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Mechanical and Industrial Engineering

Robust Control Barrier Functions for Uncertain Systems with Set-Membership Estimation and Learning

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Interval Methods in Control Seminar || October 11, 2023

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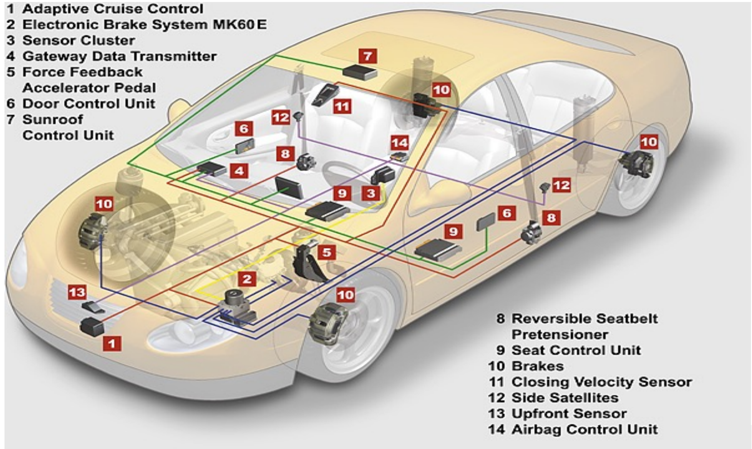
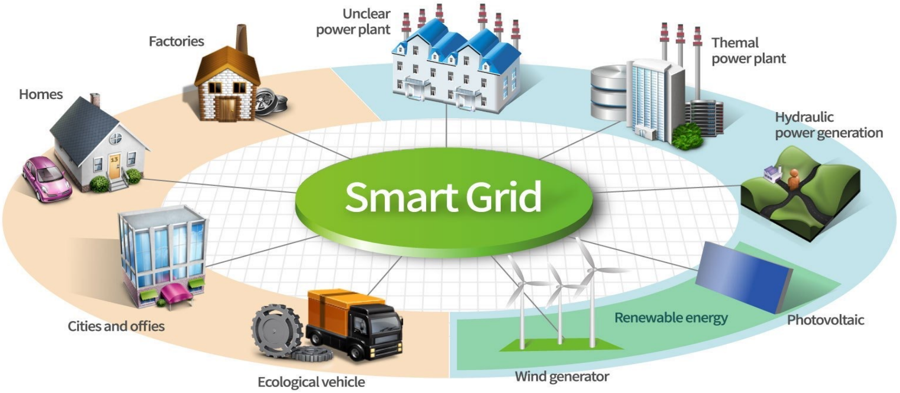
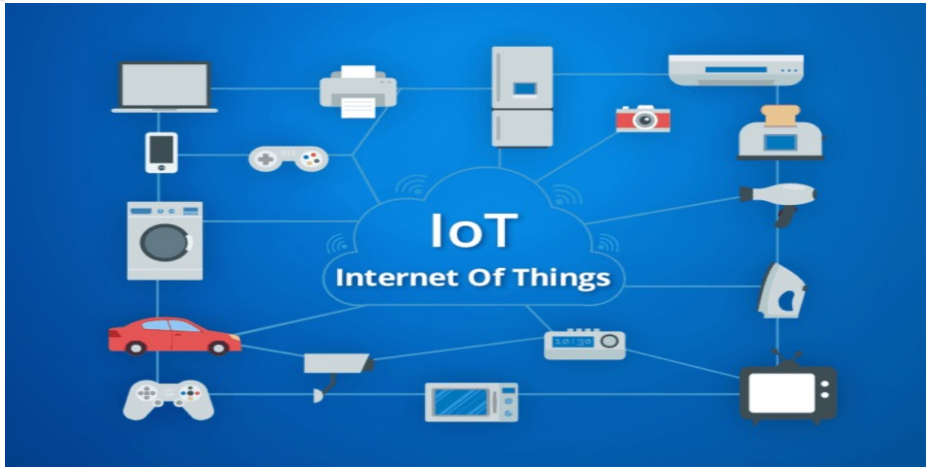
Safety-Critical Cyber-Physical Systems



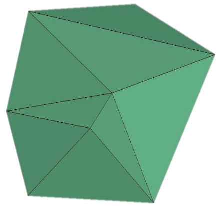




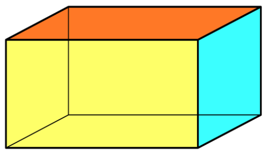
Introduction: Set-Based Methods in Control

Sets appear naturally in control systems design:
- Constraints
- Uncertainties
- Design/safety specifications

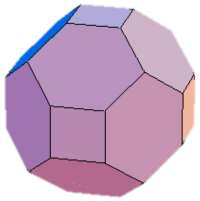
Polytope



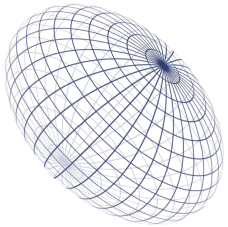
Hyperrectangle/Interval



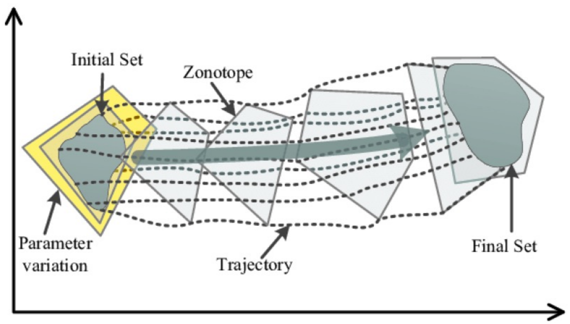
Zonotope



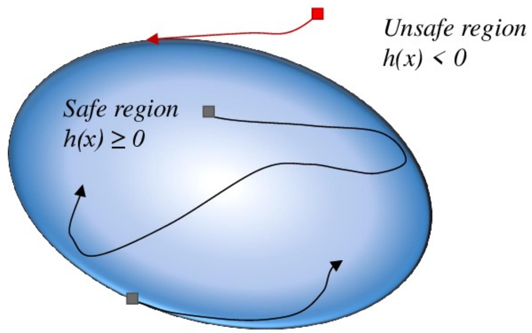
Ellipsoid



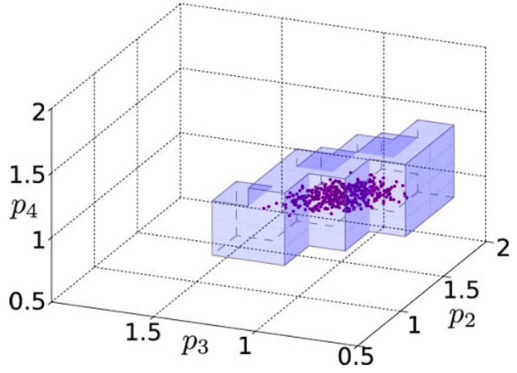
1) Reachability-Based Verification & Planning



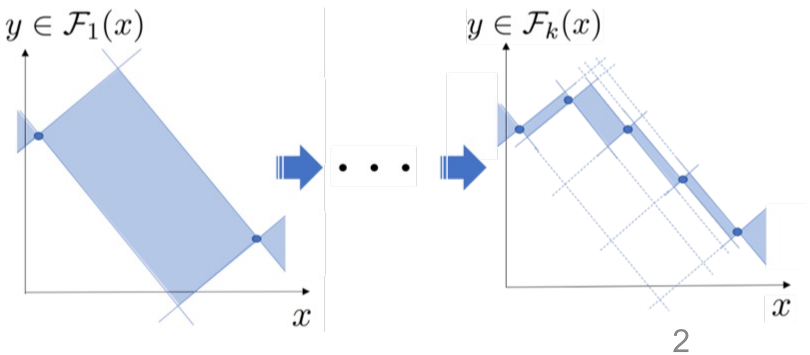
2) Set-Based Control



3) Set-Valued Estimation



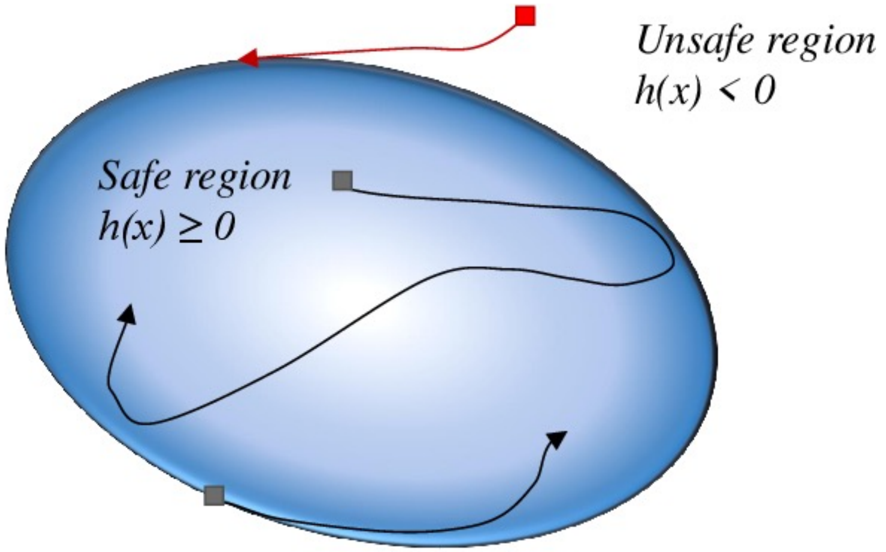
4) Set-Membership Learning



Introduction: Safety via Control Barrier Function

Controlled Invariant Set (CIS)

$$x(0) \in \mathcal{C} \Rightarrow \exists u(x(t))\,\text{s.t.}\, x(t) \in \mathcal{C}, \forall t \geq 0$$



Control Barrier Function ⊂ CIS

- Known control affine system
  $$\dot{x} = f(x) + g(x)u$$
- Find safe set
  $$\mathcal{S} \triangleq \{x \in \mathbb{R}^n : h(x) \geq 0\}$$
  such that
  $$\sup_{u \in U}[L_f h(x) + L_g h(x)u + \alpha(h(x))] \geq 0$$

$$u(x) = \arg\min_{u \in \mathcal{U}} \frac{1}{2}\|u - k(x)\|$$
$$s.t. \ \frac{\partial h}{\partial x}(x)(f(x) + g(x)u) \geq -\alpha(h(x))$$

Safety Filter

Ames et al., TAC, 2016

Introduction: Safety via Control Barrier Function

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$$s.t. \ \frac{\partial h}{\partial x}(x)(f(x) + g(x)u) \geq -\alpha(h(x))$$

Safety Filter

Ames et al., TAC, 2016

Challenges

- Systems are uncertain
  1) Uncertain, time-varying parameters

$$\dot{x}(t) = f(x(t), \theta^*(t)) + g(x(t), \theta^*(t))u(t)$$

  2) Mathematical model unavailable

$$\dot{x} = f(x, u)$$

- Given an (uncertain) safe set

$$\mathcal{S}_\theta \triangleq \{x \in \mathcal{X} \mid h(x, \theta) \geq 0\}$$

how to guarantee controlled invariance for 1) and 2)?

➔ **Robust Control Barrier Function**

Overview

A. Preliminaries on Mixed-Monotonicity

B. Robust Control Barrier Function
   - Set-Membership Parameter Estimation

C. Robust Data-Driven Control Barrier Function
   - Set-Membership Learning

Overview

A. **Preliminaries on Mixed-Monotonicity**

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Preliminaries on Mixed-Monotonicity

Yang, Mickelin & Ozay, 2019

A mapping $f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathcal{T} \subseteq \mathbb{R}^m$ is (discrete-time) mixed monotone if there exists a decomposition function $f_d : \mathcal{X} \times \mathcal{X} \to \mathcal{T}$ satisfying:

1. $f_d(x, x) = f(x)$,
2. $x_1 \geq x_2 \Rightarrow f_d(x_1, y) \geq f_d(x_2, y)$, and
3. $y_1 \geq y_2 \Rightarrow f_d(x, y_1) \leq f_d(x, y_2)$.

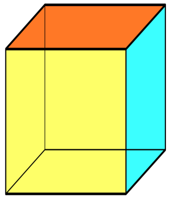
Coogan & Arcak 2015

Then, if $\underline{x} \leq x \leq \overline{x}$,

- $f_d(\underline{x}, \overline{x}) \leq f(x) \leq f_d(\overline{x}, \underline{x})$

Enables interval bounding of nonlinear functions

Intervals



- Decomposition functions are not unique!

Remainder-Form Decomposition Function

$$\overline{f}_{d,i}(z,\hat{z}) = \min_{\mathbf{m}\in\mathbf{M}^c} f_i(\zeta_{\mathbf{m}}(z,\hat{z})) + \mathbf{m}^\top(\zeta_{\mathbf{m}}(\hat{z},z) - \zeta_{\mathbf{m}}(z,\hat{z})),$$

$$\underline{f}_{d,i}(\hat{z},z) = \max_{\mathbf{m}\in\mathbf{M}^c} f_i(\zeta_{\mathbf{m}}(\hat{z},z)) + \mathbf{m}^\top(\zeta_{\mathbf{m}}(z,\hat{z}) - \zeta_{\mathbf{m}}(\hat{z},z)),$$

$$\zeta_{\mathbf{m},j}(z,\hat{z}) = \begin{cases} \hat{z}_j, & \text{if } \mathbf{m}_j \geq \max((\overline{J}^f_C)_{ij}, 0), \\ z_j, & \text{if } \mathbf{m}_j \leq \min((\underline{J}^f_C)_{ij}, 0), \end{cases}$$

DT: $\mathbf{M}^c_i = \{\mathbf{m} \in \mathbb{R}^{n_z} \mid \mathbf{m}_j = \max((\overline{J}^f_C)_{ij}, 0) \ \vee \ \mathbf{m}_j = \min((\underline{J}^f_C)_{ij}, 0), \forall j \in \mathbb{N}_{n_z}\}$

CT: $\mathbf{M}^c_i = \{\mathbf{m} \in \mathbb{R}^{n_z} \mid \mathbf{m}_j = \max((\overline{J}^f_C)_{ij}, 0) \ \vee \ \mathbf{m}_j = \min((\underline{J}^f_C)_{ij}, 0), \forall j \in \mathbb{N}_{n_z}, j \neq i, \mathbf{m}_i = 0\}$

Guarantees

- Tightest in the family (that includes Yang et al. 2019)
- Tractable/Computable in closed form
- Applicable to non-smooth, semi-continuous functions

Often outperforms all variations of natural inclusion in interval arithmetic

Khajenejad, M. and **Yong, S.Z.** IEEE TAC, 2023.

Embedding Systems and Framer Property

For a system $x_t^+ = g(x_t, w_t)$ with a pair of decomposition functions, $\underline{g}_d(\cdot), \overline{g}_d(\cdot)$, its embedding system can be defined as:

$$\begin{bmatrix} \underline{x}_t^+ \\ \overline{x}_t^+ \end{bmatrix} = \begin{bmatrix} \underline{g}_d([(\underline{x}_t)^\top \ \underline{w}^\top]^\top, [(\overline{x}_t)^\top \ \overline{w}^\top]^\top) \\ \overline{g}_d([(\overline{x}_t)^\top \ \overline{w}^\top]^\top, [(\underline{x}_t)^\top \ \underline{w}^\top]^\top) \end{bmatrix}.$$

Then, the solution has a framer property: $\underline{x}_t \leq x_t \leq \overline{x}_t, \forall t, \forall w_t \in \mathcal{W}$.

Provides interval framers for (CT and DT) systems by construction

Khajenejad, M. and **Yong, S.Z.** IEEE TAC, 2023.

Interval Observer

Continuous-time (CT) or discrete-time (DT) system with bounded noise:

$$x_t^+ = f(x_t) + W w_t,$$
$$y_t = h(x_t) + V v_t.$$

- Interval uncertainties: $w \in [\underline{w}, \overline{w}]$, $v \in [\underline{v}, \overline{v}]$, $x_0 \in [\underline{x}_0, \overline{x}_0]$.
- $f(x)$ and $h(x)$ are differentiable with known Jacobian bounds.

Interval Observer

1. Find equivalent system (adds additional degrees of freedom)

2. Write embedding system → Interval observer

3. Find linear comparison system

4. Apply stability/gain minimization results to obtain observer gains.

Let $L, N \in \mathbb{R}^{n\times l}$ and $T \in \mathbb{R}^{n\times n}$ such that $T + NC = I_n$, then we can write $\mathcal{G}$ equivalently as:

$$\begin{aligned}\xi_t^+ &= (TA - LC - NA_2)x_t + T\phi(x_t) - N\rho(x_t, w_t) \\ &\quad + (TW - NB_2)w_t - L\psi(x_t) + L(y_t - Vv_t), \\ x_t &= \xi_t + Ny_t - NVv_t,\end{aligned}$$

with JSS decompositions.

Framer error ($\varepsilon_t \triangleq \overline{x}_t - \underline{x}_t$) dynamics $\tilde{\mathcal{G}}$ for DT case:

$$\begin{aligned}\varepsilon_t^+ &= |TA - LC - NA_2|\varepsilon_t + |T|\Delta_d^\phi + |N|\Delta_d^\rho + |L|\Delta_d^\psi \\ &\quad + |TW - NB_2|\Delta w + (|LV| + |NV|)\Delta v + |MNV|\Delta v \\ &\leq (|TA - LC - NA_2| + |T|\overline{F}_x^\phi + |N|\overline{F}_x^\rho + |L|\overline{F}_x^\psi)\varepsilon_t \\ &\quad + (|TW - NB_2| + |N|\overline{F}_w^\rho)\Delta w + (|LV| + |NV|)\Delta v, \\ &\triangleq \tilde{A}\varepsilon_t + \tilde{B}\begin{bmatrix}\Delta w \\ \Delta v\end{bmatrix} \\ z_t &= \varepsilon_t = \tilde{C}\varepsilon_t + \tilde{D}\begin{bmatrix}\Delta w \\ \Delta v\end{bmatrix}\end{aligned}$$

Interval Observer

Khajenejad, M. and **Yong, S.Z.** IEEE L-CSS, 2022.

Pati, T., **Yong, S.Z.**, et al. IEEE L-CSS, 2022.

$\mathcal{H}_\infty$ -Optimal

- Minimize $\mathcal{H}_\infty$ gain
- Leads to a mixed-integer nonconvex program
- With additional constraints → SDP or MISDP

L_1 -Robust

- Minimize L_1 gain
- Positive framer error system
- Leads to a mixed-integer linear program (MILP)
- With additional constraints → LP

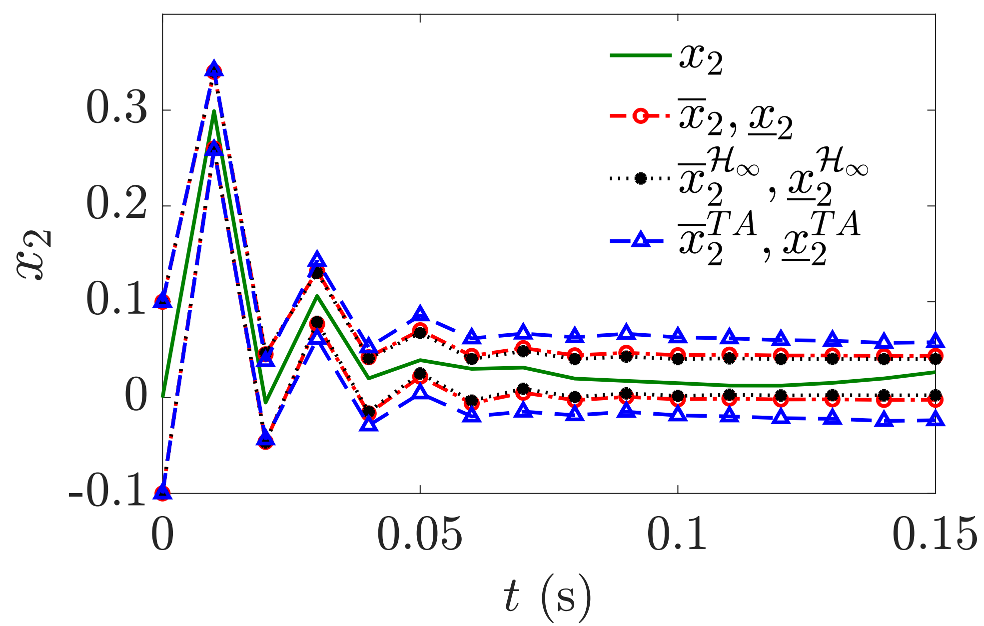
Extensions:

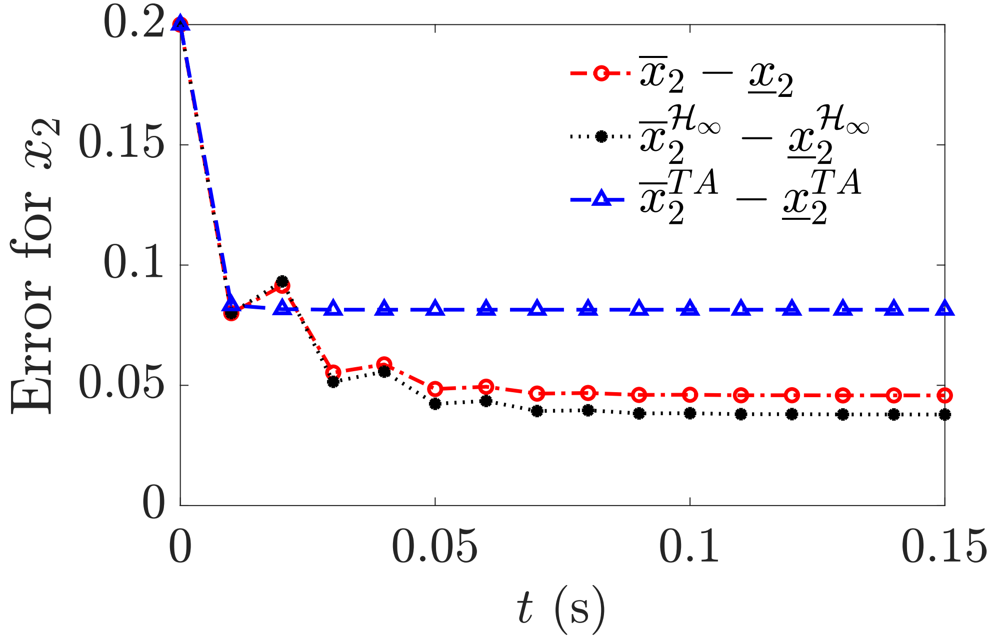
- State and input interval observers
- Hybrid interval observers

Simulation: DT Example

$$x_{t+1} = \begin{bmatrix} 0 & 1 \\ 0.3 & 0 \end{bmatrix} x_t + \begin{bmatrix} 0.05 \\ 0 \end{bmatrix} [1 - x_{t,1}^2] + w_t,$$

$$y_t = x_{t,1} + v_t.$$





Simulation: CT Example

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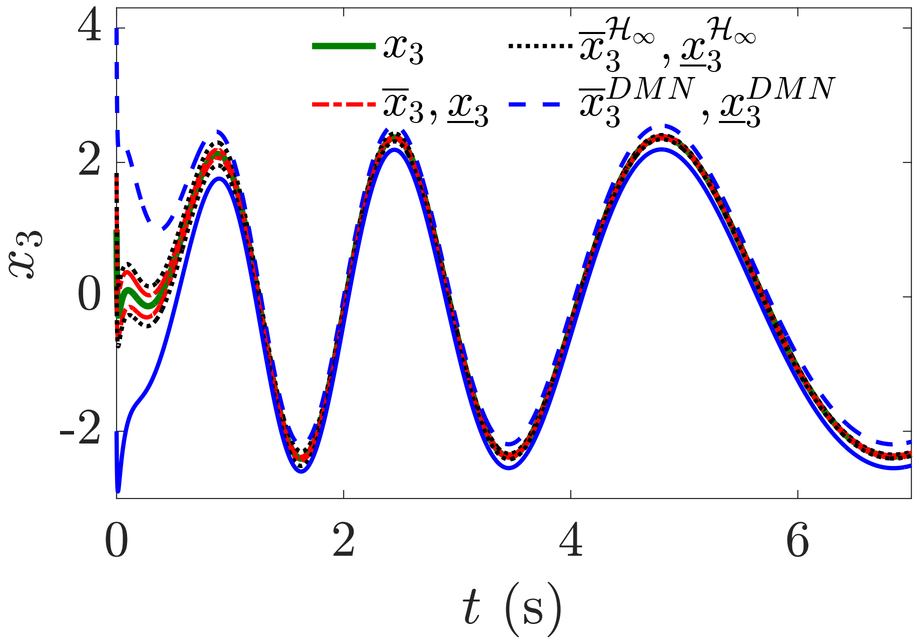
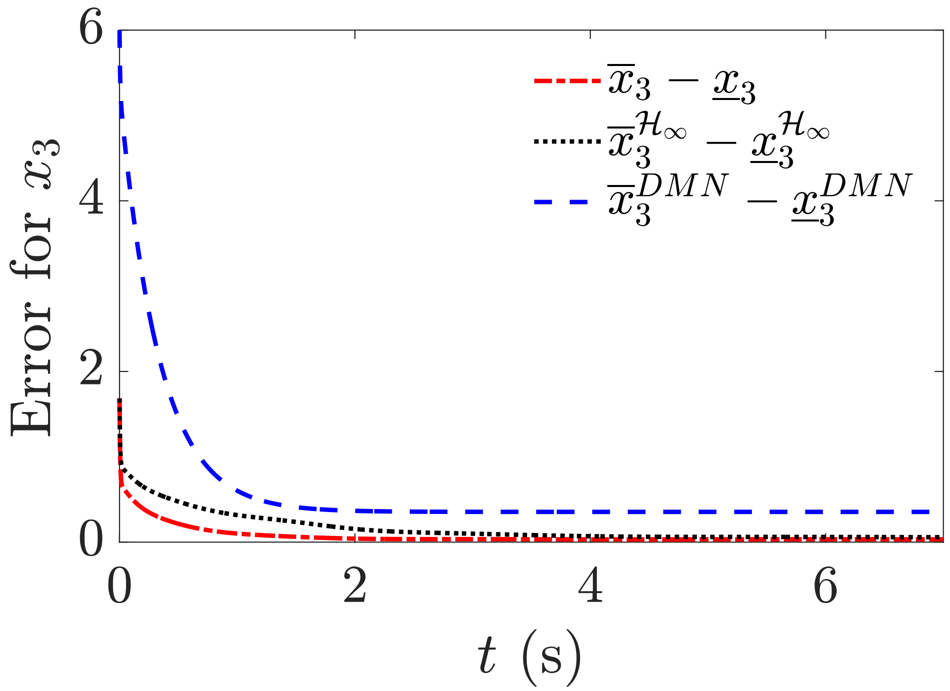
$$\dot{x}_1 = x_2 + w_1, \quad \dot{x}_2 = b_1 x_3 - a_1 \sin(x_1) - a_2 x_2 + w_2,$$
$$\dot{x}_3 = -a_3(a_2 x_1 + x_2) + \frac{a_1}{b_1}(a_4 \sin(x_1) + \cos(x_1) x_2) - a_4 x_3 + w_3.$$





Overview

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Motivation and Literature Review

Challenge: Extend control barrier function [Ames et al., 2016] to guarantee safety under parametric uncertainties

- The degradation of safety [Kolathaya & Ames, 2018], analysis on robustness [Xu et al., 2015].
- Additive Uncertainty: [Jankovic, 2018; Breeden & Panagou, 2021]
- Parametric Uncertainty: Adaptive CBF [Taylor & Ames, 2020]; Robust adaptive CBF [Lopez et al., 2020]; Unmatched CBF [Lopez & Slotine, 2023]; Adaptive CBF with persistence of excitation [Black, Arabi & Panagou, 2021]

➔ **Do not apply for time-varying and nonlinear parametric uncertainties!**

Motivation and Literature Review

Challenge: Reduce conservatism of robust CBFs via set-membership parameter estimation, i.e., to find

$$\hat{\Theta}(t) \text{ such that } \theta^*(t) \in \hat{\Theta}(t) \subseteq \Theta$$

- Set-membership identification at sampled times [Lopez et al., 2020]
- Disturbance observer + robust CBFs [Das & Murray, 2022; Wang & Xu, 2023]
- Adaptive CBF with persistence of excitation [Black, Arabi & Panagou, 2021]

➔ **Do not apply for time-varying and nonlinear parametric uncertainties!**

Problem Statement: Uncertain System Dynamics

Consider a control affine system with time-varying, nonlinear parametric uncertainty:

$$\dot{x}(t) = f(x(t), \theta^*(t)) + g(x(t), \theta^*(t))u(t). \tag{1}$$

- State: $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$,
- Input: $u(t) \in \mathcal{U} \subset \mathbb{R}^m$,
- Unknown parameter: $\theta^*(t) \in \Theta \subset \mathbb{R}^p$, and
- Unknown parameter variation: $\dot{\theta}^*(t) \in \Theta_d \subset \mathbb{R}^p$.

Problem Statement: Uncertainty-Dependent Safe Set

Definition 1 (Uncertainty-Dependent Safety Set)

A superlevel set $\mathcal{S}_\theta$ defined on a continuously differentiable function $h : \mathcal{X} \times \Theta \to \mathbb{R}$ parametrized by θ:

$$\mathcal{S}_\theta \triangleq \{x \in \mathcal{X} \mid h(x, \theta) \geq 0\}, \tag{2}$$

$$\partial\mathcal{S}_\theta \triangleq \{x \in \mathcal{X} \mid h(x, \theta) = 0\}, \tag{3}$$

$$int(\mathcal{S}_\theta) \triangleq \{x \in \mathcal{X} \mid h(x, \theta) > 0\}. \tag{4}$$

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Problem Statement

Problem 1 (Robust Safety)

Given system (1) and $\mathcal{S}_{\theta^*}$, construct a robust CBF to guarantee robust controlled invariance of all possible safety sets, i.e., $\mathcal{S}_\theta$ for all $\theta \in \Theta$ and $\dot{\theta} \in \Theta_d$.

- Thus, safe for all unknown time-varying $\theta^*(t)$ and $\dot{\theta}^*(t)$, $\forall t \geq 0$.

Problem 2 (Tractable & Less Conservative Robust CBF Conditions)

Given system (1) and $\mathcal{S}_{\theta^*}$, find sufficient and/or necessary rCBF conditions that are computationally tractable and less conservative.

- Computational tractable → linear in decision variables (i.e., control input), no semi-infinite ('for all') constraints
- Less conservative → with respect to estimated parameter set $\theta^*(t) \in \hat{\Theta}(t) \subseteq \Theta$

Approach: rCBF

Definition 2 (Robust Control Barrier Function (rCBF))

For system in (1), a continuously differentiable function $h : \mathcal{X} \times \Theta \to \mathbb{R}$ is an rCBF for $\mathcal{S}_{\theta^*}$ (cf. Definition 1), if there exists a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that

$$\sup_{u \in \mathcal{U}} \dot{h}(x, u, \theta, \dot{\theta}) \geq -\alpha(h(x, \theta)), \tag{5}$$

for all $x \in \mathcal{X}$, $\theta \in \Theta$, $\dot{\theta} \in \Theta_d$, and $t \geq 0$, where

$$\dot{h}(x, u, \theta, \dot{\theta}) \triangleq \frac{\partial h}{\partial x}(x, \theta)(f(x, \theta) + g(x, \theta)u) + \frac{\partial h}{\partial \theta}(x, \theta)\dot{\theta}. \tag{6}$$

Moreover, for any $x \in \mathcal{S}_\Theta \triangleq \bigcap_{\theta \in \Theta} \mathcal{S}_\theta$, we define the safe input set:

$$K_{\mathcal{S}_\Theta}(x) = \{u \in \mathcal{U} \mid \dot{h}(x, u, \theta, \dot{\theta}) \geq -\alpha(h(x, \theta)), \forall \theta \in \Theta, \dot{\theta} \in \Theta_d\}. \tag{7}$$

Approach: rCBF

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Theorem 1 (Robust Safety)

If h is an rCBF on $\mathcal{S}_\Theta$ and $\frac{\partial h}{\partial x}(x,\theta) \neq 0, \forall x \in \partial\mathcal{S}_\Theta$, then any Lipschitz continuous controller

$$u(x) \in K_{\mathcal{S}_\Theta}(x)$$

for the system (1) renders the set $\mathcal{S}_\Theta$ robustly safe, i.e., it also renders

$$h(x,\theta^*) \geq 0, \forall x \in \mathcal{S}_\Theta \subseteq \mathcal{S}_{\theta^*}$$

$$u(x) = \arg\min_{u \in \mathcal{U}} \frac{1}{2}\|u - k(x)\|$$

$$s.t. \ \frac{\partial h}{\partial x}(x,\theta)(f(x,\theta) + g(x,\theta)u) + \frac{\partial h}{\partial \theta}(x,\theta)\dot{\theta} \geq -\alpha(h(x,\theta)), \forall \theta \in \Theta, \dot{\theta} \in \Theta_d$$

Safety Filter

Pati, T. and **Yong, S.Z.** IFAC World Congress, 2023.

Approach: Tractable rCBF

Assumption 1

Uncertainty parameter sets are known intervals/hyperrectangles:

$$\theta^* \in \mathbb{I}\Theta \triangleq [\underline{\theta}, \overline{\theta}], \quad \dot{\theta}^* \in \mathbb{I}\Theta_d \triangleq [\underline{\theta}_d, \overline{\theta}_d].$$

Functions $h(x, \theta)$ and $\alpha \in \kappa_\infty$ are such that

$$\dot{h}(x, u, \theta, \dot{\theta}) + \alpha(h(x, \theta))$$

is a differentiable function in θ and $\dot{\theta}$ with known Jacobian bounds:

$$J(\theta, \dot{\theta}) \in \mathbb{I}J \triangleq [\underline{J}, \overline{J}], \forall \theta \in \mathbb{I}\Theta, \dot{\theta} \in \mathbb{I}\Theta_d.$$

Approach: Tractable rCBF

- Assumption 1 $\Rightarrow \exists$ mixed-monotone decomposition functions h_d, $\tilde{f}_d$, $\tilde{g}_d$, and $\tilde{h}_d$ for $h(x(t),\theta)$, $\tilde{f}(\theta) \triangleq \frac{\partial h}{\partial x} f(x(t),\theta)$, $\tilde{g}(\theta) \triangleq \frac{\partial h}{\partial x} g(x(t),\theta)$, $\tilde{h}(\theta) \triangleq \frac{\partial h}{\partial \theta}(x(t),\theta)$

Definition 4 (rCBF-MM)

Let Assumption 1 hold. Then, $h : \mathcal{X} \times \Theta \to \mathbb{R}$ *is an rCBF-MM if:*

$$\left\{ \begin{array}{ll} \sup\limits_{u^+,u^-} & \tilde{g}_d(\underline{\theta},\overline{\theta})u^+ - \tilde{g}_d(\overline{\theta},\underline{\theta})u^- \\ \text{s.t.} & u^+ - u^- \in \mathcal{U}, u^+ \geq 0, u^- \geq 0 \end{array} \right\} \geq \begin{array}{l} -\alpha(h_d(\underline{\theta},\overline{\theta})) \\ -\tilde{f}_d(\underline{\theta},\overline{\theta}) - \Delta, \end{array} \quad (8)$$

with $\Delta \triangleq \min\{\tilde{h}_d(\underline{\theta},\overline{\theta})\underline{\theta}_d, \tilde{h}_d(\underline{\theta},\overline{\theta})\overline{\theta}_d, \tilde{h}_d(\overline{\theta},\underline{\theta})\underline{\theta}_d, \tilde{h}_d(\overline{\theta},\underline{\theta})\overline{\theta}_d\}$,

- $K^{MM}_{\mathcal{S}_\Theta}(x) = \{u = u^+ - u^- \in \mathcal{U} \mid (8)\ \forall\, x \in \mathcal{S}_\Theta \triangleq \bigcap_{\theta\in\Theta} \mathcal{S}_\theta\}$

Approach: Tractable rCBF

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Theorem 2 (Sufficient Condition for rCBF-MM)

If h is a rCBF-MM on $\mathcal{S}_\Theta$ (cf. Definition 4) and $\frac{\partial h}{\partial x}(x,\theta)g(x,\theta) \neq 0$, $\forall x \in \partial\mathcal{S}_\Theta$, then any Lipschitz continuous controller

$$u(x) \in K^{MM}_{\mathcal{S}_\Theta}(x)$$

for the system (1) renders the set $\mathcal{S}_\Theta$ robustly safe.

Proof Sketch: Use mixed-monotonicity property and interval arithmetic for lower bounding functions:

- $\tilde{f}(x,\theta) \geq \tilde{f}_d(\underline{\theta},\overline{\theta})$,
- $\tilde{g}(x,\theta)u = \tilde{g}(x,\theta)(u^+ - u^-) \geq \tilde{g}_d(\underline{\theta},\overline{\theta})u^+ - \tilde{g}_d(\overline{\theta},\underline{\theta})u^-$,
- $\tilde{h}(x,\theta)\dot{\theta} \geq \min\{\tilde{h}_d(\underline{\theta},\overline{\theta})\underline{\theta}_d, \tilde{h}_d(\underline{\theta},\overline{\theta})\overline{\theta}_d, \tilde{h}_d(\overline{\theta},\underline{\theta})\underline{\theta}_d, \tilde{h}_d(\overline{\theta},\underline{\theta})\overline{\theta}_d\}$,
- $h(x,\theta) \geq h_d(\underline{\theta},\overline{\theta})$.

Pati, T. and **Yong, S.Z.** IFAC World Congress, 2023.

Approach: Tractable rCBFs + rCLFs

Alternative tractable CBFs:

- Concave dependence on parametric uncertainty ➔ rCBF-C via vertex enumeration
- Linear dependence on parametric uncertainty ➔ rCBF-L via robust optimization/dual linear programming or rCBF-C

Analogous tractable CLFs:

- General parametric uncertainty ➔ rCLF-MM via mixed-monotonicty
- Convex dependence on parametric uncertainty ➔ rCLF-C via vertex enum.
- Linear dependence on parametric uncertainty ➔ rCLF-L via robust opt. or robust adaptive CLF (raCLF)

Pati, T. and **Yong, S.Z.,** 2023, submitted.

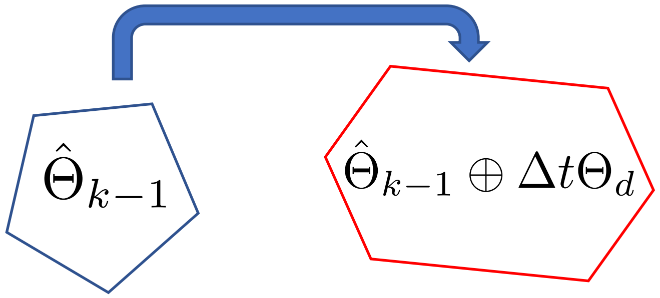
Approach: Set-Membership Parameter Estimation

Method 1: Polyhedral Intersections (via Computational Geometry Tools)

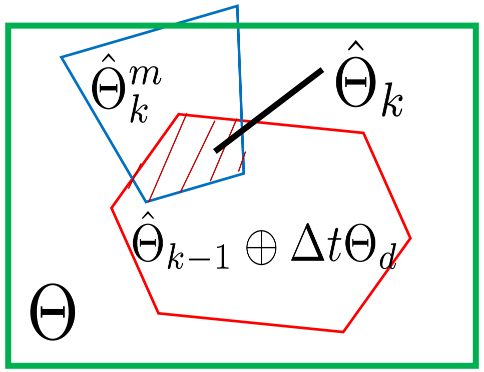
- Generalization of SMID in [Lopez et al., 2020] to allow time-varying parameters

$$
\begin{aligned}
\hat{\Theta}_{PI,k} = (&\hat{\Theta}_{PI,k-1} \oplus \Delta t\Theta_d) \; \cap \Theta \cap \\
&\{-F(x(i))\theta \leq -\hat{\dot{x}}(i) + f(x(i) + g(x(i))u(i) + \epsilon, \\
&\quad F(x(i))\theta \leq \hat{\dot{x}}(i) - f(x(i) - g(x(i))u(i) + \epsilon\}
\end{aligned}
$$

$$
\hat{\Theta}_{PI}(t) = \hat{\Theta}_{PI,\lfloor t/\Delta t \rfloor} \; \oplus \; \Delta t\Theta_d.
$$



Parameter “Propagation”



Measurement Update

Pati, T. and **Yong, S.Z.,** 2023, submitted.

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Approach: Set-Membership Parameter Estimation

Method 2: Interval Observers

- Leverage mixed-monotone embedding systems [Pati, **Yong**, et al., 2023]

Augmented system dynamics:

$$\begin{aligned} \dot{z} &= \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} f(x,\theta) + g(x,\theta)u \\ w \end{bmatrix} \triangleq \hat{f}(z) + Ww, \\ \tilde{\zeta} &= \begin{bmatrix} x \\ \hat{\dot{x}} \end{bmatrix} = \begin{bmatrix} x \\ f(x,\theta) + g(x,\theta)u + v \end{bmatrix} \triangleq h(z) + Vv \end{aligned}$$

Interval Observer

$$\begin{aligned} \dot{\underline{\xi}} &= (TA-LC)^{\uparrow}\underline{z} - (TA-LC)^{\downarrow}\,\overline{z} + L\zeta + T^{\oplus}\omega_d(\underline{z},\overline{z}) - T^{\ominus}\omega_d(\overline{z},\underline{z}) \\ &\quad +(TW)^{\oplus}\underline{w} - (TW)^{\ominus}\overline{w} + (TA-LC)N\zeta, \\ \dot{\overline{\xi}} &= (TA-LC)^{\uparrow}\overline{z} - (TA-LC)^{\downarrow}\,\overline{z} + L\zeta + T^{\oplus}\omega_d(\overline{z},\underline{z}) - T^{\ominus}\omega_d(\underline{z},\overline{z}) \\ &\quad +(TW)^{\oplus}\overline{w} - (TW)^{\ominus}\underline{w} + (TA-LC)N\zeta, \end{aligned}$$

$$\begin{aligned} \underline{z} &= \underline{\xi} + N\zeta, \\ \overline{z} &= \overline{\xi} + N\zeta, \end{aligned}$$

$$\begin{aligned} M^{\uparrow} &\triangleq M^{d} + M^{nd,\oplus} \\ M^{\downarrow} &\triangleq M^{nd,\ominus} \end{aligned}$$

Parameter Set Estimate

$$\hat{\Theta}_{IO}(t) = \{\theta \in \Theta \mid W^{\top}\underline{z}(t) \leq \theta \leq W^{\top}\overline{z}(t)\}$$

Approach: Set-Membership Parameter Estimation

Proposition 1

The interval observer is correct by construction and L_1-robust, if there exist Q, $\tilde{T}$, $\tilde{N}$, $\tilde{L}$, q, and γ that solve the following mixed-integer program (MIP):

$$
\begin{aligned}
&(\gamma^*, q^*, Q^*, \tilde{T}^*, \tilde{L}^*, \tilde{N}^*) \in \\
&\quad \arg\min_{\{\gamma, q, Q, \tilde{T}, \tilde{L}, \tilde{N}\}} \ \gamma \\
&\quad s.t. \ \ \mathbf{1}_{1\times(n+p)} \begin{bmatrix}\Omega & \Lambda\end{bmatrix} < \begin{bmatrix}\sigma & \gamma\mathbf{1}_{1\times n_w}\end{bmatrix}, \\
&\qquad\quad \tilde{T} + \tilde{N}C = Q, \ \ q > 0, \ \gamma > 0,
\end{aligned}
$$

where $Q \triangleq diag(q)$, $\Lambda \triangleq |\tilde{T}W|$, $\sigma \triangleq -\mathbf{1}_{1\times n}$, $\Omega \triangleq M^m + |\tilde{T}|\overline{F}^{\omega}$ *with* $M \triangleq TA - LC$ *and* $\overline{F}^{\omega} \triangleq ((\overline{J}^{\omega})^{\oplus} + (\underline{J}^{\omega})^{\ominus})$.
Then, $T^* = (Q^*)^{-1}\tilde{T}^*, L^* = (Q^*)^{-1}\tilde{L}^*$ *and* $N^* = (Q^*)^{-1}\tilde{N}^*$.

Proof Sketch: Leverage CT mixed-monotone embedding systems with appropriate equivalent system transformation, as well as positivity of error system [Pati, Yong, et al., 2023]

Pati, T. and **Yong, S.Z.,** 2023, submitted.

Comparison: Safety via Adaptive/Robust CBF

$$\mathcal{S} \triangleq \{x \in \mathbb{R}^n : h(x, \Theta) \geq 0\}$$

Adaptive CBF

$$\dot{x} = f(x) + F(x)\theta^* + g(x)u$$

$$\sup_{u \in U}[L_f h(x,\hat{\theta}) + L_F h(x,\hat{\theta})\Lambda(x,\hat{\theta}) + L_g h(x)u + \alpha(h(x,\hat{\theta}) - \frac{1}{2}\tilde{\vartheta}^\top \Gamma^{-1}\tilde{\vartheta})] \geq 0$$

$$\dot{\hat{\theta}} = \Gamma F(x)\left(\frac{\partial h}{\partial x}(x,\hat{\theta})\right)^\top$$

- θ^* is constant
- Dynamics is linear in θ^*
- Only guarantees that $h(x, \hat{\theta}) \geq 0$ and not $h(x, \theta^*) \geq 0$

Lopez et al., LCSS, 2020.

Similarly, in Taylor & Ames, ACC, 2022; Lopez & Slotine, ACC, 2023.

Robust CBF

$$\dot{x} = f(x,\theta^*) + g(x,\theta^*)u$$

$$\sup_{u \in U}[L_f h(x,\theta) + L_g h(x,\theta)u + \frac{\partial h}{\partial \theta}(x,\theta)\dot{\theta} + \alpha(h(x,\theta))] \geq 0, \forall \theta \in \Theta, \dot{\theta} \in \Theta_d$$

+ Robust Optimization, Concavity or Mixed-Monotonicity

- $\theta^* \in \Theta$, $\dot{\theta}^* \in \Theta_d$ can be <u>time-varying</u>
- Dynamics can be <u>nonlinear in θ^*</u>
- Guarantees that $h(x, \theta^*) \geq 0$

Pati, T. and **Yong, S.Z.** IFAC World Congress, 2023.

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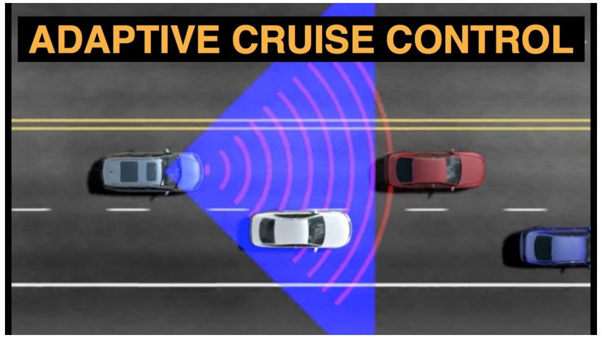
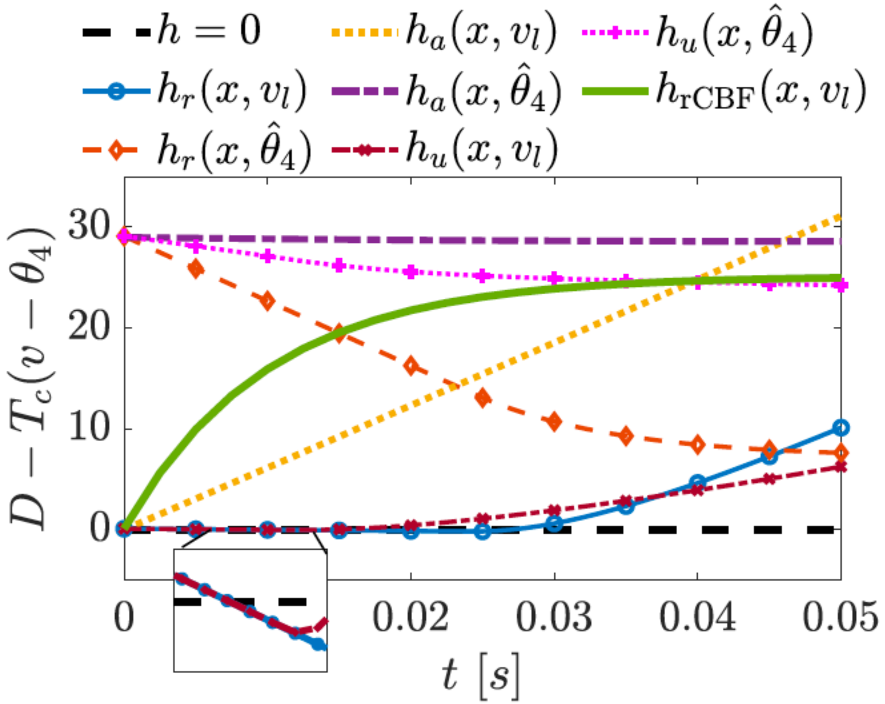
Simulation: Adaptive Cruise Control

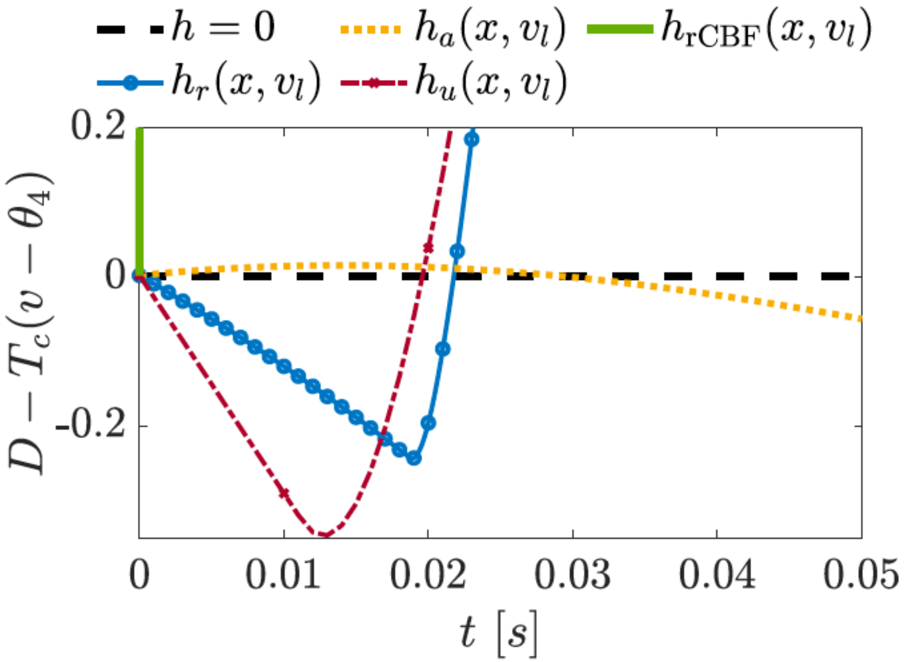


Time-to-Collision Safety: $D \geq T_c(v - v_l),\ v_l$ unknown

h_a: Adaptive CBF [Taylor & Ames, 2020]
h_r: Robust adaptive CBF (Lopez et al., 2020]
h_u: Unmatched CBF [Lopez & Slotine, 2023]
h_{rCBF}: Robust CBF

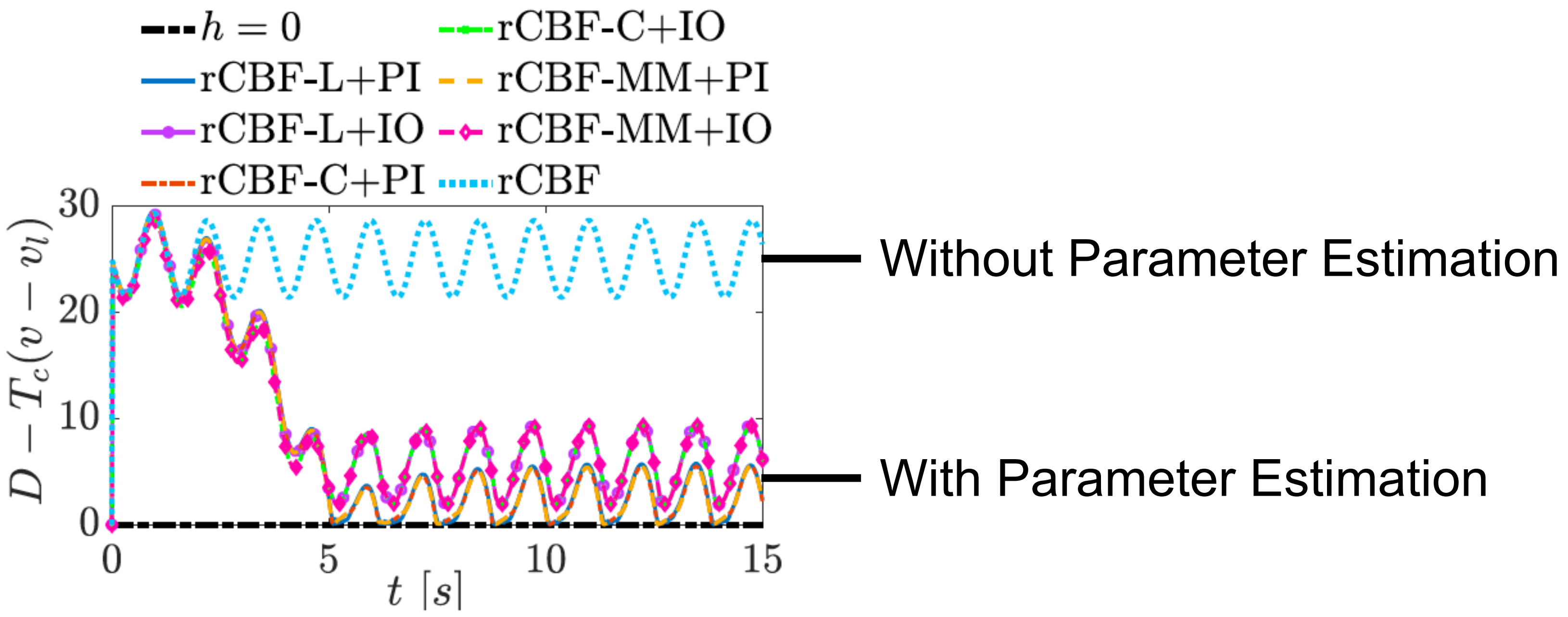


(a) Time-invariant v_l, i.e., $\dot{v}_l = 0$



(b) Time-varying v_l, i.e., $\dot{v}_l \neq 0$

Simulation: Adaptive Cruise Control



Parameter Estimation	rCBF-L + rCLF-L	rCBF-C + rCBF-C	rCBF-MM + rCBF-MM	rCBF-MM + raCLF
None	74.9 (20.6%)	64.3 (3.5%)	64.6 (4%)	64.4 (3.7%)
PI	110.4 (77.1%)	97.1 (56%)	98.3 (58%)	96.9 (56%)
IO	83.6 (21.5%)	67.9 (9.4%)	68.2 (10%)	70.4 (13%)

Pati, T. and **Yong, S.Z.,** 2023, submitted.

Overview

A. Preliminaries on Mixed-Monotonicity

B. Robust Control Barrier Function
  - Set-Membership Parameter Estimation

C. Robust Data-Driven Control Barrier Function
  - Set-Membership Learning

Motivation and Literature Review

Challenge: Extend control barrier function [Ames et al., 2016] to guarantee safety with no mathematical model but only prior state trajectory data

- Neural Networks, e.g., [Choi et al. 2020; Taylor et al., 2020]
- Gaussian Process, e.g., [Jagtap et al., 2020; Dhiman et al., 2023]

➔ Either no guarantees or only probabilistic guarantees

- Control Certificate Function under Lipschitz continuity [Taylor et al., 2021]

➔ Lipschitz continuity assumption may be strong
➔ Computationally expensive Second-Order Cone Programs (SOCPs)

Problem Statement: Uncertain System Dynamics

Consider an unknown nonlinear system:

$$\dot{x} = f(x, u)$$

- State: $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$,
- Input: $u(t) \in \mathcal{U} \subset \mathbb{R}^m$,

with safe set $\mathcal{S} \triangleq \{x \in \mathbb{R}^n : h(x) \geq 0\}$

- $h(x)$ is known
- $f(x, u)$ are unknown but continuous

$\Rightarrow \dot{h}(x, u)$ is unknown but trajectory data is available

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Problem Statement: Continuity Assumptions

Assumption

The function $\dot{h} : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is

1. globally Lipschitz continuous,
2. globally componentwise Lipschitz continuous, or
3. differentiable w.r.t. x and u with globally bounded Jacobians.

Problem: Robust Data-Driven CBF

Given a unknown system $\dot{x}$, CBF candidate $h : \mathcal{R}^n \to \mathcal{R}$ satisfying one of the continuity assumptions and an *a priori* data set, find sufficient conditions for the robust controlled invariance of the safe set $\mathcal{S} \triangleq \{x \in \mathcal{X} \mid \exists u \in \mathcal{U} \text{ s.t. } h(x) \geq 0\}$ (with state feedback).

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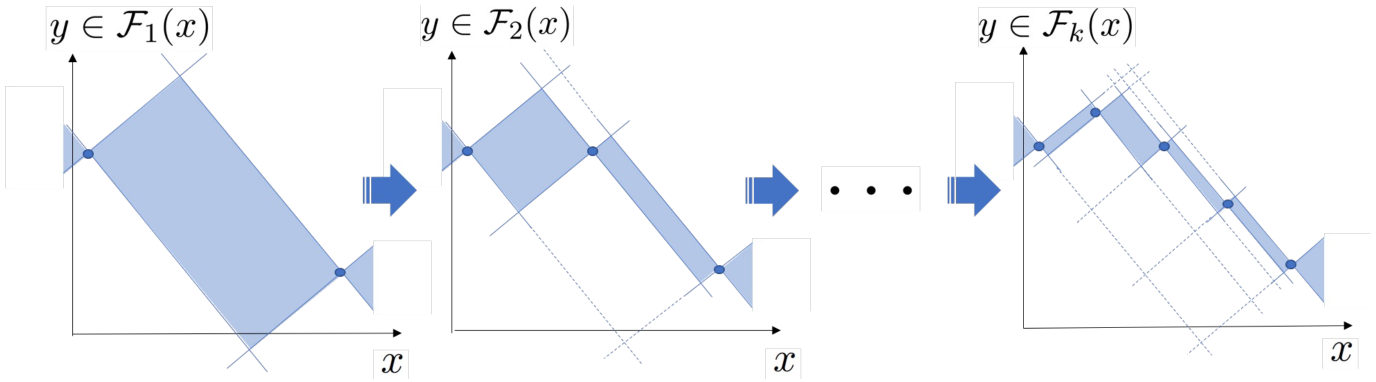
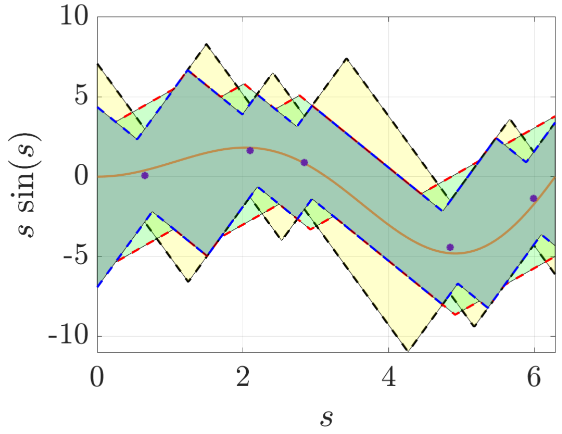
Idea: Set-Membership Learning

- Set-membership prediction, Lipschitz interpolation, kinky inference
  - Non-parametric learning approach with continuity assumption
    - Lipschitz continuous → Piecewise affine bounding functions
    - Hölder continuous → Piecewise nonconvex bounding functions
    - Differentiable with bounded Jacobians
      - Piecewise affine bounding functions
      - Less conservative than Lipschitz approach





Jin, Z., Khajenejad, M. and **Yong, S.Z.** LCSS, 2022.

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Idea: Set-Membership Learning

- Lower bound $\dot{h}(x, u)$ using data, directly from continuity definitions or interval/mixed-monotone bounding

Assumption

The function $\dot{h} : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is

1. globally Lipschitz continuous,
2. globally componentwise Lipschitz continuous, or
3. differentiable w.r.t. x and u with globally bounded Jacobians.

1. $\dot{h} \geq \dot{h}_i - L_x \|x - x_i\|_p - L_u \|u - u_i\|_p$
2. $\dot{h} \geq \dot{h}_i - L_x^\top |x - x_i| - L_u^\top |u - u_i|$
3. $\dot{h} \geq \dot{h}_i + \underline{J}_x \Delta x_i^+ - \overline{J}_x \Delta x_i^- + \underline{J}_u \Delta u_i^+ - \overline{J}_u \Delta u_i^-$
   where $\Delta x_i \triangleq x - x_i$ and $\Delta u_i \triangleq u - u_i$.

$$\dot{h}(x, u) \geq \underline{\dot{h}}_J(x, u) \geq \underline{\dot{h}}_{CL}(x, u) \geq \underline{\dot{h}}_L(x, u)$$

Jin, Z., Khajenejad, M. and **Yong, S.Z.** LCSS, 2022 & LSCC, 2023.

Approach: Robust Data-Driven CBF

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CBF-DD-L

Robust CBF condition (Lipschitz continuous):

$$\sup_{u\in\mathcal{U}} \max_{i\in\mathbb{Z}_N^+} \dot{h}_i - L_x\|x - x_i\|_p - L_u\|u - u_i\|_p \geq -\alpha(h(x)),$$

for all $x \in \mathcal{X}$ and $t \geq 0$.

CBF-DD-CL

Robust CBF condition (Componenentwise Lipschitz continuous):

$$\sup_{u\in\mathcal{U}} \max_{i\in\mathbb{Z}_N^+} \dot{h}_i - L_x^\top|x - x_i| - L_u^\top|u - u_i| \geq -\alpha(h(x)),$$

for all $x \in \mathcal{X}$ and $t \geq 0$.

CBF-DD-J1

Robust CBF condition (Bounded Jacobians v1):

$$\sup_{u\in\mathcal{U}} \max_{i\in\mathbb{Z}_N^+} \begin{array}{l} \dot{h}_i + \underline{J}_x\Delta x_i^+ - \overline{J}_x\Delta x_i^- \\ +\underline{J}_u\Delta u_i^+ - \overline{J}_u\Delta u_i^- \end{array} \geq -\alpha(h(x)),$$

for all $x \in \mathcal{X}$ and $t \geq 0$, where $\Delta x_i \triangleq x - x_i$ and $\Delta u_i \triangleq u - u_i$.

CBF-DD-J2

Robust CBF condition (Bounded Jacobians v2):

$$\sup_{\substack{u\in\mathcal{U},\\ u_i^\oplus, u_i^\ominus}} \max_{i\in\mathbb{Z}_N^+} \dot{h}_i + \underline{J}_x\Delta x_i^+ - \overline{J}_x\Delta x_i^- + \underline{J}_u u_i^\oplus - \overline{J}_u u_i^\ominus \geq -\alpha(h(x))$$
$$\text{s.t. } u_i^\oplus \geq 0,\ u_i^\ominus \geq 0,\ u_i^\oplus - u_i^\ominus = u - u_i$$

for all $x \in \mathcal{X}$ and $t \geq 0$, where $\Delta x_i \triangleq x - x_i$.

➔ **Involves piecewise affine functions** → Mixed-Integer Quadratic Programs

Jin, Z., Khajenejad, M. and **Yong, S.Z.** LCSS, 2023.

Approach: Complexity Reduction Strategies

1. **Parallel Computing** via decomposition into multiple quadratic programming (QP) or analytical subproblems

CBF-DD-sub

Consider a data point $(\dot{h}_i, x_i, u_i)$ in the data set $\mathcal{D} = \{(\dot{h}_i, x_i, u_i)\}_1^N$, we can find the u_i that is closest to the u in the safe input set $\mathcal{U}_i(x)$ by solving the following optimization problem:

$$u_i^*(x) = \arg\min_u \tfrac{1}{2}\|u - k(x)\|_2^2$$
$$s.t.\ u \in \mathcal{U}_i^{\phi}(x).$$

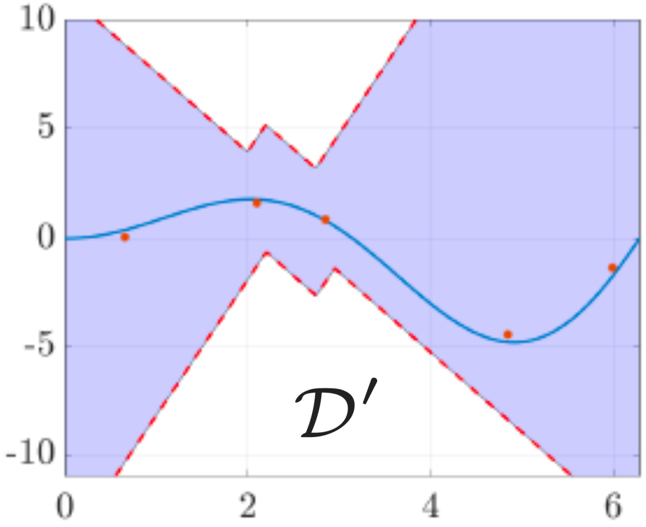
with $\mathcal{U}_i^{\phi}(x), \phi \in \{L, CL, J1, J2\}$ based on the given continuity case.

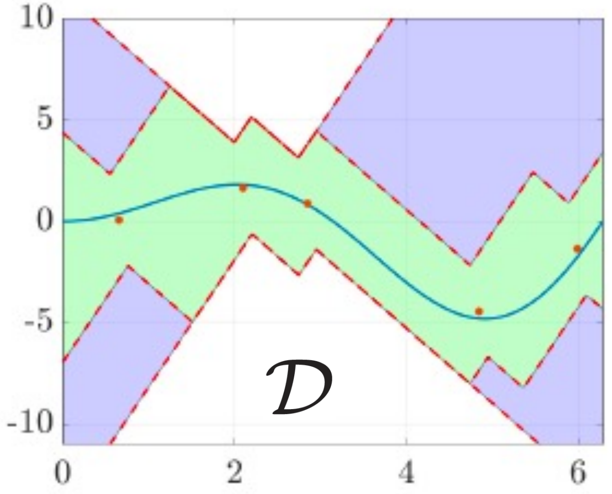
$$u(x) = \arg\min_{u \in \{u_1^*(x), \dots, u_N^*(x)\}} \tfrac{1}{2}\|u - k(x)\|_2^2.$$

Jin, Z., Khajenejad, M. and **Yong, S.Z.** LCSS, 2023.

Approach: Complexity Reduction Strategies

2. **Downsampling** via the use of a subset of "nearby" data by leveraging monotonicity of the learning approach
   - kNN and clustering method (cf. [Jin, Khajenejad & **Yong**, 2020])





Monotonicity

The safe input sets in Definitions satisfy monotonicity, in the sense that given two data sets $\mathcal{D}$ and $\mathcal{D}'$ and their corresponding safe input sets $K_S(x)$ and $K'_S(x)$, $\mathcal{D}' \subseteq \mathcal{D}$ implies that $K'_S(x) \subseteq K_S(x)$.

Approach: Lipschitz Constants/Jacobian Bounds

What if the Lipschitz constants or Jacobian bounds are unknown?

Estimation of Lipschitz Constant

The Lipschitz constant from sampled data set $\overline{\mathcal{D}} = \{(\tilde{s}_j^o, \tilde{y}_{j+1}^o) | j = n_y, \cdots, N-1\}$ can be estimated by:

$$\hat{L}_p^{(i)} = \max_{j \neq k} \frac{|(\tilde{y}_j^o)^{(i)} - (\tilde{y}_k^o)_k^{(i)}| - 2\varepsilon_v}{||\tilde{s}_j^o - \tilde{s}_k^o||_p + 2\varepsilon_s}.$$

Estimation of Jacobian Bounds

The Jacobian bounds from the data set $\overline{\mathcal{D}} = \{(\tilde{s}_j, \tilde{y}_{j+1}) | j = n_y, \cdots, N-1\}$ by solving the MILP:

$$\min_{J_u, J_l} \sum_{i=1}^{m} \overline{g}^{(i)}(J_u, J_l, \overline{\Delta s}_{j,\ell}, \underline{\Delta s}_{j,\ell}) - \underline{g}^{(i)}(J_u, J_l, \overline{\Delta s}_{j,\ell}, \underline{\Delta s}_{j,\ell})$$

subject to $\forall j, \ell \in \{n_y, \ldots, N-1\},\ j \neq \ell :$

$$\tilde{y}_{j+1} - \tilde{y}_{\ell+1} \leq \overline{g}(J_u, J_l, \overline{\Delta s}_{j,\ell}, \underline{\Delta s}_{j,\ell}) + 2\varepsilon_v,$$

$$\tilde{y}_{j+1} - \tilde{y}_{\ell+1} \geq \underline{g}(J_u, J_l, \overline{\Delta s}_{j,\ell}, \underline{\Delta s}_{j,\ell}) - 2\varepsilon_v,$$

$$J_u \geq J_l$$

Estimation with high probability:

Proposition (PAC Learning)

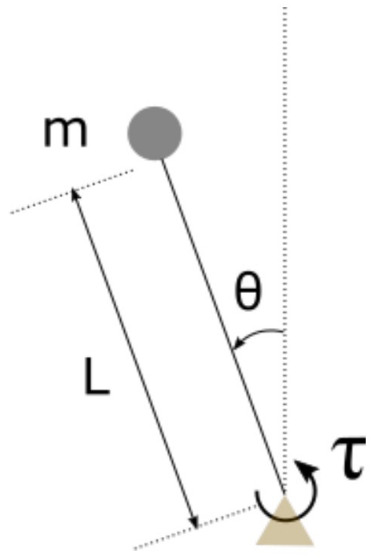
Let $\epsilon, \delta \in \mathbb{R}^+$. Suppose N samples drawn from some $\mathcal{P}$ satisfies $N \geq \frac{1}{\epsilon} \ln \frac{1}{\delta}$.
Then, with a probability greater than $1 - \delta$, the probability of an error $\mathtt{err}_{\mathcal{P}}(\hat{\theta})$ is less than ϵ.

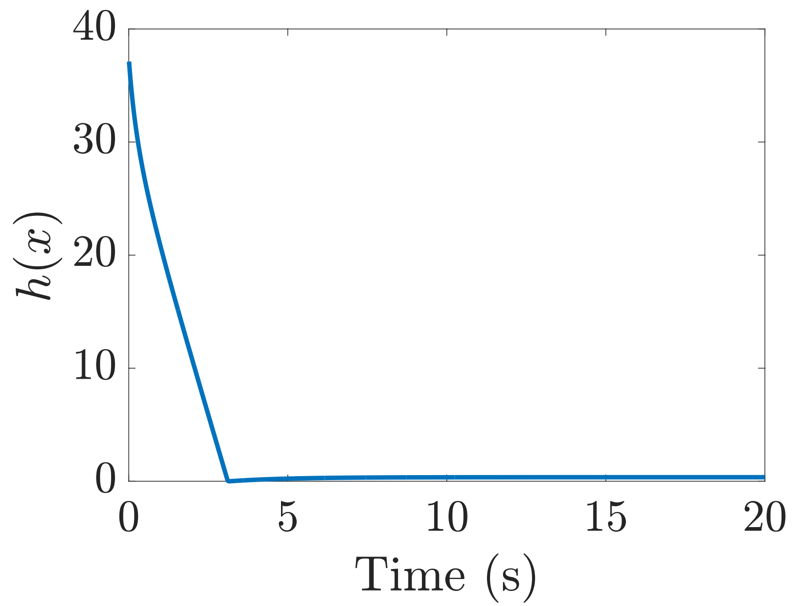
Simulations: Inverted Pendulum & Adaptive Cruise Control

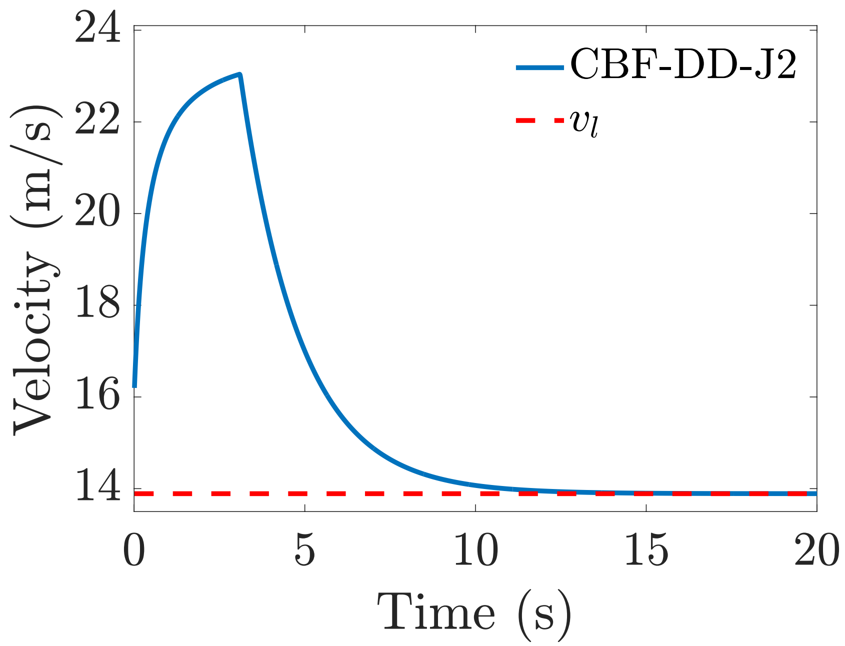
TABLE I: CPU time comparison for different methods.

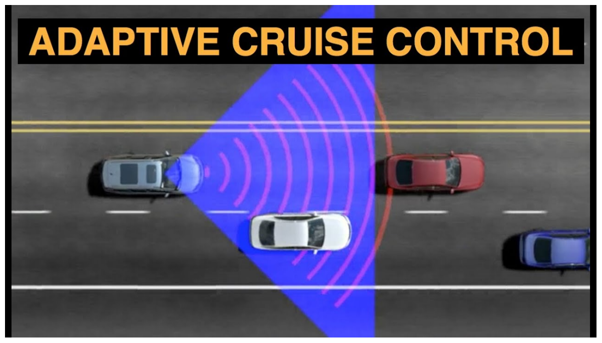
Method	L	CL	J1	J2	SOCP [16]
CPU time (s)	3029	3054	3154	2724	2.84×10^5

Taylor, Dorobantu, Dean, Recht, Yue, and Ames, CDC, 2021.









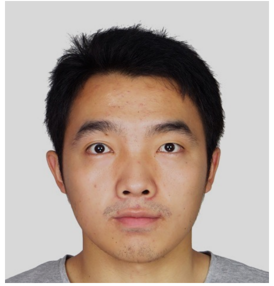
Summary

- Robust control barrier functions for uncertain systems:
  - Uncertain parameter-varying systems
    - Leveraged mixed-monotonicity, concave bounding and robust optimization
    - Set-membership parameter estimation using computational geometry and mixed-monotonicity based interval observers
  - Unknown continuous systems with only prior trajectory data
    - Set-membership learning under various continuity assumptions
    - Complexity reduction techniques for robust data-driven CBF

Challenges/Opportunities

- Computationally efficient and tight set-membership parameter estimation
  → Tighter zonotopic/polytopic observers for immersion/nonlinear systems
- Reliable estimation of continuity parameters—Lipschitz constant/Jacobian bounds—for set-membership learning
  → Confidence or error bounds for these constants/bounds
- Learning of control barrier functions from positive demonstrations
  → Non-parametric/set-membership learning of CBFs
  → Active learning for exploring safety boundaries while remaining safe
- Preview control barrier functions
  → Incorporation of future/preview information of (immutable) disturbances or predictions for nonlinear systems

Thank you!



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