International Online Seminar on Interval Methods in Control Engineering

Encompassing computation of the ellipsoidal image, in the singular case

Morgan Louédec

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Context - Set propagation

Principe:
Set of initial states $\mathcal{S}_0 \subset \mathbb{R}^n$
Nonlinear mapping $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^n$
Enclosing Set $\mathcal{S}_{\text{out}} \subset \mathbb{R}^n$ such that
$\boldsymbol{g}(\mathcal{S}_0) \subseteq \mathcal{S}_{\text{out}}$

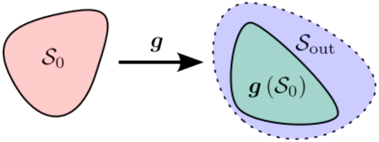


Figure 1: Set representation

Interest:

- guaranteed prediction algorithm
- a tool for mathematical proofs

Limitations:

- wrapping effect
- computational complexity

Common shapes of sets

- boxes
- zonotopes
- ellipsoids

Context - Ellipsoids

Definition

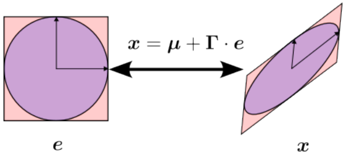
A non-degenerated ellipsoid is a subset of $\mathbb{R}^n$ described by the quadratic form

$$\mathcal{E}(\boldsymbol{\mu}, \boldsymbol{\Gamma}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n | (\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Gamma}^{-T} \boldsymbol{\Gamma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \leq 1 \right\} \quad (1)$$

with $\boldsymbol{\Gamma} \in \mathbb{R}^{n \times n}$, the center $\boldsymbol{\mu} \in \mathbb{R}^n$ and the positive definite matrix $\boldsymbol{\Gamma}\boldsymbol{\Gamma}^T$.

An ellipsoid is an affine transformation of the unit sphere:

$$\begin{aligned} \mathcal{E}(\boldsymbol{\mu}, \boldsymbol{\Gamma}) &= \{ \boldsymbol{x} \in \mathbb{R}^n | \exists \boldsymbol{e} \in \mathbb{R}^n, \boldsymbol{x} = \boldsymbol{\mu} + \boldsymbol{\Gamma} \cdot \boldsymbol{e}, \|\boldsymbol{e}\|_2 \leq 1 \} \\ &= \boldsymbol{\mu} + \boldsymbol{\Gamma} \cdot \mathcal{E}(0, \boldsymbol{I}_n) \quad (2) \end{aligned}$$



Existing method - Linearisation

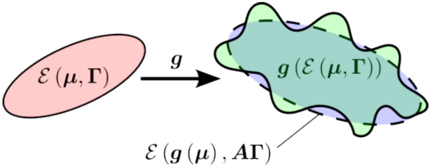


Figure 3: Estimation by linearisation

Propagation by linearisation at the point $\boldsymbol{\mu}$:

$$\boldsymbol{\mu}_l = \boldsymbol{g}(\boldsymbol{\mu}) \tag{3}$$

$$\boldsymbol{\Gamma}_l = \boldsymbol{A} \cdot \boldsymbol{\Gamma} \tag{4}$$

$$\boldsymbol{A} = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}(\boldsymbol{\mu}) \tag{5}$$

Propagation by extended Kalman filter

$$\hat{\boldsymbol{x}}_{k+1} = \boldsymbol{g}(\hat{\boldsymbol{x}}_k) \tag{6}$$

$$\boldsymbol{G}_{k+1} = \boldsymbol{A}_k \cdot \boldsymbol{G}_k \cdot \boldsymbol{A}_k^T + \boldsymbol{G}_\alpha \tag{7}$$

$$\boldsymbol{A}_k = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}}_k) \tag{8}$$

Resemblance

$$\boldsymbol{\mu} \leftrightarrow \hat{\boldsymbol{x}}_k \tag{9}$$

$$\boldsymbol{\Gamma}\boldsymbol{\Gamma}^T \leftrightarrow \boldsymbol{G}_k \tag{10}$$

Existing method - Theorem

Theorem - Rauh et al. 2022 [1]

Consider the ellipsoid $\mathcal{E}(\boldsymbol{\mu}, \boldsymbol{\Gamma})$ and the nonlinear mapping $\boldsymbol{g}$. The matrix $\boldsymbol{A} = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}(\boldsymbol{\mu})$ is supposed invertible. A set enclosing $\boldsymbol{g}(\mathcal{E}(\boldsymbol{\mu}, \boldsymbol{\Gamma}))$ and written $\mathcal{E}(\boldsymbol{\mu}_{\text{out}}, \boldsymbol{\Gamma}_{\text{out}})$ is given by

$$\boldsymbol{\Gamma}_{\text{out}} = (1 + \rho) \cdot \boldsymbol{A} \cdot \boldsymbol{\Gamma} \tag{11}$$

$$\boldsymbol{\mu}_{\text{out}} = \boldsymbol{g}(\boldsymbol{\mu}) \tag{12}$$

where

$$\rho = \max_{\|\tilde{\boldsymbol{x}}\| \leq 1} \left\| \tilde{\boldsymbol{b}}(\tilde{\boldsymbol{x}}) \right\| \tag{13}$$

$$\tilde{\boldsymbol{b}}(\tilde{\boldsymbol{x}}) = \boldsymbol{\Gamma}^{-1} \cdot \boldsymbol{A}^{-1} \left(\boldsymbol{g}(\boldsymbol{\Gamma} \cdot \tilde{\boldsymbol{x}} + \boldsymbol{\mu}) - \boldsymbol{g}(\boldsymbol{\mu}) \right) - \tilde{\boldsymbol{x}} \tag{14}$$

Existing method - Illustration

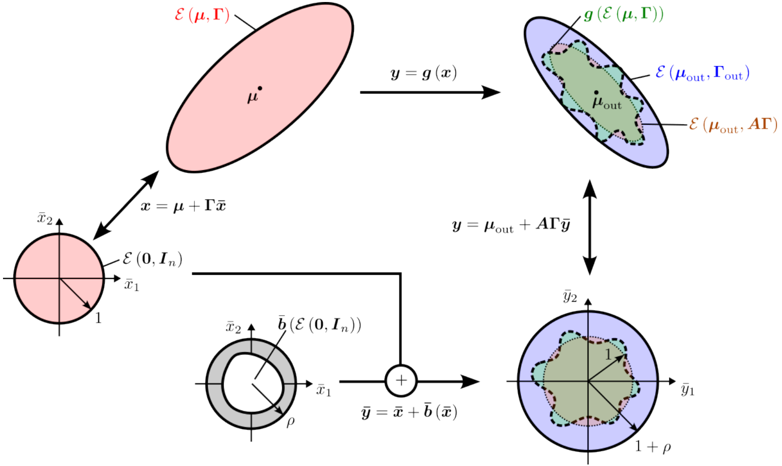


Figure 4: Theorem illustration

Singularity

When $\boldsymbol{A\Gamma}$ is not invertible:

- The ellipsoid $\mathcal{E}\left(\boldsymbol{\mu}_{\text{out}}, \boldsymbol{A\Gamma}\right)$ is *degenerated*
- Inflating the ellipsoid won't guaranty the enclosure $\boldsymbol{g}\left(\mathcal{E}\left(\boldsymbol{\mu}, \boldsymbol{\Gamma}\right)\right)$

Common Examples:

- Projections
- Fluid damping (quadratic)

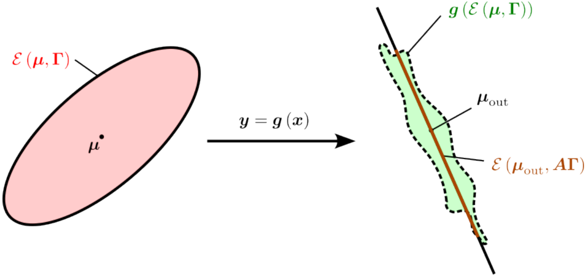


Figure 5: Singularity example

Singularity - Degenerated ellipsoid

No quadratic form, $\boldsymbol{\Gamma}$ not invertible but there is an affine transformation:

$$\mathcal{E}(\boldsymbol{\mu}, \boldsymbol{\Gamma}) = \boldsymbol{\mu} + \boldsymbol{\Gamma} \cdot \mathcal{E}(0, \boldsymbol{I}_n) \tag{15}$$

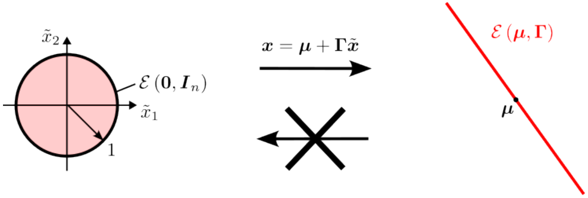


Figure 6: Degenerated example

Singularity - Investigated solution

Start from $\mathcal{E}(\boldsymbol{\mu}_{\text{out}}, \boldsymbol{A\Gamma})$ to find an ellipsoid $\mathcal{E}(\boldsymbol{\mu}_{\text{out}}, \boldsymbol{\Gamma}_s)$ similar to $\boldsymbol{g}(\mathcal{E}(\boldsymbol{\mu}, \boldsymbol{\Gamma}))$

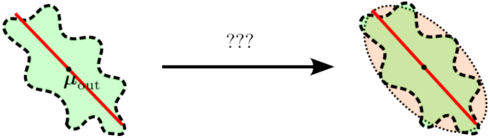


Figure 7: Find lost dimensions

Particular case, planar projection: one can keep the degenerated ellipsoid $\mathcal{E}(\boldsymbol{\mu}_{\text{out}}, \boldsymbol{A\Gamma})$

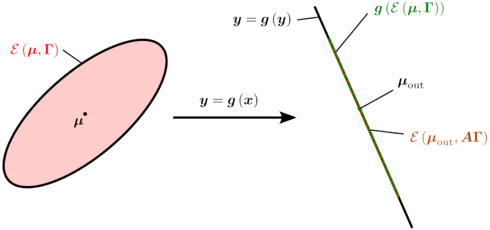


Figure 8: Projection example

Singularity - Singular Value Decomposition

Singular value decomposition

$$\boldsymbol{A}\boldsymbol{\Gamma} = \boldsymbol{U} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{V}^T \qquad (16)$$

with $\boldsymbol{U} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{V} \in \mathbb{R}^{n \times n}$ orthonormals and $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$ diagonal.

The diagonal elements of $\boldsymbol{\Sigma}$ are the singular values $\sigma_i \geq 0$.

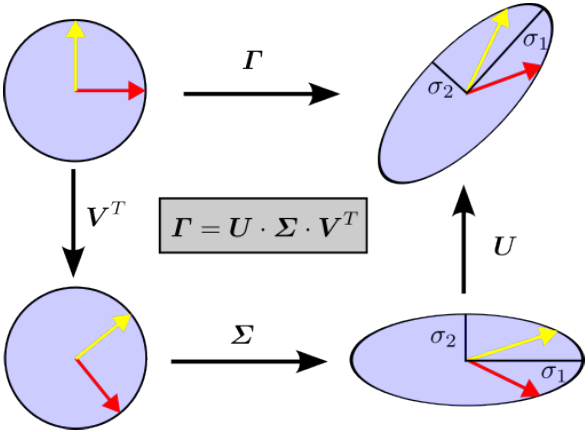


Figure 9: Example in 2 dimensions

Singularity - Adding singular values

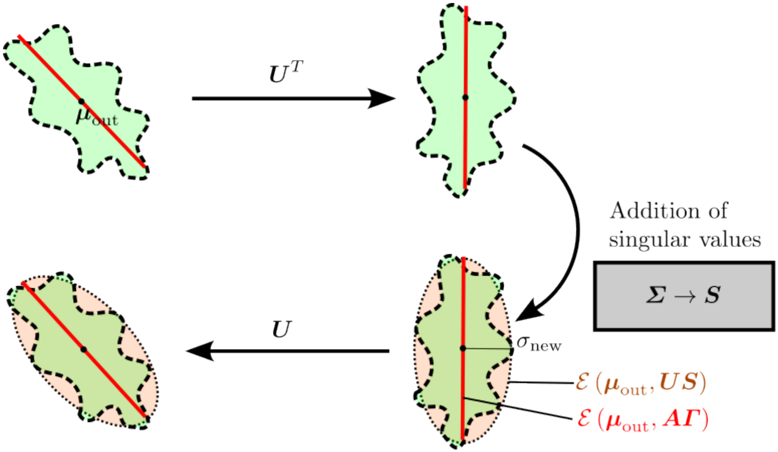


Figure 10: Meet the shape of the set

Singularity - Proposed solution

$\mathcal{E}\left(\boldsymbol{\mu}_{\text{out}}, \boldsymbol{\Gamma}_{\text{out}}\right)$ is given by

$$\boldsymbol{\Gamma}_{\text{out}} = (1+\rho)\cdot \boldsymbol{\Gamma}_s \qquad (17)$$

$$\boldsymbol{\mu}_{\text{out}} = \boldsymbol{g}\left(\boldsymbol{\mu}\right) \qquad (18)$$

where

$$\rho = \max_{\|\tilde{\boldsymbol{x}}\|\leq 1} \left\|\tilde{\boldsymbol{b}}\left(\tilde{\boldsymbol{x}}\right)\right\| \qquad (19)$$

$$\tilde{\boldsymbol{b}}\left(\tilde{\boldsymbol{x}}\right) = \boldsymbol{W}\left(\boldsymbol{g}\left(\boldsymbol{\Gamma}\cdot \boldsymbol{e} + \boldsymbol{\mu}\right) - \boldsymbol{g}\left(\boldsymbol{\mu}\right)\right) - \boldsymbol{Z}\tilde{\boldsymbol{x}} \qquad (20)$$

and where the matrices $\boldsymbol{\Gamma}_s$, $\boldsymbol{W}$ and $\boldsymbol{Z}$ from $\mathbb{R}^{n\times n}$ are given by:

- (General case), if $\boldsymbol{A\Gamma}$ is invertible, then

$$\boldsymbol{\Gamma}_s = \boldsymbol{A\Gamma} \qquad (21)$$

$$\boldsymbol{W} = \boldsymbol{\Gamma}_s^{-1} \qquad (22)$$

$$\boldsymbol{Z} = \boldsymbol{I}_n \qquad (23)$$

Singularity - Proposed solution

- (Singular case), if $\boldsymbol{A\Gamma}$ is not invertible, then

$$\boldsymbol{\Gamma}_s = \boldsymbol{US} \tag{24}$$

$$\boldsymbol{W} = \boldsymbol{S}^* \boldsymbol{U}^T \tag{25}$$

$$\boldsymbol{Z} = \boldsymbol{S}^* \boldsymbol{S} \boldsymbol{V}^T \tag{26}$$

with the diagonal matrices

$$\boldsymbol{S} = \mathrm{diag}\left(s_1, s_2, \ldots, s_n\right) \tag{27}$$

$$\boldsymbol{S}^* = \mathrm{diag}\left(s_1^*, s_2^*, \ldots, s_n^*\right) \tag{28}$$

where

$$s_i = \begin{cases} \sigma_i & \text{if } \sigma_i > 0 \\ \max\limits_{\boldsymbol{x} \in \mathcal{E}(\boldsymbol{\mu}, \boldsymbol{\Gamma})} \left| \boldsymbol{e}_i^T \boldsymbol{U}^T \left(\boldsymbol{g}\left(\boldsymbol{x}\right) - \boldsymbol{g}\left(\boldsymbol{\mu}\right)\right) \right| & \text{if } \sigma_i = 0 \end{cases}$$

$$s_i^* = \begin{cases} 1/s_i & \text{if } s_i \neq 0 \\ 0 & \text{else} \end{cases} \tag{29}$$

Singularity - Proposed Solution

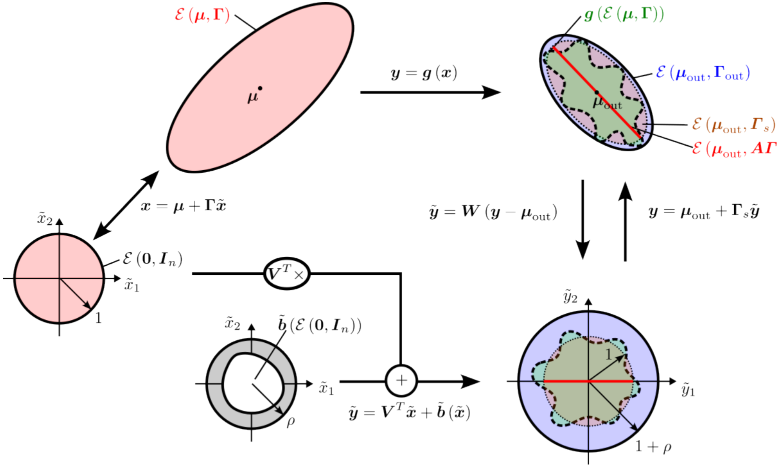


Figure 11: New theorem illustration

Application - Consensus

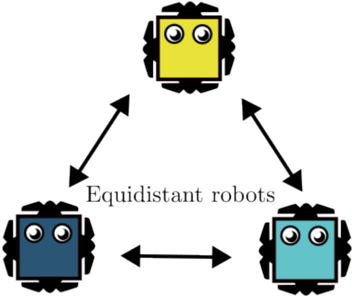


Figure 12: Example of formation control by consensus

Continuous system with three robots and periodic position measurement:

$$\dot{x}_i = v_i, \tag{30}$$

$$\dot{v}_i = u_i, \tag{31}$$

$$y_{i,k} = x_i(t_k), \tag{32}$$

$$t_k = k \cdot \delta_t \tag{33}$$

with $i \in \mathcal{I}_3$, $\mathcal{I}_3 = \{1, 2, 3\}$, $k \in \mathbb{N}$, $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$, $u_i \in \mathbb{R}$ and the period between the measurements $\delta_t > 0$

Application - Consensus

The robots must find a consensus

$$\lim_{t\to\infty} \left\| x_i(t) - x_j(t) \right\| = 0, \quad (34)$$

$$\lim_{t\to\infty} \left\| v_i(t) - v_j(t) \right\| = 0 \quad (35)$$

for all $i, j \in \mathcal{I}_3$

Consensus protocol [Zheng et al.] [2]:

$$\dot{u}_i(t) = \sum_{j \neq i} \left(y_{j,k} - x_i(t) \right) - c \cdot v_i(t),\ t \in (t_k, t_k + 1] \quad (36)$$

with the feedback gain $c > 0$.

Application - Propagation

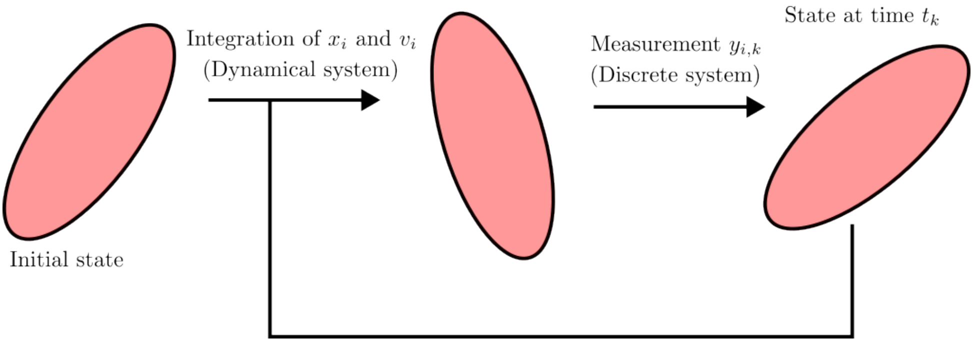


Figure 13: Propagation process

Application - Conditioning

State vector

$$\boldsymbol{z} = [z_i]_{i \in [1,9]}\,, \quad (37)$$

$$z_1(t) = x_2(t) - x_1(t), \quad (38)$$

$$z_2(t) = v_2(t) - v_1(t), \quad (39)$$

$$z_3(t) = x_3(t) - x_1(t), \quad (40)$$

$$z_4(t) = v_3(t) - v_1(t), \quad (41)$$

$$z_5(t) = x_3(t) - x_2(t), \quad (42)$$

$$z_6(t) = v_3(t) - v_2(t), \quad (43)$$

$$z_{7,k} = y_{2,k} - y_{1,k}\,, \quad (44)$$

$$z_{8,k} = y_{3,k} - y_{1,k}\,, \quad (45)$$

$$z_{9,k} = y_{3,k} - y_{2,k}\,. \quad (46)$$

Initial set of state

$$\|\boldsymbol{z}(0)\| < e, \quad (47)$$

$$z_{6+j,0} = z_j(0), \text{ for } j \in [1:3] \quad (48)$$

with $e > 0$, equivalent to $\boldsymbol{z}(0) \in \mathcal{E}(0, \Gamma_0)$ with the singular matrix

$$\Gamma_0 = e \cdot \begin{bmatrix} & & \boldsymbol{I}_6 & & & & \boldsymbol{0}_{6,3} \\ 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 0 & \boldsymbol{0}_{3,3} \\ 0 & 0 & 0 & 0 & 1 & 0 & \end{bmatrix} \quad (49)$$

Application - Singuliarity

Cause of the singularity:

- The initial state is a degenerated ellipsoid
- The measurements are linear projections

By the propagation of the degenerated ellipsoid, on can find a matrix $\boldsymbol{\Gamma}_{t_{\text{end}}}$ such that

$$\boldsymbol{z}(t_{\text{end}}) \in \mathcal{E}\left(0, \boldsymbol{\Gamma}_{t_{\text{end}}}\right) \quad (50)$$

with $t_{\text{end}} > 0$

One can then verify that the system is contracting

$$t_{\text{end}} > 0,\ \mathcal{E}\left(0, \boldsymbol{\Gamma}_{t_{\text{end}}}\right) \subseteq \mathcal{E}\left(0, \boldsymbol{\Gamma}_{0}\right) \quad (51)$$

Conclusion

The proposed solution in the singular case can be used to

- consider degenerated ellipsoids
- study systems with projections or other singular mappings

This tool is being tested for the stability analysis of n-dimensional non-linear hybrid systems

Bibliography

- Andreas Rauh and Luc Jaulin.
  A computationally inexpensive algorithm for determining outer and inner enclosures of nonlinear mappings of ellipsoidal domains.
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- Yuanshi Zheng, Qi Zhao, Jingying Ma, and Long Wang.
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