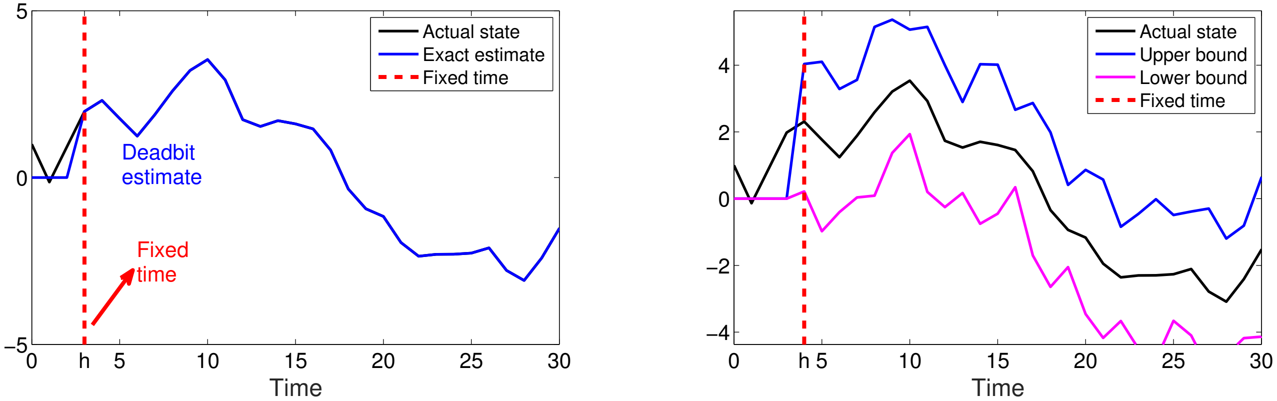

Use of past values of the input and the output of the studied system (+ Frames of disturbances) = deadbit($\rightarrow$ interval) estimate

R. Engel, G. Kreisselmeier, A Continuous-Time Observer Which Converges in Finite Time. IEEE Trans. on Automatic Control, 47, pp. 1202-1204, 2002

Advantages of proposed interval technique

Classical designs		Proposed technique
×	No monotonicity property	✓
×	No information on the bound of the initial conditions	✓

Part I: Time-Invariant Discrete-Time Systems*

*T.N. Dinh, F. Mazenc, T. Raïssi, "Finite-time guaranteed state estimation for discrete-time systems with disturbances," in Proceedings of the 4th conference on Control and Fault-Tolerant Systems, Casablanca, Morocco, pp. 342-347, 2019

Notations & Lemmas:

- Operators $\leq$ must be understood componentwise
- $Q \in \mathbb{R}^{n\times n}$, $Q^+ = \max\{Q, 0\}$, $Q^- = Q^+ - Q$, $|Q| = Q^+ + Q^-$
- Let $d \in \mathbb{R}^n$ be a vector such that $\underline{d} \leq d \leq \overline{d}$. If $\mathcal{M} \in \mathbb{R}^{m\times n}$ constant, then $\mathcal{M}^+\underline{d} - \mathcal{M}^-\overline{d} \leq \mathcal{M}d \leq \mathcal{M}^+\overline{d} - \mathcal{M}^-\underline{d}$
- For any nonlinear discrete-time system with disturbances $x(k+1) = Ax(k) + F(u(k), y(k)) + d(k)$, we can relate the state at time m_1 to the state at an earlier time m_2 as follows

$$x(m_1) = A^{m_1-m_2}x(m_2) + \sum_{\ell=m_2}^{m_1-1} A^{m_1-\ell-1}[F(u(\ell), y(\ell)) + d(\ell)]$$

Considered system

Consider the following discrete-time system

$$(\mathcal{S}) : \begin{cases} x(k+1) = Ax(k) + F(u(k), y(k)) + d(k) \\ y(k) = Cx(k) + v(k) \end{cases}$$

Assumptions:

- $\underline{d} \leq d(k) \leq \overline{d}$, $\underline{v} \leq v(k) \leq \overline{v}$ $\rightarrow$ realistic
- The pair (A, C) is observable and A is invertible

(A, C) observable $\rightarrow \exists\ L \in \mathbb{R}$ s.t. $H = A + LC$ admits a spectral radius smaller than the modulus of any eigenvalue of A

$\Rightarrow$ we can prove that there is an integer $h > 0$ such that the matrix

$$H^{-h} - A^{-h}$$

is invertible $\rightarrow$ we define $E_h = \left(H^{-h} - A^{-h}\right)^{-1}$

A is not invertible $\rightarrow$ decompose $Ax + F(u, y)$ in an alternative way so that the new matrix A is invertible

Key idea

$H = A + LC$, $y = Cx + v$ $\rightarrow$ two equivalent representations of $(\mathcal{S})$

$$x(k+1) = Ax(k) + F(u(k), y(k)) + d(k),$$
$$x(k+1) = Hx(k) + F(u(k), y(k)) - Ly(k) + d(k) + Lv(k)$$

Use of past values of the input and the output

Key idea

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Use of past values of the input and the output

Recall $x(m_1) = A^{m_1-m_2}x(m_2) + \sum_{\ell=m_2}^{m_1-1} A^{m_1-\ell-1}[F(u(\ell), y(\ell)) + d(\ell)]$

$$x(k) = A^h x(k-h) + \sum_{\ell=k-h}^{k-1} A^{k-\ell-1}[F(u(\ell), y(\ell)) + d(\ell)]$$
$$x(k) = H^h x(k-h) + \sum_{\ell=k-h}^{k-1} H^{k-\ell-1}[F(u(\ell), y(\ell)) - Ly(\ell) + d(\ell) + Lv(\ell)]$$

Key idea

$H = A + LC$, $y = Cx + v$ $\rightarrow$ two equivalent representations of $(\mathcal{S})$

$$x(k+1) = Ax(k) + F(u(k), y(k)) + d(k),$$
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Recall $x(m_1) = A^{m_1-m_2}x(m_2) + \sum_{\ell=m_2}^{m_1-1} A^{m_1-\ell-1}[F(u(\ell), y(\ell)) + d(\ell)]$

$$A^{-h}x(k) = x(k-h) + \sum_{\ell=k-h}^{k-1} A^{k-\ell-1-\mathbf{h}}[F(u(\ell), y(\ell)) + d(\ell)]$$

$$H^{-h}x(k) = x(k-h) + \sum_{\ell=k-h}^{k-1} H^{k-\ell-1-\mathbf{h}}[F(u(\ell), y(\ell)) - Ly(\ell) + d(\ell) + Lv(\ell)]$$

$$\Rightarrow \underbrace{\left(H^{-h} - A^{-h}\right)}_{E_h^{-1}} x(k) = \sum_{\ell=k-h}^{k-1} H^{k-\ell-1-\mathbf{h}}[\cdots] - \sum_{\ell=k-h}^{k-1} A^{k-\ell-1-\mathbf{h}}[\cdots]$$

Exact estimation when disturbances are known

Theorem 1. *Let $L \in \mathbb{R}^{n\times q}$ and $h \in \mathbb{N}$, $h \geq 1$ be such that the matrix $H^{-h} - A^{-h}$ is invertible. Then, for a given input $u(k)$, any solution $x(k)$ of the system $(\mathcal{S})$ which exists over $\mathbb{N}$ satisfies, for all $k \geq h$,*

$$
\begin{aligned}
x(k) = & -E_h \sum_{\ell=k-h}^{k-1} A^{k-h-\ell-1} F(u(\ell), y(\ell)) \\
& +E_h \sum_{\ell=k-h}^{k-1} H^{k-h-\ell-1} [F(u(\ell), y(\ell)) - Ly(\ell)] \\
& -E_h \sum_{\ell=k-h}^{k-1} A^{k-h-\ell-1} d(\ell) + E_h \sum_{\ell=k-h}^{k-1} H^{k-h-\ell-1} (d(\ell) + Lv(\ell))
\end{aligned}
$$

with $E_h = \left(H^{-h} - A^{-h}\right)^{-1}$

Drawback: The formula may contain many terms because h may be large and thus many values have to be stored $\rightarrow$ an alternative solution which is based on dynamic extensions

Deadbit estimate

Theorem 2. *Consider the dynamic extensions*

$$\widehat{x}(k+1) = A\widehat{x}(k) + F(u(k), y(k)) + d(k)$$

and

$$x_*(k+1) = Hx_*(k) + F(u(k), y(k)) - Ly(k) + d(k) + Lv(k).$$

Consider a solution $x(k)$ *of* $(\mathcal{S})$ *defined over* $\mathbb{N}$*. Then, for all* $k \geq h$*,*

$$x(k) = E_h \left[H^{-h} x_*(k) - x_*(k-h) - A^{-h}\widehat{x}(k) + \widehat{x}(k-h) \right]$$

Remark 1. *Notice that* x_* *is a classical observer for the system* $(\mathcal{S})$ *when disturbances are known*

Approximate estimation when disturbances are unknown

Recall $\mathcal{M}^{+}\underline{d} - \mathcal{M}^{-}\overline{d} \leq \mathcal{M}d \leq \mathcal{M}^{+}\overline{d} - \mathcal{M}^{-}\underline{d}$

Theorem 3. *For all integer $k \geq h$,*

$$\underline{x}(k) \leq x(k) \leq \overline{x}(k)$$

with

$$\begin{aligned}
\overline{x}(k) = & -E_h \sum_{\ell=k-h}^{k-1} A^{k-h-\ell-1} F(u(\ell), y(\ell)) \\
& + E_h \sum_{\ell=k-h}^{k-1} H^{k-h-\ell-1} [F(u(\ell), y(\ell)) - Ly(\ell)] \\
& + \left(\sum_{\ell=1}^{h} F_h A^{\ell} + G_h H^{\ell} \right)^{+} \overline{d} - \left(\sum_{\ell=1}^{h} F_h A^{\ell} + G_h H^{\ell} \right)^{-} \underline{d} \\
& + \left(G_h \sum_{\ell=1}^{h} H^{\ell} L \right)^{+} \overline{v} - \left(G_h \sum_{\ell=1}^{h} H^{\ell} L \right)^{-} \underline{v}
\end{aligned}$$

where $F_h = -E_h A^{-(h+1)}, \quad G_h = E_h H^{-(h+1)}$

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where $F_h = -E_h A^{-(h+1)}, \quad G_h = E_h H^{-(h+1)}$

Interval estimate

Theorem 4. *Let us introduce several dynamic extensions:*

$$z_a(k+1) = Az_a(k) + F(u(k), y(k)),$$
$$z_h(k+1) = Hz_h(k) + F(u(k), y(k)) - Ly(k),$$

Then, for all $k \geq h$, the inequalities

$$\underline{\Upsilon}(Z_k) \leq x(k) \leq \overline{\Upsilon}(Z_k),$$

with $Z = (z_a, z_h)$ and the bounds $\overline{\Upsilon}$, $\underline{\Upsilon}$ are an estimated interval for the system $(\mathcal{S})$ given by

$$\overline{\Upsilon}(Z_k) = E_h[z_a(k-h) - A^{-h}z_a(k) + H^{-h}z_h(k) - z_h(k-h)]$$
$$+ \left(\sum_{\ell=1}^{h} F_h A^\ell + G_h H^\ell\right)^+ \overline{d} - \left(\sum_{\ell=1}^{h} F_h A^\ell + G_h H^\ell\right)^- \underline{d}$$
$$+ \left(G_h \sum_{\ell=1}^{h} H^\ell L\right)^+ \overline{v} - \left(G_h \sum_{\ell=1}^{h} H^\ell L\right)^- \underline{v}$$

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$$+ \left(G_h \sum_{\ell=1}^{h} H^{\ell}L\right)^{+} \underline{v} - \left(G_h \sum_{\ell=1}^{h} H^{\ell}L\right)^{-} \overline{v}$$

Numerical example

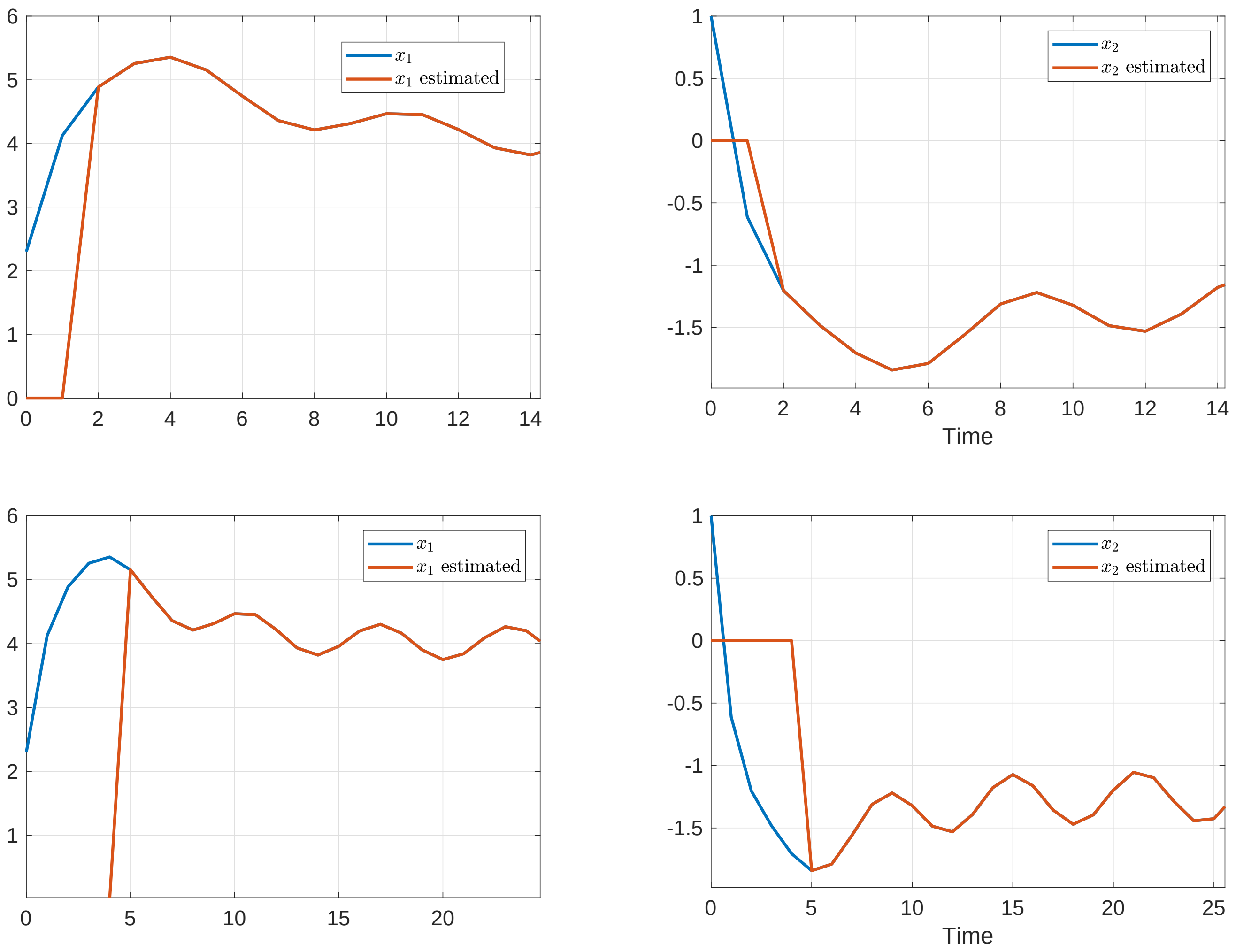
Consider the system

$$x_1(k+1) = \frac{5}{4}x_1(k) + x_2(k) + \frac{1}{4}u_1(k) + \frac{1}{9}\sin(k)$$
$$x_2(k+1) = -\frac{3}{8}x_1(k) + \frac{1}{8}u_2(k) + \frac{1}{9}\sin(k)$$
$$y(k) = x_1(k) + \frac{1}{9}\sin(k^2)$$

The choice $L = \left[-\frac{7}{8}\ \frac{11}{32}\right]^\top$ gives $H = A + LC = \begin{bmatrix} \frac{3}{8} & 1 \\ -\frac{1}{32} & 0 \end{bmatrix}$

We can prove easily that $H^{-h} - A^{-h}$ is invertible for all $h \geq 2 \rightarrow$ All assumptions are satisfied

Simulations



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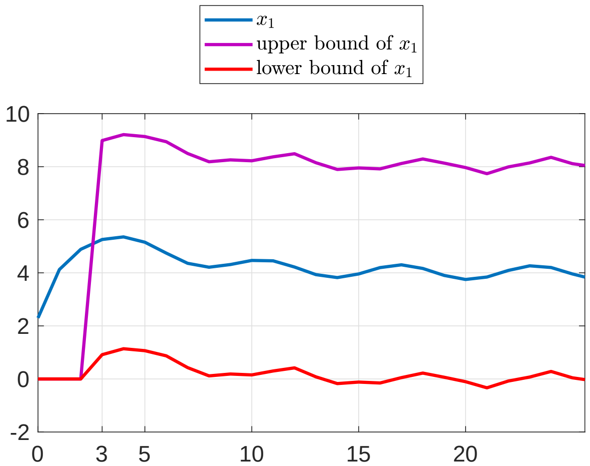

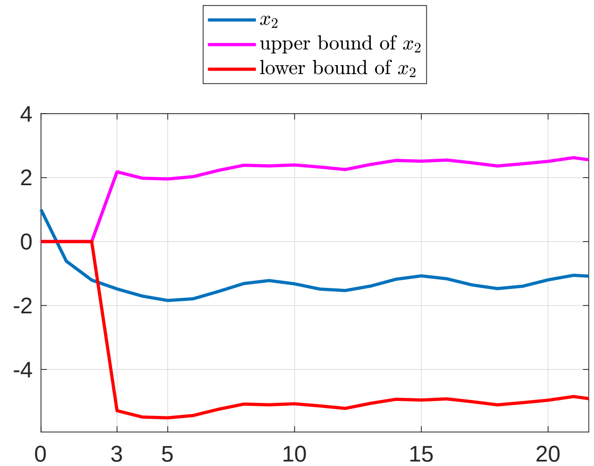

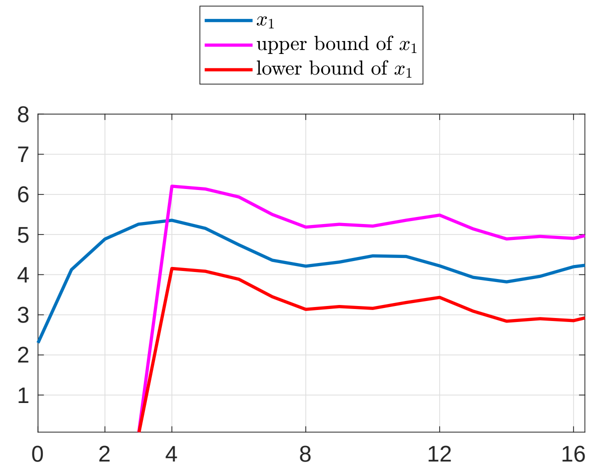

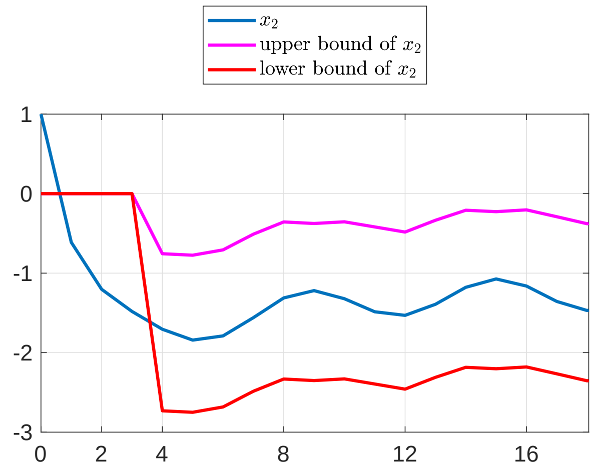

Part II: Time-Varying Discrete-Time Systems*

*T.N. Dinh, F. Mazenc, Z. Wang, T. Raïssi, "On Fixed-Time Interval Estimation of Discrete-Time Nonlinear Time-Varying Systems With Disturbances," in Proceedings of the American Control Conference, Denver, CO, USA, pp. 2605-2610, 2020

Lemma

Consider $x(k+1) = A(k)x(k) + \beta(k, u(k), y(k)) + d(k)$. We can relate the state at time m_1 to the state at an earlier time m_2:

$$x(m_1) = \Phi_A(m_1, m_2)x(m_2) + \sum_{\ell=m_2}^{m_1-1} \Phi_A(m_1, \ell+1)[\beta(\ell, u(\ell), y(\ell)) + d(\ell)]$$

where $\Phi_A(m_1, m_2)$ discrete-time state transition matrix:

$$\Phi_A(m_1, m_2) = \begin{cases} A(m_1-1)A(m_1-2)\cdots A(m_2), & m_1 > m_2 \geq 0 \\ I, & m_1 = m_2 \end{cases}$$

$A(m_1-1), A(m_1-2)..., A(m_2)$ invertible $\rightarrow$ $\Phi_A(m_1, m_2)$ nonsingular

$$\Phi_A(m_1, m_2) = \Phi_A^{-1}(m_2, m_1) \qquad \Phi_A(m_1, \ell)\Phi_A(\ell, m_2) = \Phi_A(m_1, m_2)$$

Considered system

Consider the following discrete-time system

$$(\mathcal{S}_k) : \begin{cases} x(k+1) = A(k)x(k) + \beta(k, u(k), y(k)) + d(k) \\ y(k) = C(k)x(k) + v(k) \end{cases}$$

Assumptions:

(i) $\underline{d} \leq d(k) \leq \overline{d}$, $\underline{v} \leq v(k) \leq \overline{v}$ $\rightarrow$ realistic

(ii) $\forall k$, the pair $(A(k), C(k))$ is observable and $A(k)$ is invertible. Furthermore, there exists $L(k)$ such that the matrix $H(k) = A(k) + L(k)C(k)$ is both Schur stable and invertible

(ii)$\Rightarrow$ we can prove that there exist matrices $L(k), L(k-1)..., L(k-h)$ and an integer $h_* \in \mathbb{N}$ s.t. $\forall h \in \mathbb{N}$, $h > h_*$ the matrix

$$\Phi_H^{-1}(k, k-h) - \Phi_A^{-1}(k, k-h)$$

is invertible $\rightarrow$ we define $E_h(k) = \left(\Phi_H^{-1}(k, k-h) - \Phi_A^{-1}(k, k-h)\right)^{-1}$

$A(k)$ is not invertible $\rightarrow$ decompose $A(k)x + \beta(k, u(k), y(k))$ in an alternative way so that the new matrix $A(k)$ is invertible

Key idea

$H(k) = A(k) + L(k)C(k)$, $y(k) = C(k)x(k) + v(k)$ $\rightarrow$ two equivalent representations of $(\mathcal{S}_k)$

$$x(k+1) = A(k)x(k) + \beta(k, u(k), y(k)) + d(k),$$
$$x(k+1) = H(k)x(k) + \beta(k, u(k), y(k)) - L(k)y(k) + d(k) + L(k)v(k)$$

Use of past values of the input and the output

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Use of past values of the input and the output

Recall $x(m_1) = \Phi_A(m_1, m_2)x(m_2) + \sum_{\ell=m_2}^{m_1-1} \Phi_A(m_1, \ell+1)[\beta(\ell, u(\ell), y(\ell)) + d(\ell)]$

$$x(k) = \Phi_A(k, k-h)x(k-h) + \sum_{\ell=k-h}^{k-1} \Phi_A(k, \ell+1)[\beta(\ell, u(\ell), y(\ell)) + d(\ell)]$$

$$x(k) = \Phi_H(k, k-h)x(k-h) + \sum_{\ell=k-h}^{k-1} \Phi_H(k, \ell+1)[\beta(\ell, u(\ell), y(\ell)) - L(\ell)y(\ell) + d(\ell) + L(\ell)v(\ell)]$$

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Use of past values of the input and the output

Recall $\Phi_A(m_1, m_2) = \Phi_A^{-1}(m_2, m_1)$ $\quad \Phi_A(m_1, \ell)\Phi_A(\ell, m_2) = \Phi_A(m_1, m_2)$

$$\Phi_A^{-1}(k, k-h)x(k) = x(k-h) + \sum_{\ell=k-h}^{k-1} \Phi_A(k\text{-}\mathbf{h}, \ell+1)[\beta(\ell, u(\ell), y(\ell)) + d(\ell)]$$

$$\Phi_H^{-1}(k, k-h)x(k) = x(k-h) + \sum_{\ell=k-h}^{k-1} \Phi_H(k\text{-}\mathbf{h}, \ell+1)[\beta(\ell, u(\ell), y(\ell)) - L(\ell)y(\ell) + d(\ell) + L(\ell)v(\ell)]$$

$$\Rightarrow \underbrace{\left(\Phi_H^{-1}(k, k-h) - \Phi_A^{-1}(k, k-h)\right)}_{E_h^{-1}(k)} x(k) = \sum_{\ell=k-h}^{k-1} \Phi_H(k\text{-}\mathbf{h}, \ell+1)[\cdots] - \sum_{\ell=k-h}^{k-1} \Phi_A(k\text{-}\mathbf{h}, \ell+1)[\cdots]$$

Exact estimation when disturbances are known

Theorem 5. *Let* $L(k) \in \mathbb{R}^{n\times q}$ *and* $h \in \mathbb{N}$, $h \geq 1$ *be such that the matrix* $\Phi_H^{-1}(k,k-h) - \Phi_A^{-1}(k,k-h)$ *is invertible for all* $k \in \mathbb{N}$. *Then, for all* $k \geq h$,

$$\begin{aligned} x(k) = &-E_h(k) \sum_{\ell=k-h}^{k-1} \Phi_A(k-h,\ell+1)\beta(\ell,u(\ell),y(\ell)) \\ &+ E_h(k) \sum_{\ell=k-h}^{k-1} \Phi_H(k-h,\ell+1)[\beta(\ell,u(\ell),y(\ell)) - L(\ell)y(\ell)] \\ &-E_h(k) \sum_{\ell=k-h}^{k-1} \Phi_A(k-h,\ell+1)d(\ell) \\ &+E_h(k) \sum_{\ell=k-h}^{k-1} \Phi_H(k-h,\ell+1)\left(d(\ell)+L(\ell)v(\ell)\right) \end{aligned}$$

with $E_h(k) = \left(\Phi_H^{-1}(k,k-h) - \Phi_A^{-1}(k,k-h)\right)^{-1}$

Drawback: The formula may contain many terms because h may be large and thus many values have to be stored $\rightarrow$ an alternative solution which is based on dynamic extensions

Deadbit estimate

Theorem 6. *Consider the dynamic extensions*

$$\widehat{x}(k+1) = A(k)\widehat{x}(k) + \beta(k, u(k), y(k)) + d(k)$$

and

$$x_*(k+1) = H(k)x_*(k) + \beta(k, u(k), y(k)) - L(k)y(k) + d(k) + L(k)v(k)$$

Consider a solution $x(k)$ *of* $(\mathcal{S}_k)$ *defined over* $\mathbb{N}$*. Then, for all* $k \geq h$*,*

$$x(k) = E_h(k)\left[\Phi_H^{-1}(k, k-h)x_*(k) - x_*(k-h) - \Phi_A^{-1}(k, k-h)\widehat{x}(k) + \widehat{x}(k-h)\right]$$

Remark 2. *Notice that* x_* *is a classical observer for the system* $(\mathcal{S}_k)$ *when disturbances are known*

Approximate estimation when disturbances are unknown

Recall $\mathcal{M}^+\underline{d} - \mathcal{M}^-\overline{d} \leq \mathcal{M}d \leq \mathcal{M}^+\overline{d} - \mathcal{M}^-\underline{d}$

Theorem 7. *For all integer* $k \geq h$, $\underline{x}(k) \leq x(k) \leq \overline{x}(k)$ *with*

$$\begin{aligned}
\overline{x}(k) = &-E_h(k) \sum_{\ell=k-h}^{k-1} \Phi_A(k-h, \ell+1)\beta(\ell, u(\ell), y(\ell)) \\
&+ E_h(k) \sum_{\ell=k-h}^{k-1} \Phi_H(k-h, \ell+1)[\beta(\ell, u(\ell), y(\ell)) - L(\ell)y(\ell)] \\
&+ \left(\sum_{\ell=k-h}^{k-1} F_h(k)\Phi_A(k, \ell+1) + G_h(k)\Phi_H(k, \ell+1) \right)^+ \overline{d} \\
&- \left(\sum_{\ell=k-h}^{k-1} F_h(k)\Phi_A(k, \ell+1) + G_h(k)\Phi_H(k, \ell+1) \right)^- \underline{d} \\
&+ \left(G_h(k) \sum_{\ell=k-h}^{k-1} \Phi_H(k, \ell+1)L \right)^+ \overline{v} - \left(G_h(k) \sum_{\ell=k-h}^{k-1} \Phi_H(k, \ell+1)L \right)^- \underline{v}
\end{aligned}$$

where $F_h(k) = -E_h(k)\Phi_A^{-1}(k, k-h), \quad G_h(k) = E_h(k)\Phi_H^{-1}(k, k-h)$

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& + \left(\sum_{\ell=k-h}^{k-1} F_h(k)\Phi_A(k, \ell+1) + G_h(k)\Phi_H(k, \ell+1) \right)^+ \underline{d} \\
& - \left(\sum_{\ell=k-h}^{k-1} F_h(k)\Phi_A(k, \ell+1) + G_h(k)\Phi_H(k, \ell+1) \right)^- \overline{d} \\
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Interval estimate

Theorem 8. *Let us introduce several dynamic extensions:*

$$z_a(k+1) = A(k)z_a(k) + \beta(k,u(k),y(k))$$
$$z_h(k+1) = H(k)z_h(k) + \beta(k,u(k),y(k)) - L(k)y(k)$$

Then, for all $k \geq h$, the inequalities

$$\underline{\Upsilon}(Z_k) \leq x(k) \leq \overline{\Upsilon}(Z_k),$$

with $Z = (z_a, z_h)$ and the bounds $\overline{\Upsilon}$, $\underline{\Upsilon}$ are an estimated interval for the system $(\mathcal{S}_k)$ given by

$$\begin{aligned}\overline{\Upsilon}(Z_k) = {} & E_h(k)[z_a(k-h) - \Phi_A^{-1}(k,k-h)z_a(k) + \Phi_H^{-1}(k,k-h)z_h(k) - z_h(k-h)] \\ & + \left(\sum_{\ell=k-h}^{k-1} F_h(k)\Phi_A(k,\ell+1) + G_h(k)\Phi_H(k,\ell+1)\right)^+ \overline{d} \\ & - \left(\sum_{\ell=k-h}^{k-1} F_h(k)\Phi_A(k,\ell+1) + G_h(k)\Phi_H(k,\ell+1)\right)^- \underline{d} \\ & + \left(G_h(k)\sum_{\ell=k-h}^{k-1} \Phi_H(k,\ell+1)L\right)^+ \overline{v} - \left(G_h(k)\sum_{\ell=k-h}^{k-1} \Phi_H(k,\ell+1)L\right)^- \underline{v}\end{aligned}$$

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$$\begin{aligned}\underline{\Upsilon}(Z_k) = E_h(k)[z_a(k-h) - \Phi_A^{-1}(k, k-h)z_a(k) + \Phi_H^{-1}(k, k-h)z_h(k) - z_h(k-h)]\\ + \left(\sum_{\ell=k-h}^{k-1} F_h(k)\Phi_A(k, \ell+1) + G_h(k)\Phi_H(k, \ell+1)\right)^+ \underline{d}\\ - \left(\sum_{\ell=k-h}^{k-1} F_h(k)\Phi_A(k, \ell+1) + G_h(k)\Phi_H(k, \ell+1)\right)^- \overline{d}\\ + \left(G_h(k)\sum_{\ell=k-h}^{k-1} \Phi_H(k, \ell+1)L\right)^+ \underline{v} - \left(G_h(k)\sum_{\ell=k-h}^{k-1} \Phi_H(k, \ell+1)L\right)^- \overline{v}\end{aligned}$$

Numerical example

Consider the system borrowed from [Zhong et al., 2010] with

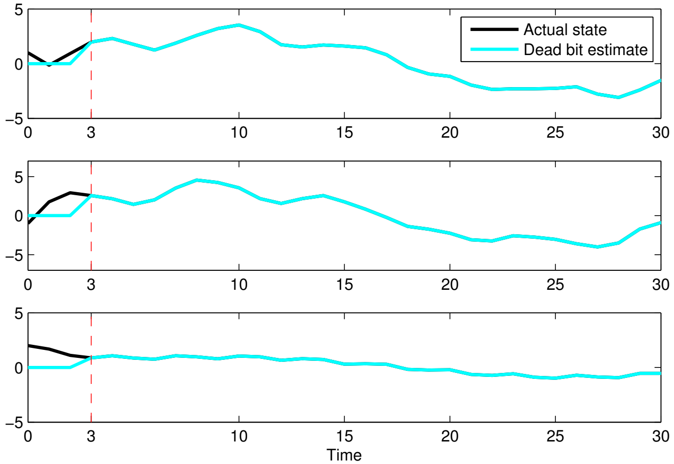
$$A(k) = \begin{bmatrix} 0.2e^{-\frac{k}{100}} & 0.6 & 0 \\ 0 & 0.5 & \sin(k) \\ 0 & 0 & 0.7 \end{bmatrix}, \beta(u(k), y(k)) = \begin{bmatrix} 0.2\sin(0.2k) \\ 1.8\sin(0.2k) \\ 0.3\sin(0.2k) \end{bmatrix},$$

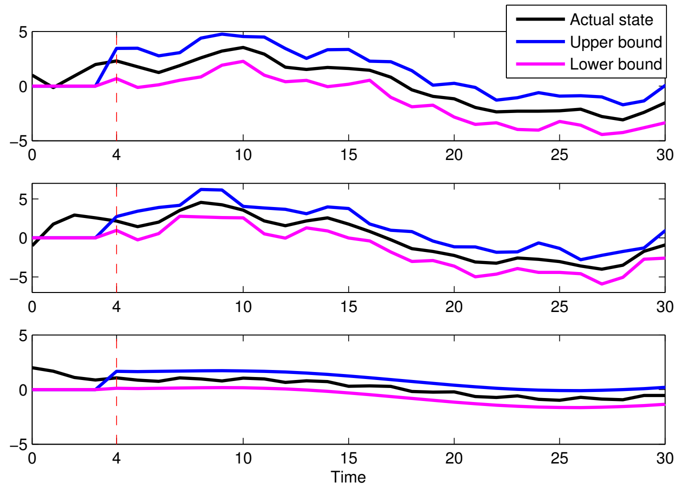
$$d(k) = \begin{bmatrix} \frac{1}{4}\sin(2k) \\ \frac{1}{4}\sin(2k) \\ \frac{1}{4}\sin(2k) \end{bmatrix} \quad \text{and} \quad v(k) = 0.3\cos(k)$$

Choose $L(k) = \left[\frac{1}{4}\ 0\ 0\right]^\top$. We can prove easily that $A(k)$, $H(k) = A(k)+L(k)C(k)$ and $E_h(k) = \Phi_H^{-1}(k, k-h) - \Phi_A^{-1}(k, k-h)$ are invertible for $\forall k \in \mathbb{N}$, $h = \{3, 4\}$ $\rightarrow$ All assumptions are satisfied

M. Zhong, D. Zhou, S.X. Ding, On Designing H_∞ Fault Detection Filter for Linear Discrete Time-Varying Systems, IEEE Transactions on Automatic Control, 55, pp. 1689-1695, 2010

Simulations





Application to Switched Linear DT Systems*

*L. Dadi, T.N. Dinh, T. Raïssi, H. Ethabet, M. Aoun, "New Finite-Time Observers design for a Discrete-Time Switched Linear System," in Proceedings of the 1st IFAC Workshop on Control of Complex Systems , Bologna, Italy, pp. 73-78, 2022

Problem statement

Consider the following discrete-time switched linear system

$$(\mathcal{S}_q) : \begin{cases} x(k+1) = A_q x(k) + B_q u(k) + d(k) \\ y(k) = C_q x(k) + v(k) \end{cases}$$

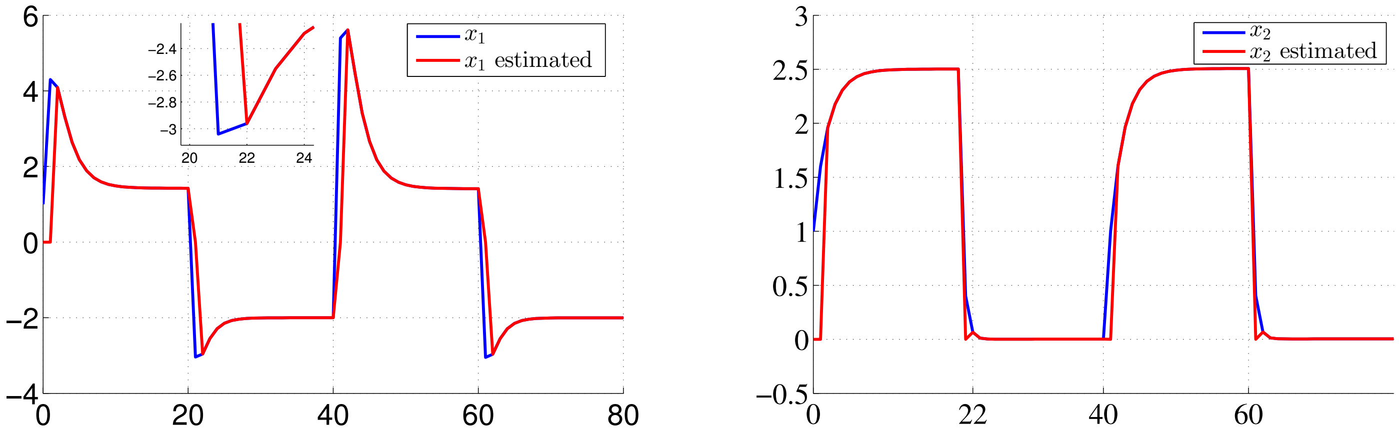
Assumptions:

- Minimal dwell time: $\exists$ known τ_a s.t. $\forall q \in \overline{1,N}, t_q - t_{q-1} \geq \tau_a$
- $\underline{d} \leq d(k) \leq \overline{d}$, $\underline{v} \leq v(k) \leq \overline{v} \rightarrow$ realistic
- $\forall q \in \overline{1,N}$, the pair (A_q, C_q) is observable and A_q is invertible
- $\exists h > 0$ and $h < \tau_a$ such that the matrix

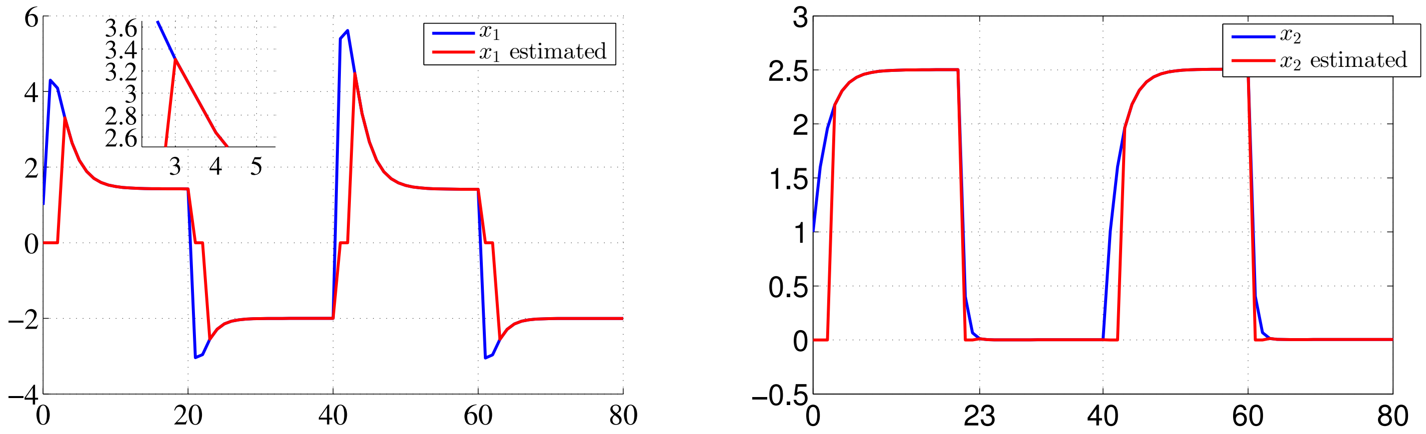
  $$H_q^{-h} - A_q^{-h}$$

  is invertible

Simulations

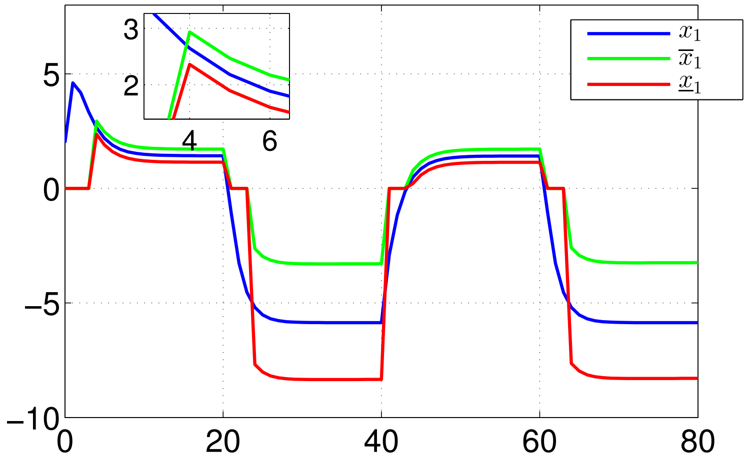


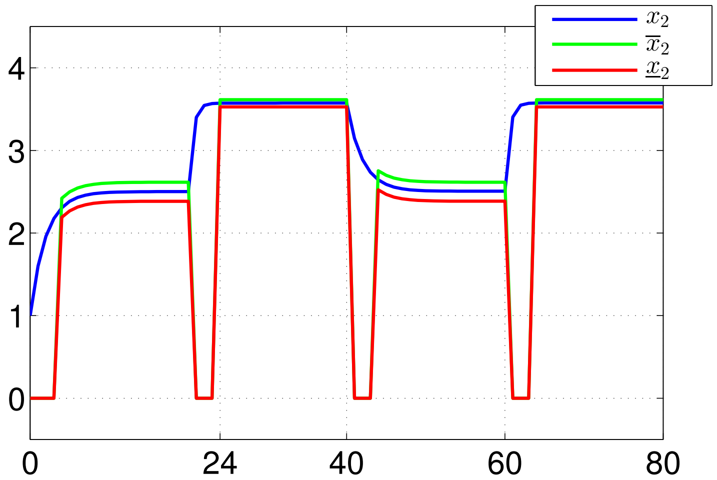
$h = 2$



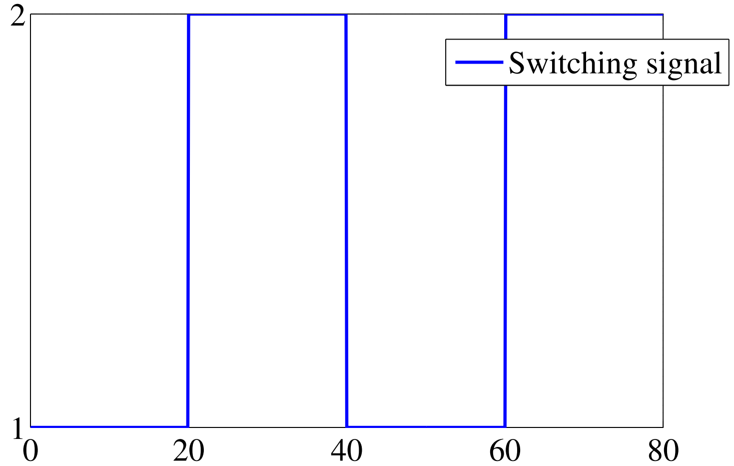
$h = 3$

Simulations





$h = 4$



Conclusion and Perspectives

Conclusion and Perspectives

- The key idea relies on the use of past values of the input and the output of the studied system
- Neither information on the bound of the initial conditions nor monotonicity property is needed in our development
- Providing exact values of the solutions in the absence of disturbances and a lower and upper bounds when the disturbances are present after a fixed time which can be tuned
- Future work:
  - Extension to more general families of nonlinear discrete-time systems can be considered
  - Adding unknown input
  - Observer-based feedback is also expected

Thank you for your attention ☺!