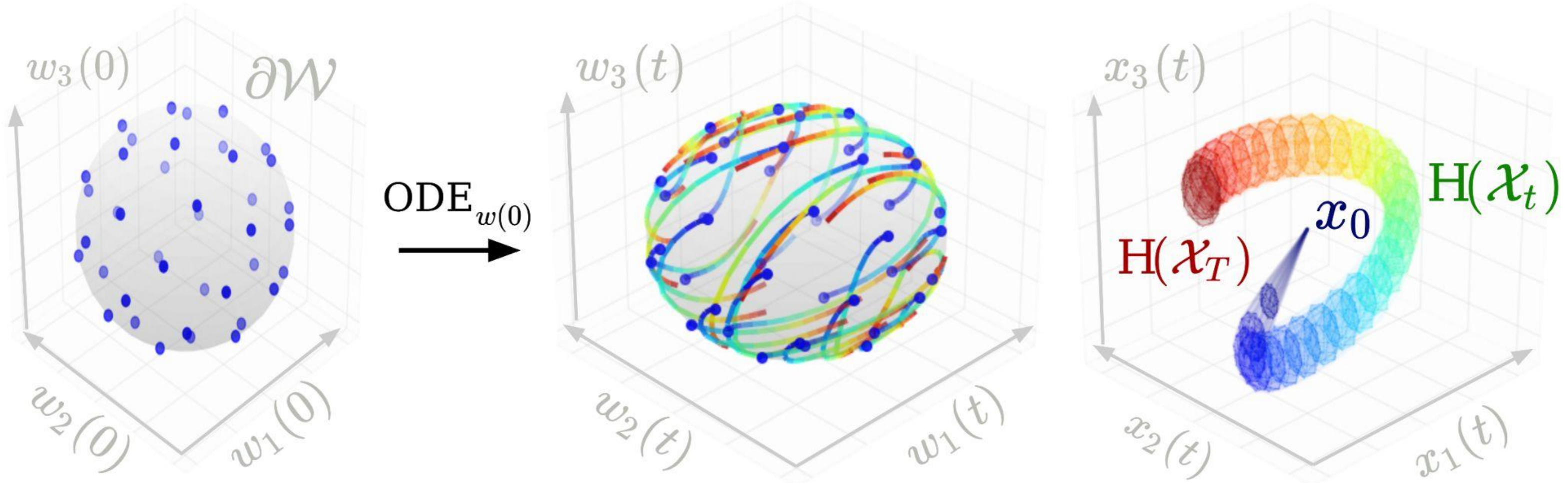

Exact Characterization of the Convex Hulls of Reachable Sets

Thomas Lew[1,2]

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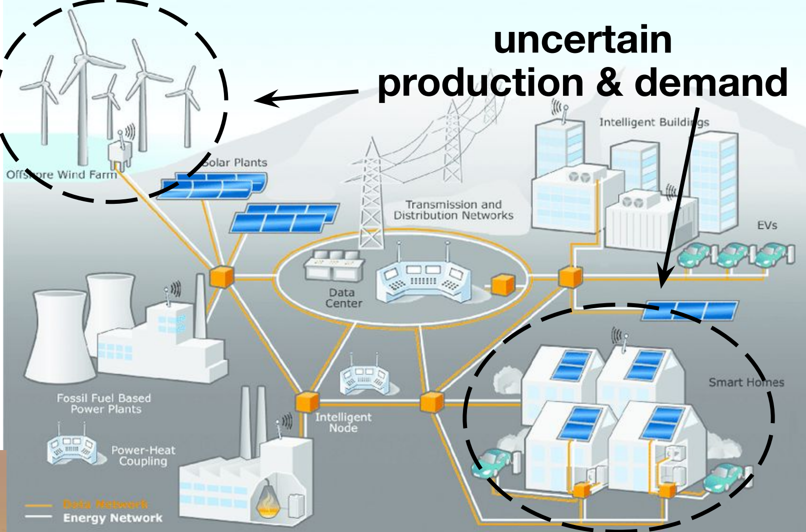
Riccardo Bonalli[3]

Marco Pavone[2]

International Online Seminar on Interval Methods in Control Engineering January 2024

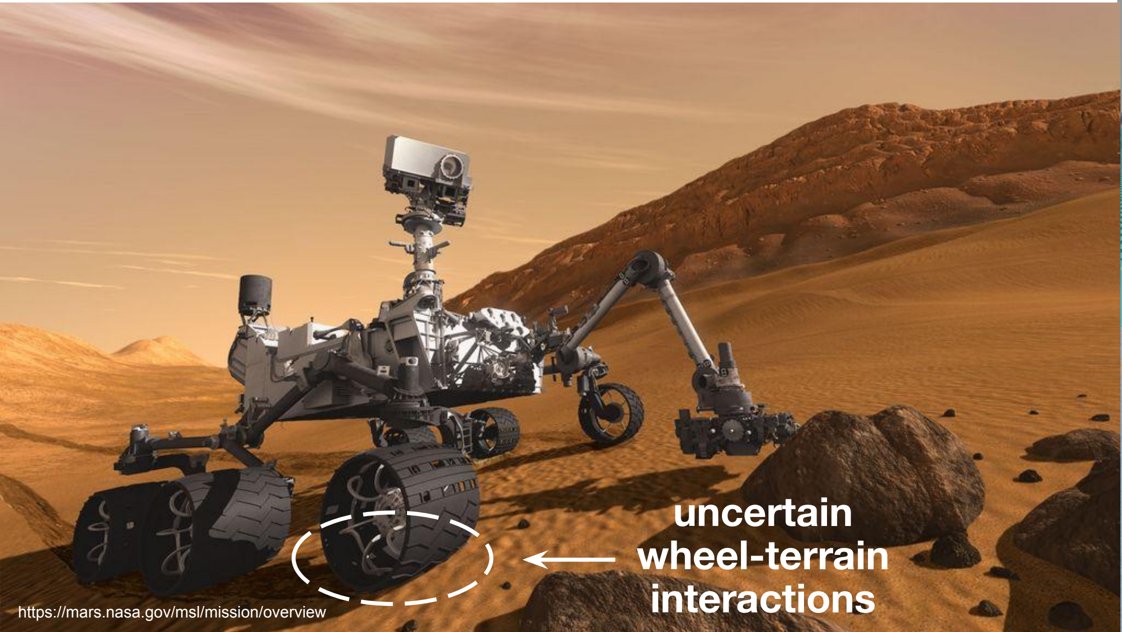
Modern control systems are

- increasingly **complex**
- used in **high-uncertainty** environments
- used in **high-stakes applications**



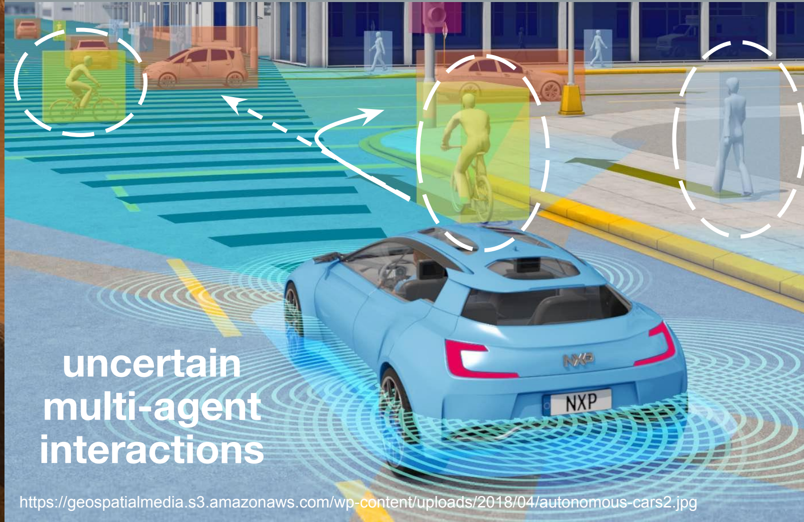
uncertain production & demand

Yasin Kabalci, Energy: A survey on smart metering and smart grid communication, 2016



uncertain wheel-terrain interactions

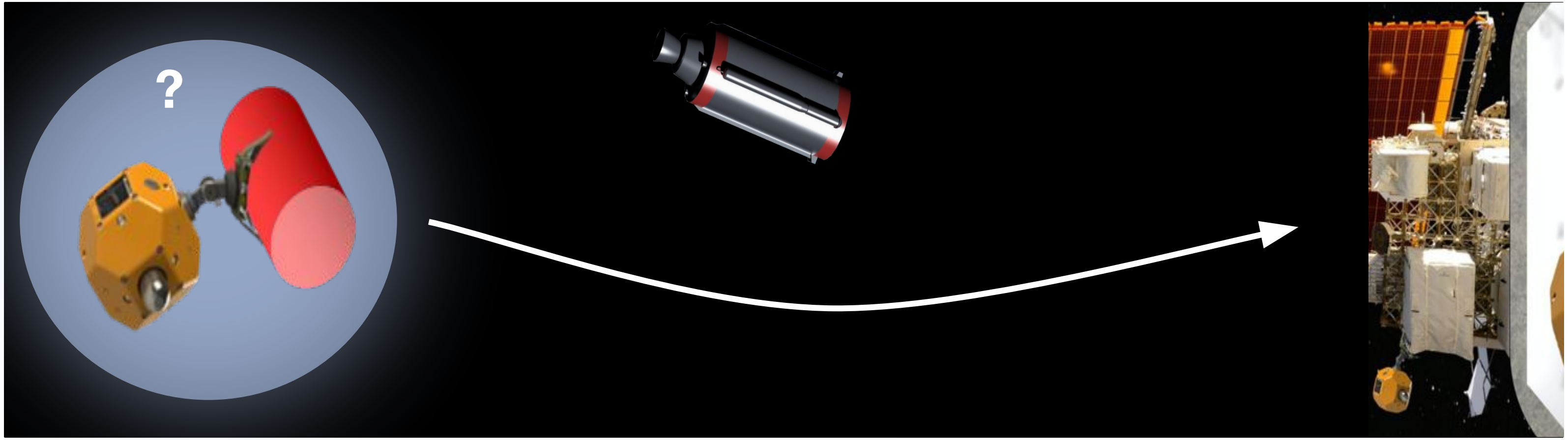
https://mars.nasa.gov/msl/mission/overview



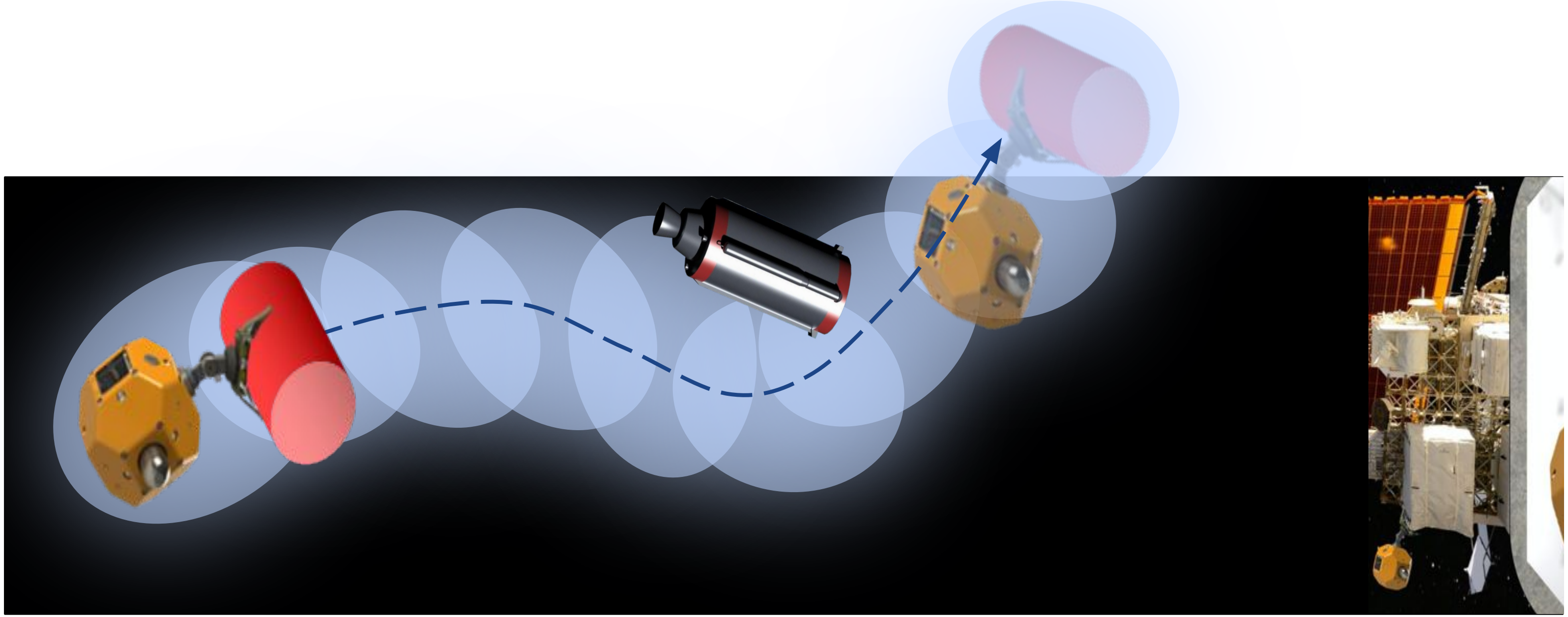
uncertain multi-agent interactions

https://geospatialmedia.s3.amazonaws.com/wp-content/uploads/2018/04/autonomous-cars2.jpg

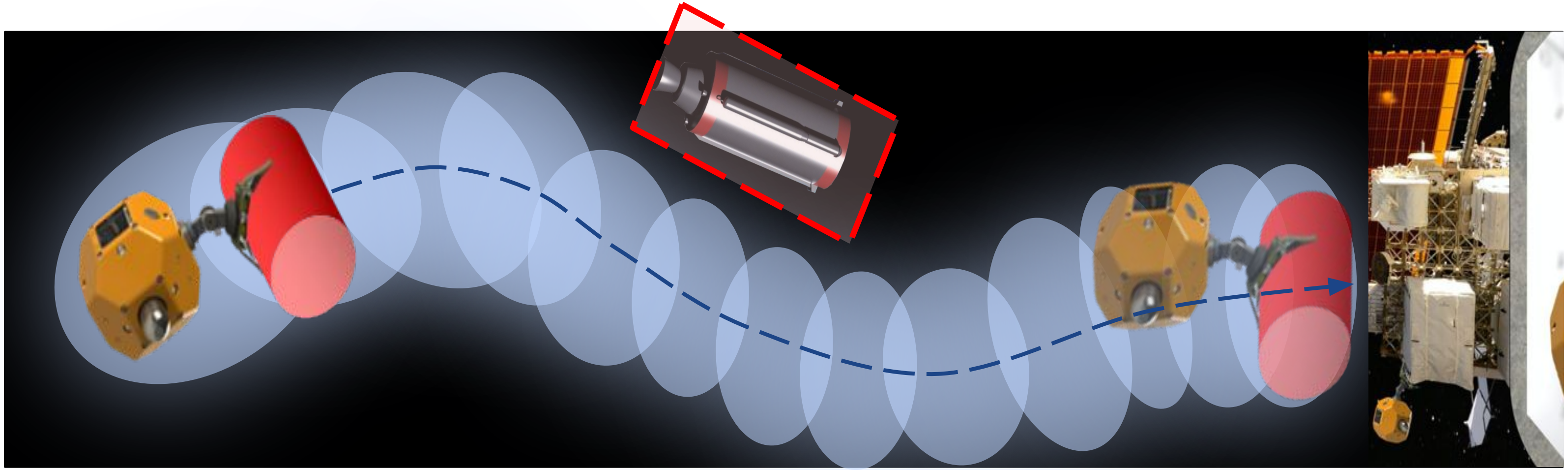
Accounting for **uncertain dynamics** to perform tasks under uncertainty



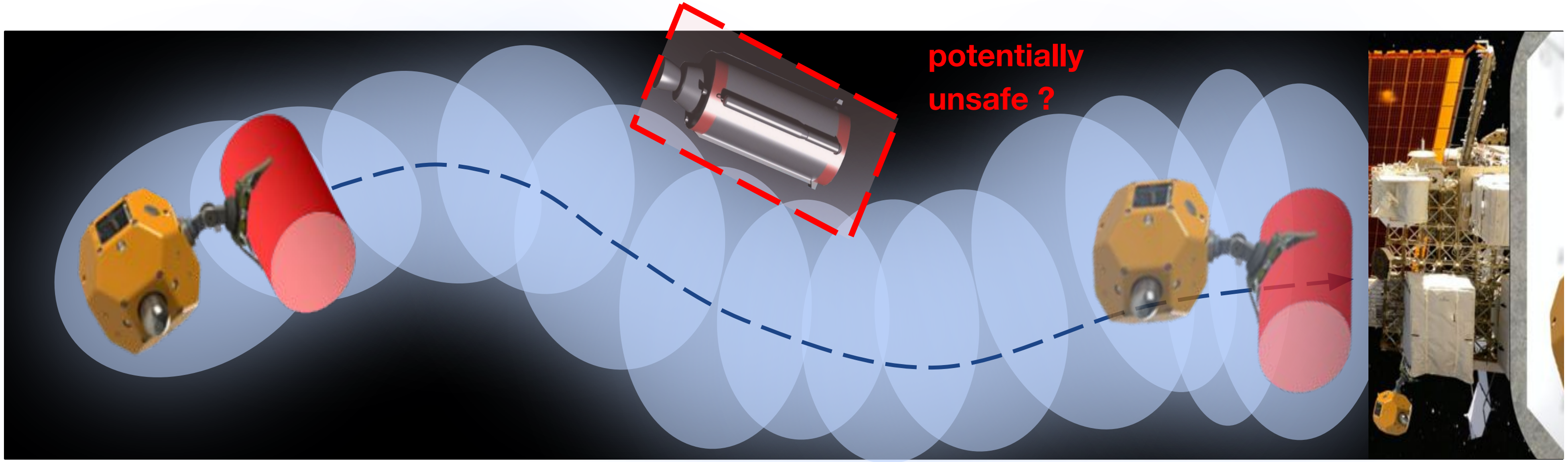
Uncertainty Propagation



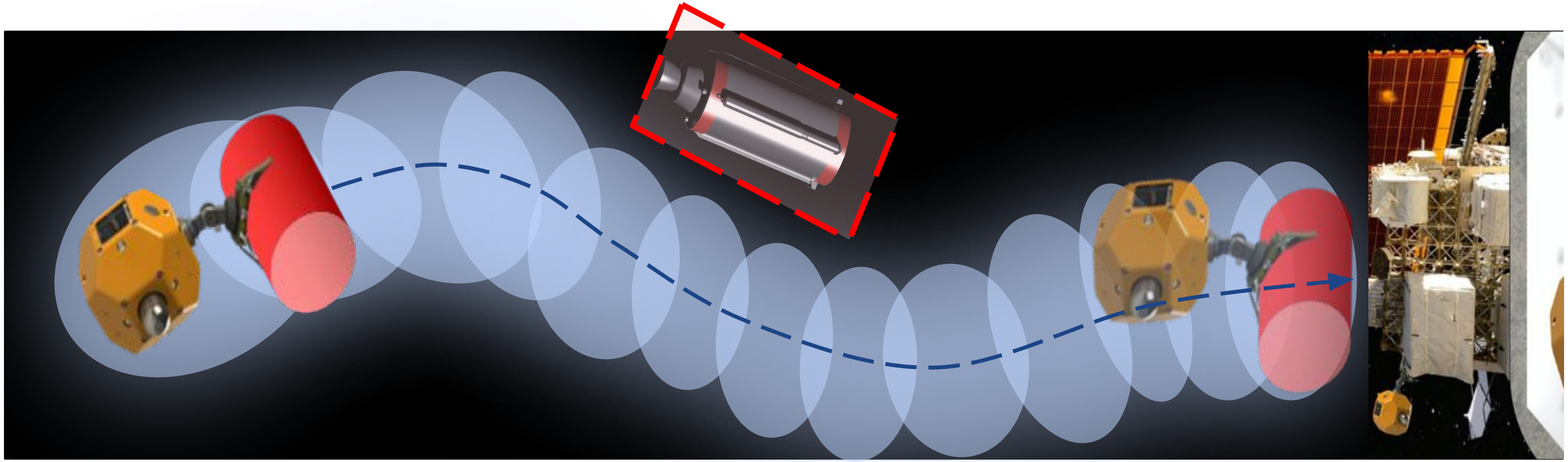
Robust Planning



Accurate Uncertainty Propagation is Needed



This talk is about **Reachability Analysis**



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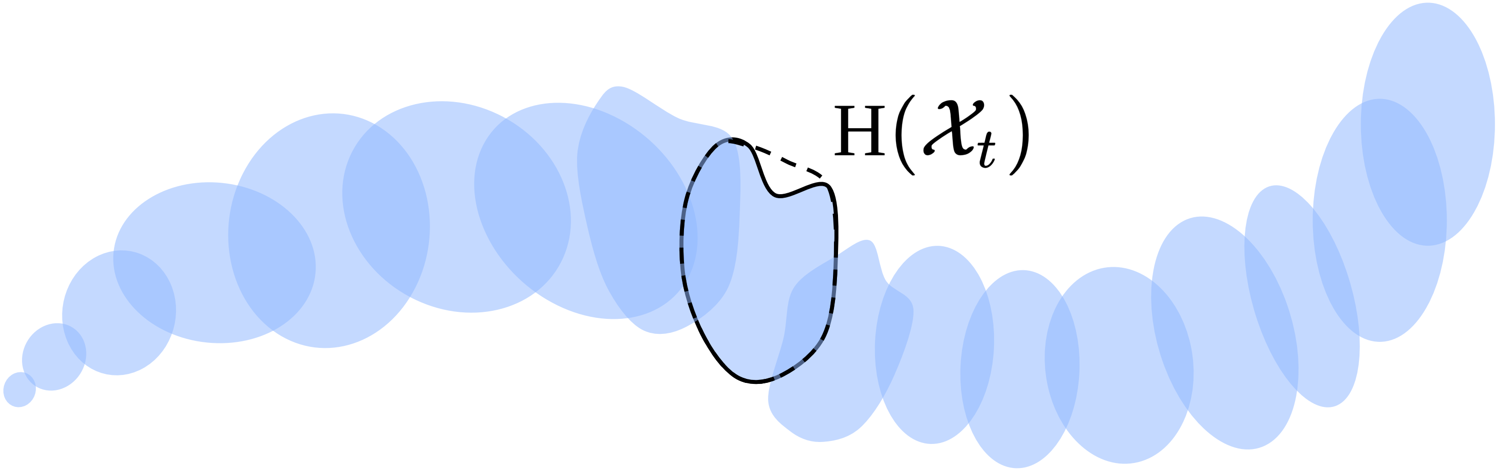
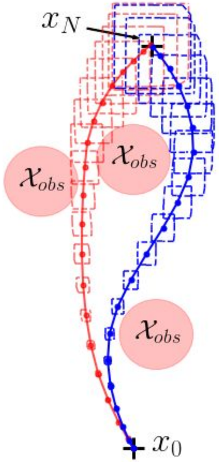


an exact characterization of the convex hulls of reachable sets

$$\mathrm{H}\left(\mathcal{X}_t\right) = \mathrm{H}\left(F(\partial\mathcal{W}, t)\right)$$

Related work

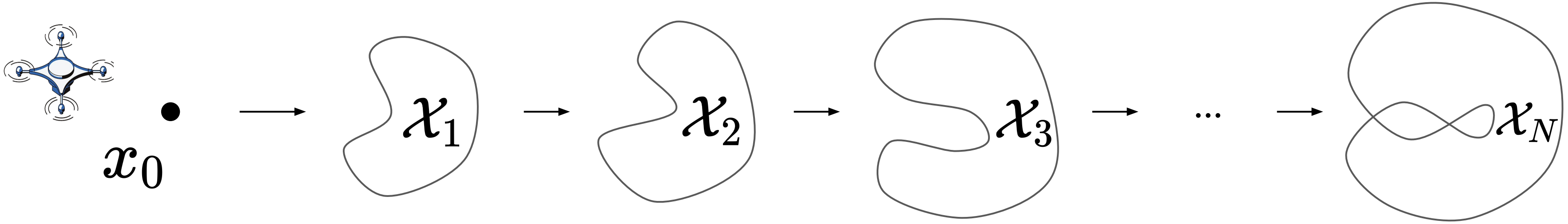


- Applications: robust model predictive control (MPC), robustness analysis of feedback loops, …
  - *For computational reasons,* ***convex over-approximations*** *are often used in practice.*

- A wide range of tools
  - Linearization-based methods [Althoff‘08] [Yu’18] [Koller’18] [Leeman‘23]
  - Differential inequalities [Scott’13] [Ramdani’19]
  - Taylor models [Berz’98] [Chen’13]
  - Hamilton-Jacobi [Chen’18]
  - Sampling-based approaches [Lew’20] [Devonport’20] [Thorpe’21]

Related work

- Applications: robust model predictive control (MPC), …
  - ○ ***Convex*** *over-approximations are often sufficient*
- A wide range of tools (linearization-based methods, …)
- **Single-step** propagation methods (e.g., [Koller et al, CDC’18])



For many problems, existing methods remain either too **slow** or too **conservative.** *Is there **additional structure** we can use?*

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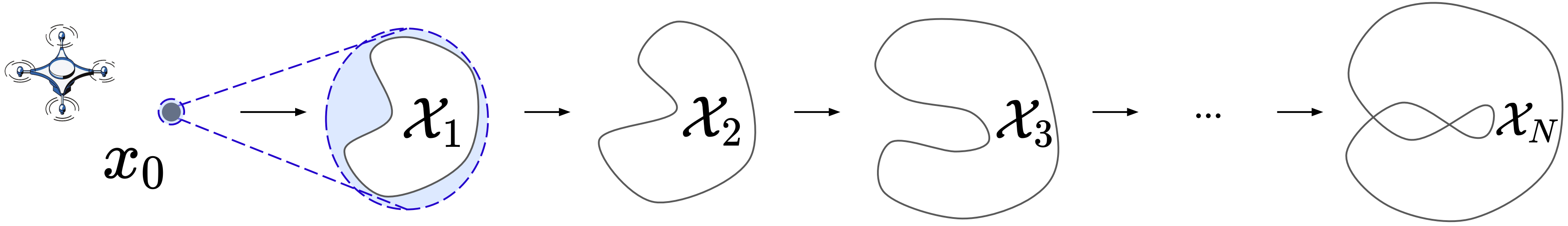

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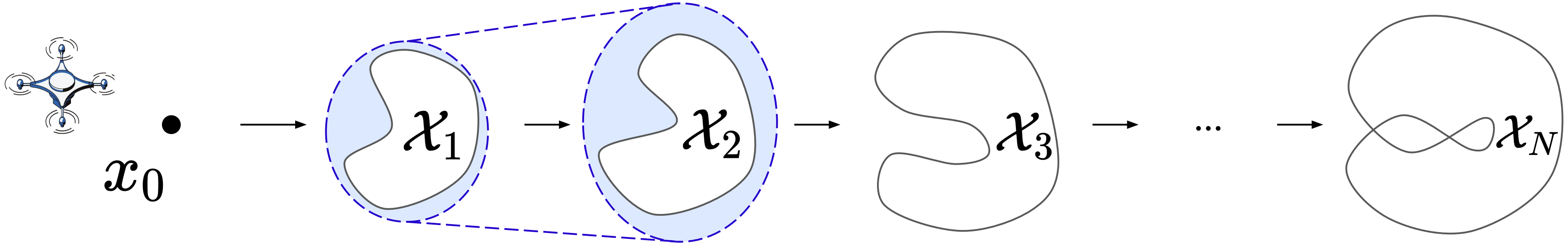

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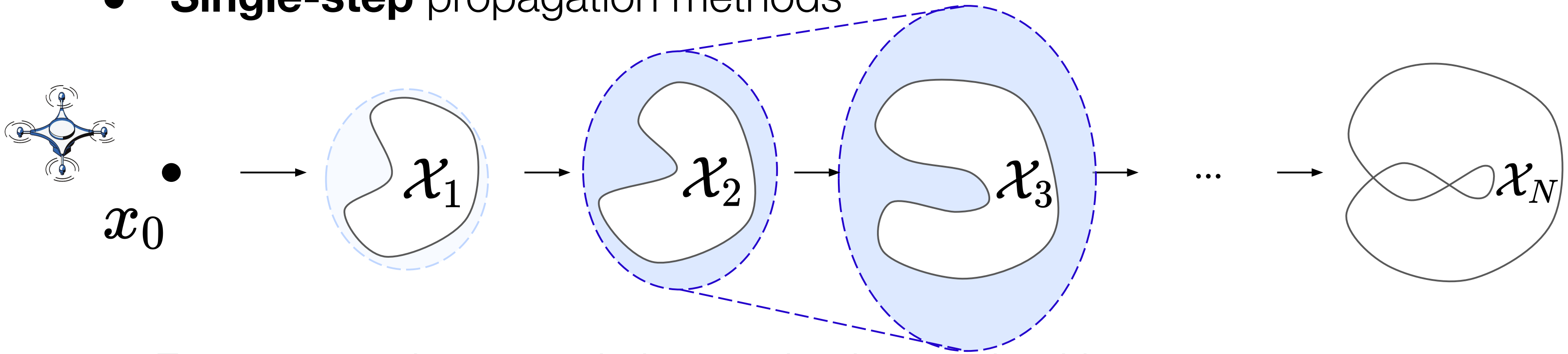

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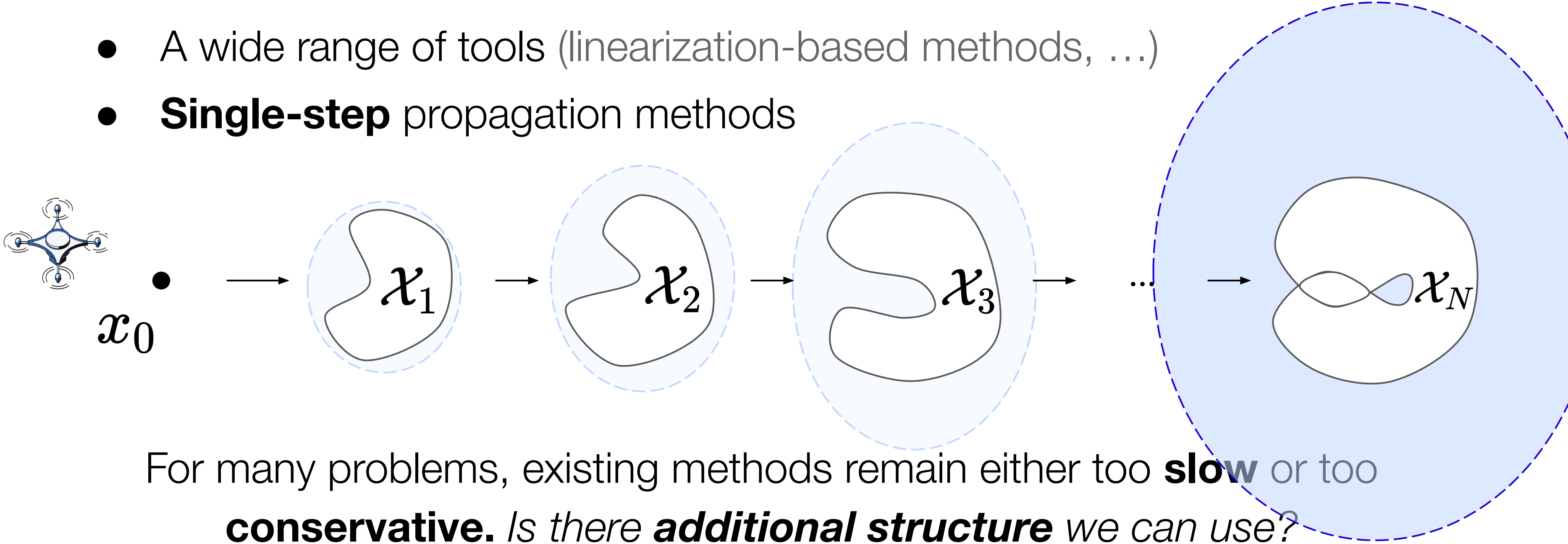


For many problems, existing methods remain either too **slow** or too **conservative.** *Is there **additional structure** we can use?*

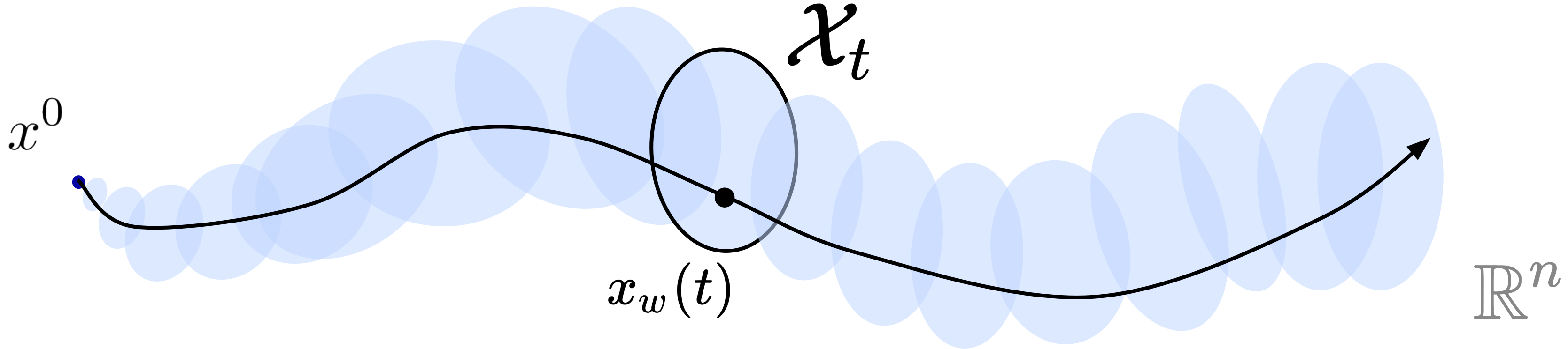
Problem Definition

Dynamical system (**ODE**):

$$\dot{x}(t) = f(x(t)) + w(t), \quad x(0) = x^0, \quad w(t) \in \mathcal{W}$$

Reachable sets:

$$\mathcal{X}_t = \left\{ x_w(t) = x^0 + \int_0^t \left(f(x_w(s)) + w(s)\right)\mathrm{d}s \ : \ w \in L^\infty([0,T],\mathcal{W}) \right\}$$



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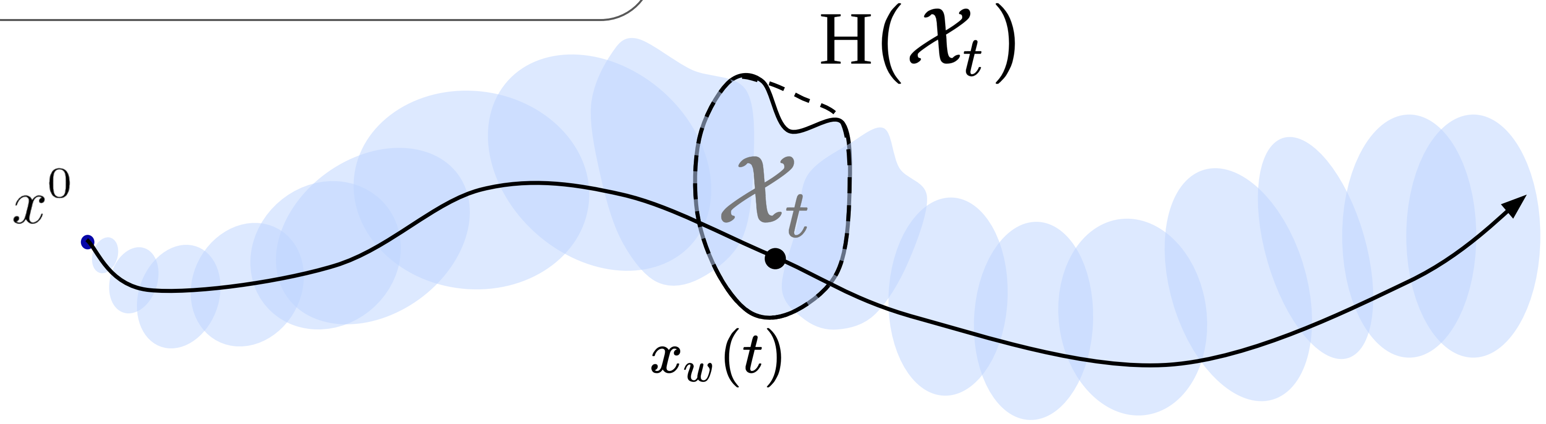
Objective: computing the **convex hulls** $\mathrm{H}(\mathcal{X}_t)$



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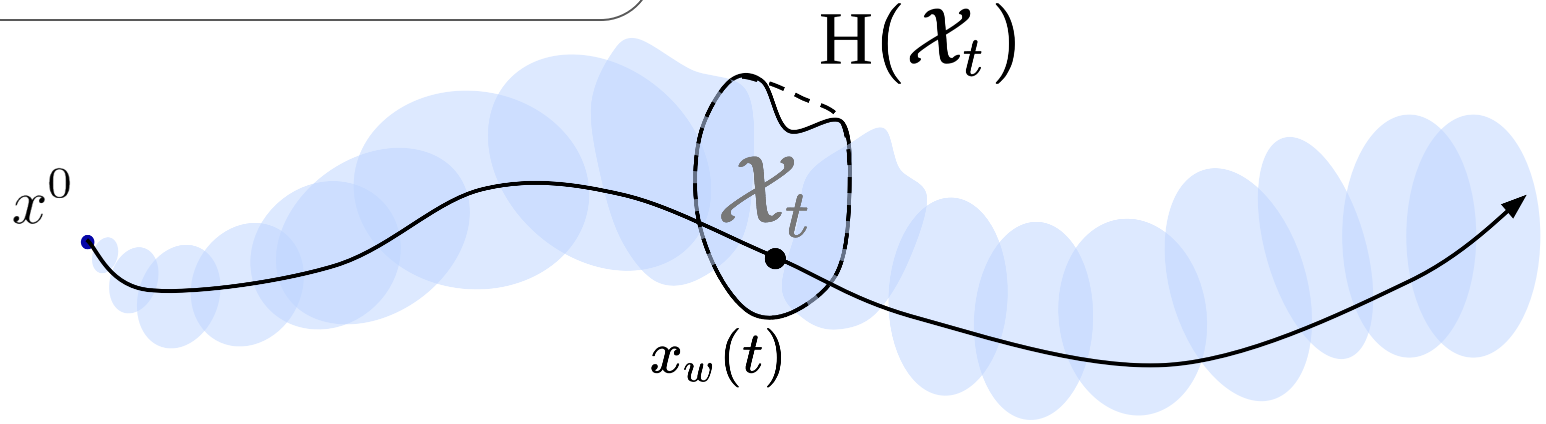
Objective: computing the **convex hulls** $\mathrm{H}(\mathcal{X}_t)$

each $\mathcal{X}_t$ is the image of an **infinite-dimensional** set



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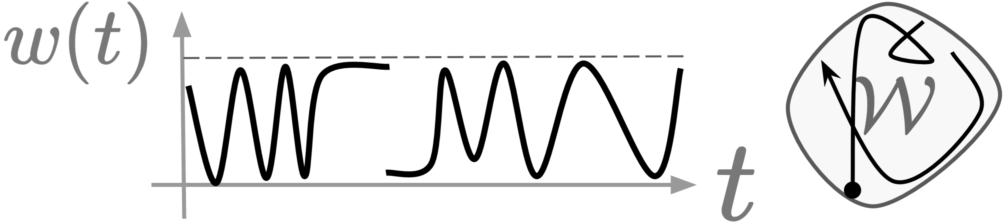

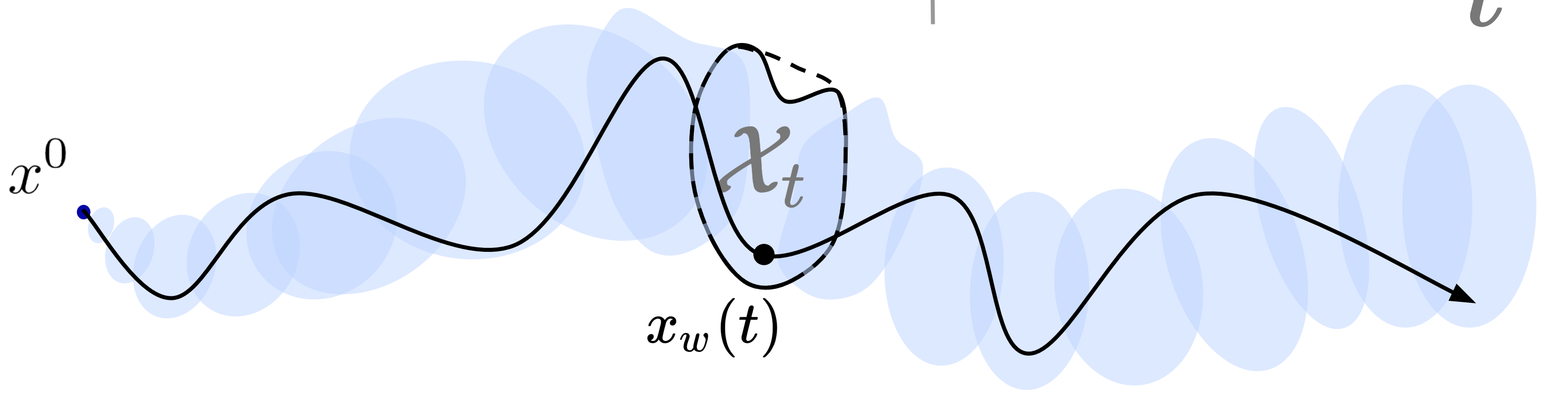

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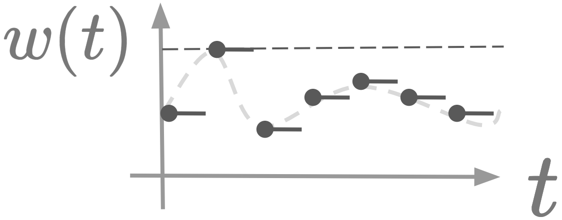
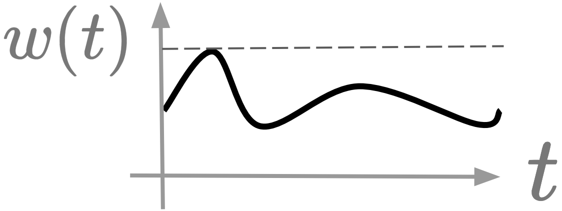
Important Remark

- Studying **discrete-time** formulations instead?
    - May lead to algorithms that are **sensitive to the choice of discretization**



- Studying **continuous-time** formulations
    - informs the design of algorithms that are **robust to the discretization**
    - gives **additional structure to use**



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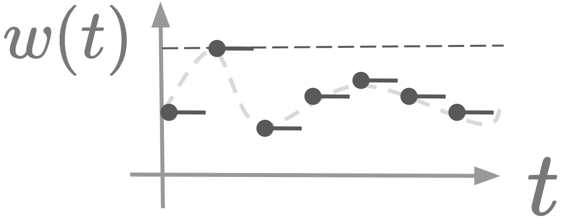

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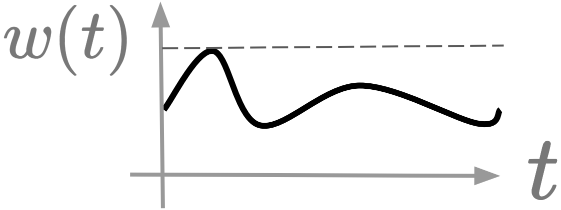


Characterization of the Convex Hulls

Reachable sets:

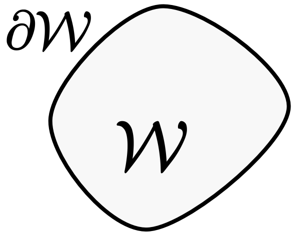
$$\mathcal{X}_t = \left\{ x_w(t) = x^0 + \int_0^t \big(f(x_w(s)) + w(s)\big)\mathrm{d}s \ : \ w \in L^\infty([0,T], \mathcal{W}) \right\}$$

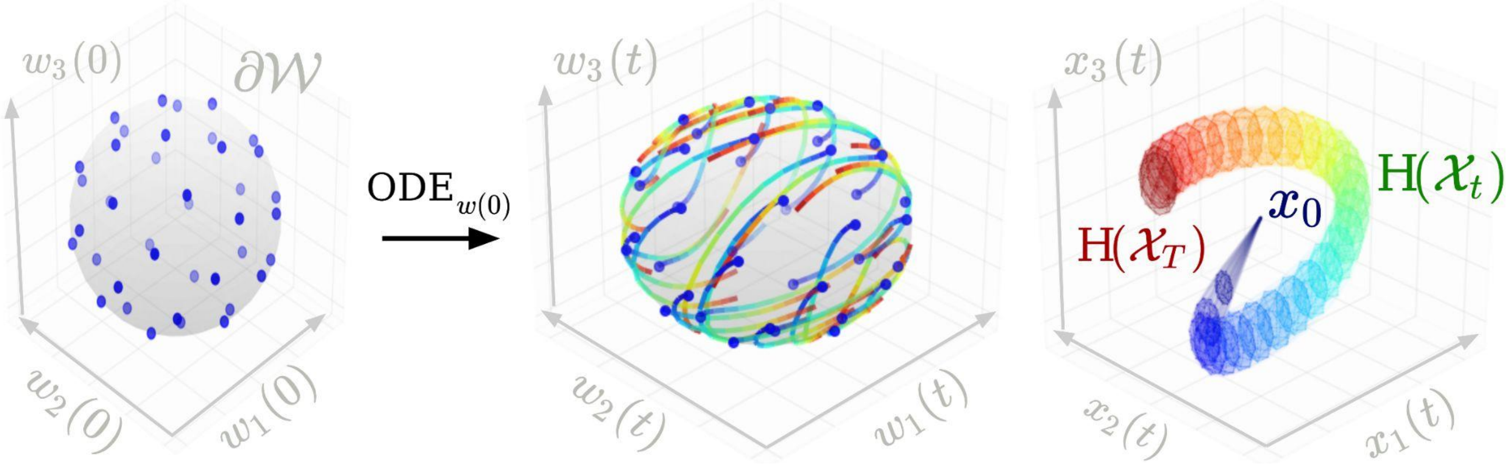
under two assumptions, for all times,

$$\mathrm{H}\left(\mathcal{X}_t\right) = \mathrm{H}\left(F(\partial\mathcal{W}, t)\right)$$

solution to an ODE

of dimension (**n-1**): **finite-dimensional**





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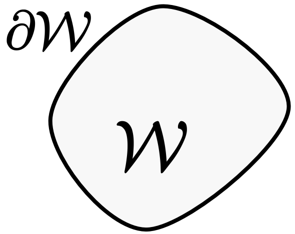
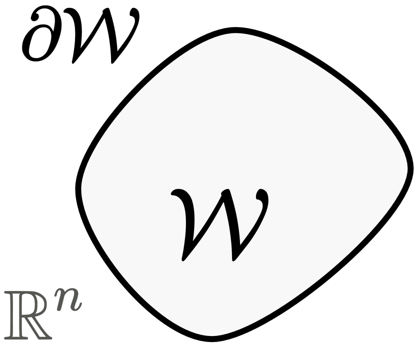


$$\dot{x}(t) = f(x(t)) + w(t), \quad x(0) = x^0, \quad w(t) \in \mathcal{W}$$

Assumption 1: f is smooth.

Assumption 2: $\mathcal{W}$ is compact and its boundary $\partial\mathcal{W}$ is an ovaloid.



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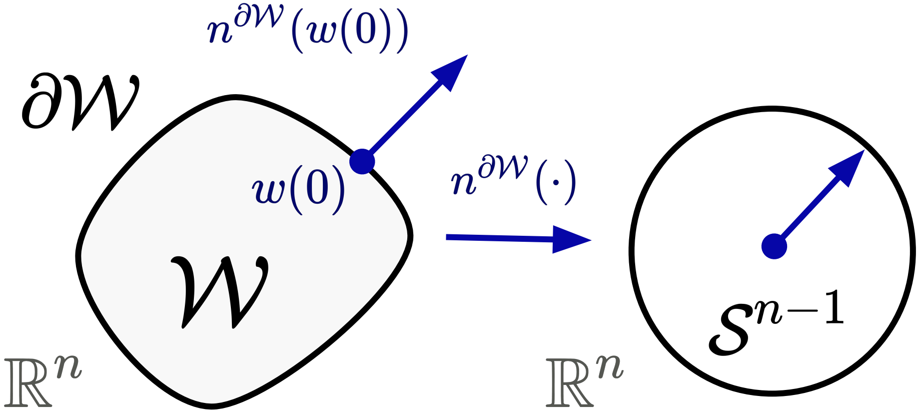


The **Gauss map**

$$n^{\partial\mathcal{W}} : \partial\mathcal{W} \to \mathcal{S}^{n-1}$$

gives the **normal** vector of $\partial\mathcal{W}$ and is a *diffeomorphism*.

Example: If $\mathcal{W}$ is a ball, then $n^{\partial\mathcal{W}}(w) = \frac{w}{\|w\|}$.

Main Result

$$\dot{x}(t) = f(x(t)) + w(t), \quad x(0) = x^0, \quad w(t) \in \mathcal{W}$$

Define the **augmented ODE**

$$\mathbf{ODE}_{w(0)} : \begin{cases} \dot{x}(t) = f(x(t)) + \left(n^{\partial\mathcal{W}}\right)^{-1}(q(t)) \\ \dot{q}(t) = -\mathrm{Proj}_{q(t)}\left(\nabla f(x(t))^\top q(t)\right) \\ (x(0), q(0)) = \left(x^0, n^{\partial\mathcal{W}}(w(0))\right) \end{cases}$$

$$\boxed{\mathrm{Proj}_v(u) = \left(I_{n\times n} - vv^\top\right)u}$$

and the **solution map** $F : \partial\mathcal{W} \times [0,T] \to \mathbb{R}^n$,

$$F(w(0), t) = x_{w(0)}(t)$$

Theorem: under assumptions 1 and 2,

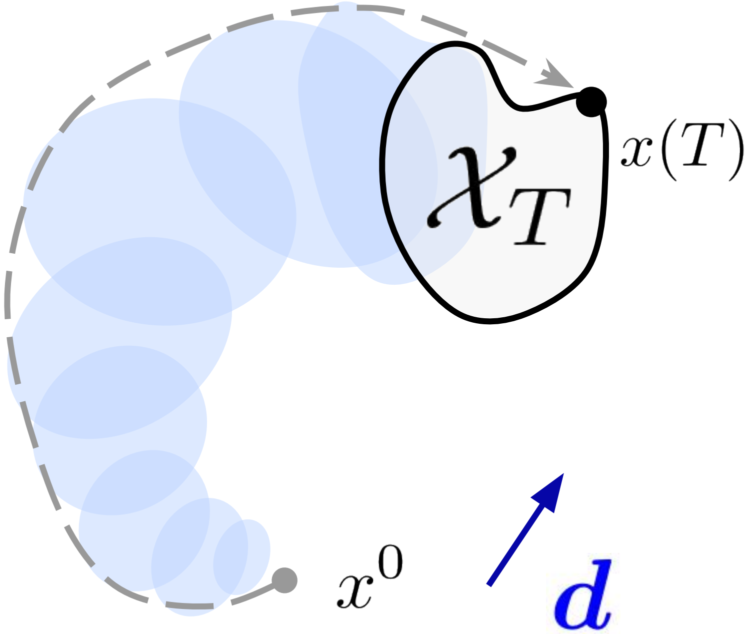
$$\mathrm{H}\left(\mathcal{X}_t\right) = \mathrm{H}\left(F(\partial\mathcal{W}, t)\right) \quad \text{for all } t \in [0,T].$$

Proof (1/4) - *define an* ***optimal control problem*** *to search over boundary states*

Let $\boldsymbol{d}$ be a search direction and define

$$(\mathbf{OCP}_{\boldsymbol{d}})$$

$$\begin{aligned} \inf_{w} \quad & -\boldsymbol{d}^\top x(T) \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t)) + w(t), \\ & x(0) = x^0. \end{aligned}$$

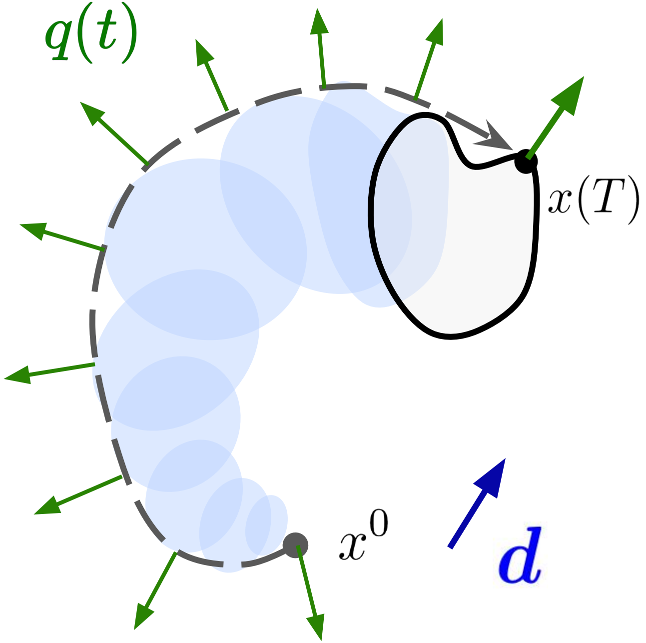


Proof (2/4) - use ***Pontryagin's Maximum Principle*** to reformulate as a boundary-value problem (BVP)

For any locally-optimal solution of $(\mathbf{OCP}_{\boldsymbol{d}})$, there exists an **adjoint vector** q such that

$$\mathbf{BVP}_{\boldsymbol{d}}: \begin{cases} \dot{x}(t) = f(x(t)) + (n^{\partial\mathcal{W}})^{-1}(q(t)) \\ \dot{q}(t) = -\mathsf{Proj}_{q(t)}\left(\nabla f(x(t))^\top q(t)\right) \\ x(0) = x^0 \\ q(T) = \boldsymbol{d} \end{cases}$$

where $\mathsf{Proj}_v(u) = (I_{n\times n} - vv^\top)\,u$ projects onto the tangent space of the unit sphere.



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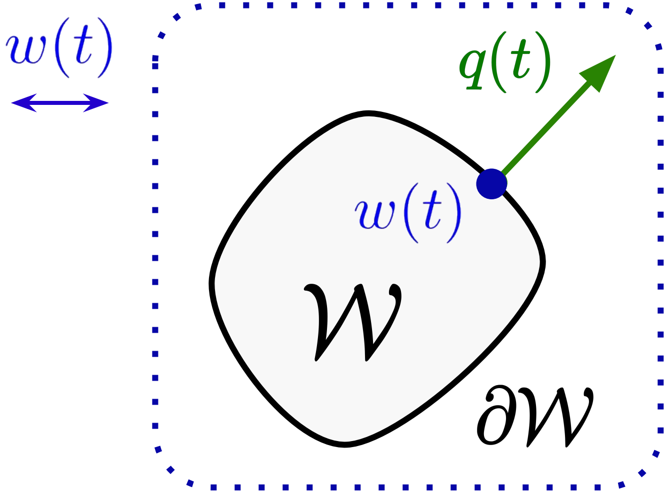

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Proof (2/4) - we have a boundary-value problem

$$\mathbf{BVP}_{\boldsymbol{d}} : \begin{cases} \dot{x}(t) = f(x(t)) + (n^{\partial\mathcal{W}})^{-1}(q(t)) \\ \dot{q}(t) = -\mathrm{Proj}_{q(t)}\left(\nabla f(x(t))^\top q(t)\right) \\ x(0) = x^0 \\ q(T) = \boldsymbol{d} \end{cases}$$

We reduced the search for solutions to from $w \in \boxed{L^\infty([0,T];\mathcal{W})}$ to $\boxed{q(0) \in \mathcal{S}^{n-1}}$

infinite-dimensional

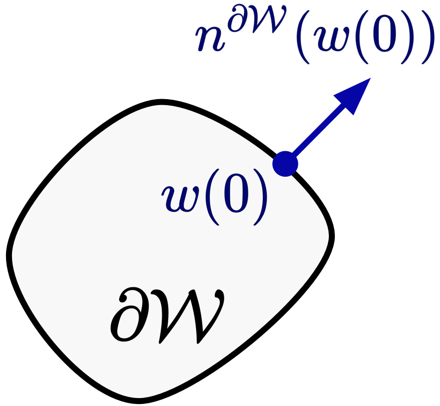
only **n** variables

However, **solving BVPs** can be **challenging**.

- [Gornov et al'15] explored this approach for reachability analysis in a related 3d setting.

Proof (3/4) - what if we knew $w(0)$?

$$\mathbf{BVP}_{\boldsymbol{d}} : \begin{cases} \dot{x}(t) = f(x(t)) + \boxed{(n^{\partial\mathcal{W}})^{-1}(q(t))} \quad \longleftrightarrow \quad w(t) \\ \dot{q}(t) = -\mathrm{Proj}_{q(t)}\left(\nabla f(x(t))^\top q(t)\right) \\ x(0) = x^0 \\ q(T) = \boldsymbol{d} \end{cases} \qquad \boxed{q(0) = n^{\partial\mathcal{W}}(w(0))}$$



$$\mathbf{ODE}_{w(0)} : \begin{cases} \dot{x}(t) = f(x(t)) + (n^{\partial\mathcal{W}})^{-1}(q(t)) \\ \dot{q}(t) = -\mathrm{Proj}_{q(t)}\left(\nabla f(x(t))^\top q(t)\right) \\ x(0) = x^0 \\ \boxed{q(0) = n^{\partial\mathcal{W}}(w(0))} \end{cases}$$

With $w(0)$, we obtain an **ODE** that can be efficiently integrated.

Proof (4/4) - try *all possible $w(0) \in \partial\mathcal{W}$*

Define the **augmented ODE**

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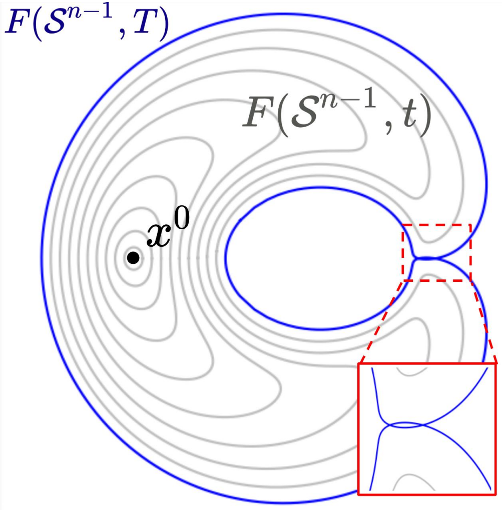
proof via convex geometry

On geometry and reachability analysis

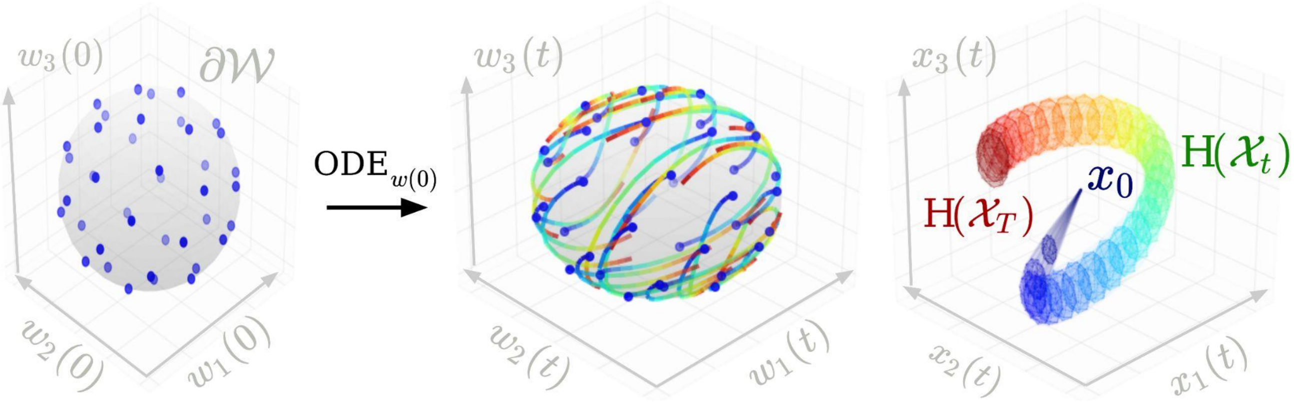
- The connection between geometry and reachability analysis via the PMP is well-known [Agrachev'04]
- Previous analysis in [Krener & Schättler '98] studies the reachable sets of systems with a small number of inputs: it relies on a **small-time assumption**
- *Studying convex hulls prevents* ***self-intersections*** *and thus does not require small-time assumptions*



Estimation Algorithm

Corollary: If the $w^i(0)$ samples δ-cover $\partial\mathcal{W}$,

$d_H(\mathrm{H}(\mathcal{X}_t), \mathrm{H}(F(w^i(0), t))) \leq \bar{L}_t\delta$ for all $t \in [0, T]$.



```
for all $i = 1, \ldots, M$ do
    $x^i \leftarrow$ Integrate($\mathbf{ODE}_{w^i(0)}$)
return $\mathrm{H}\left(\{x^i(t), i = 1, \ldots, M\}\right), t \in [0, T]$.
```

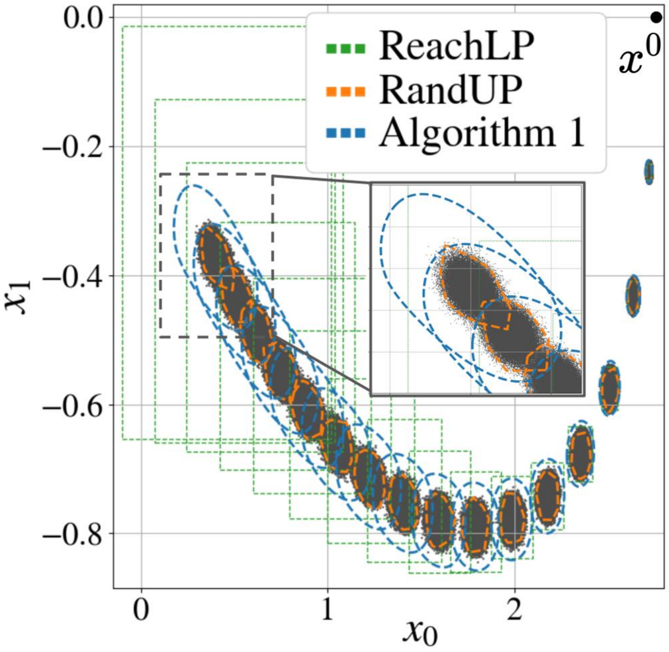
Results

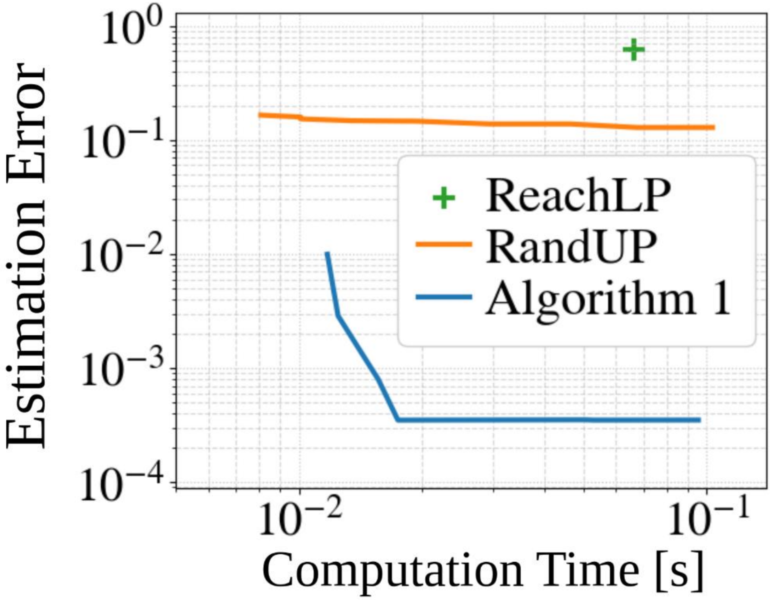
Neural feedback loop analysis

$$\dot{x}(t) = f_{\theta}^{\mathrm{nn}}(x(t)) + w(t)$$

[**ReachLP**] *Formal method* by Everett et al, IEEE Access 2021

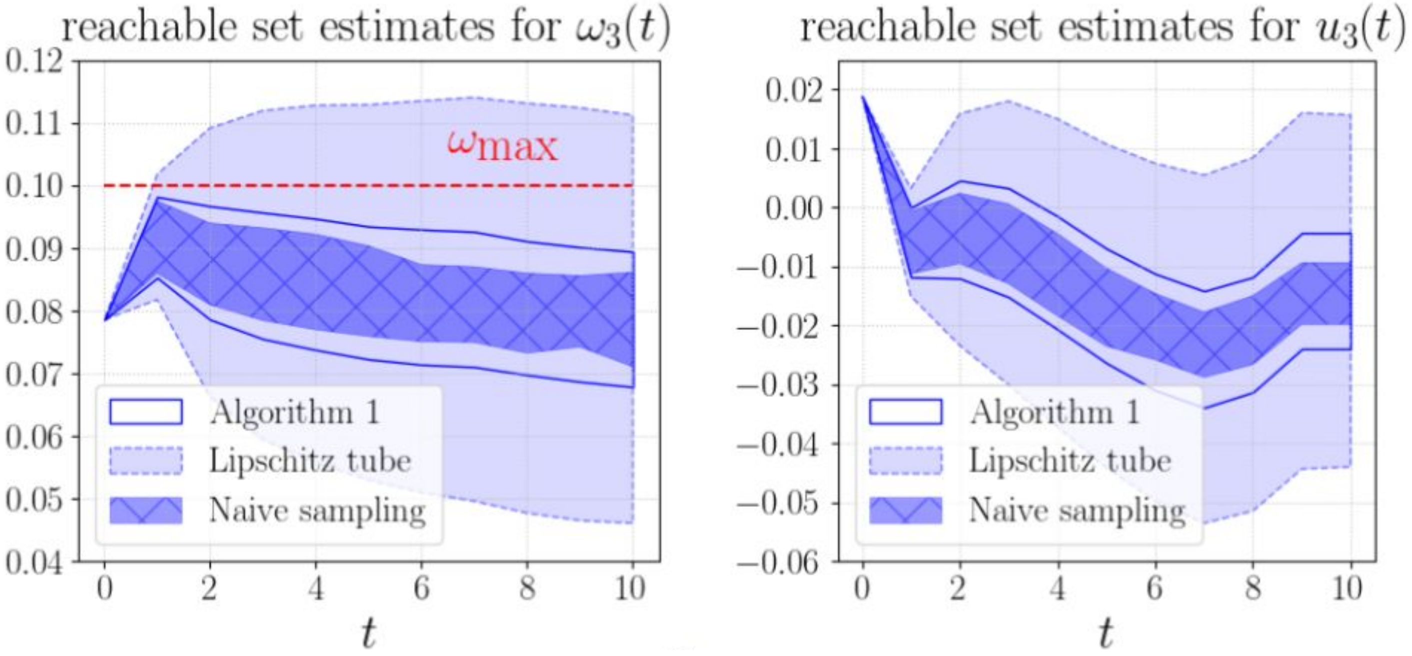
[**RandUP**] *Sampling-based method* by Lew et al, CoRL 2020





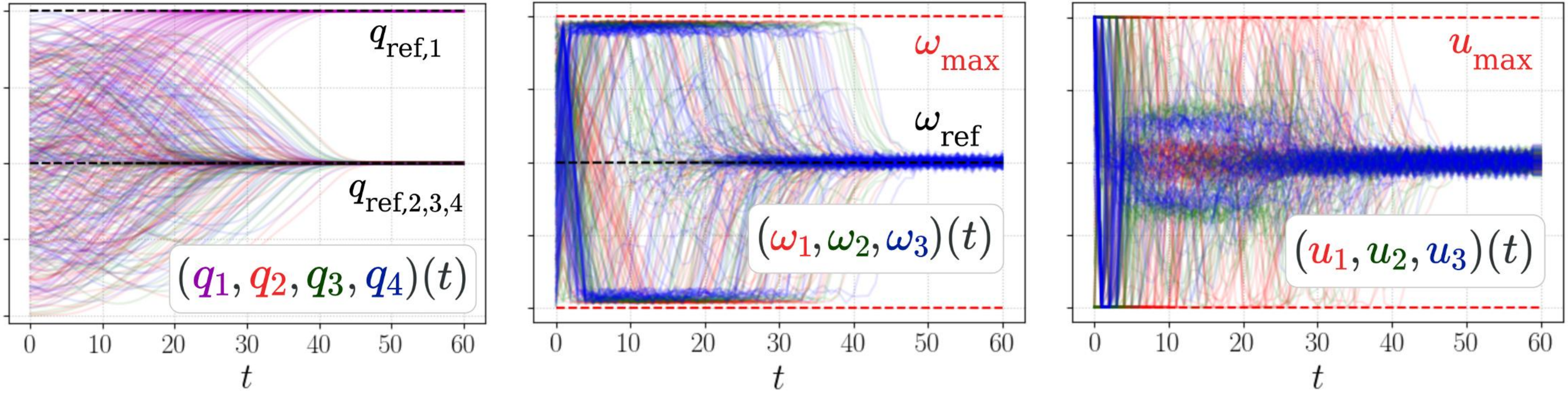
Results - spacecraft system

$$\begin{aligned}\dot{q}(t) &= \Omega(\omega(t))q(t),\\ \dot{\omega}(t) &= J^{-1}(u(t) - S(\omega(t))J\omega(t) + w(t)),\end{aligned}$$



Results - spacecraft Robust MPC

$$\dot{q}(t) = \Omega(\omega(t))q(t),$$
$$\dot{\omega}(t) = J^{-1}(u(t) - S(\omega(t))J\omega(t) + w(t)),$$
$$\|\omega(t)\|_\infty \leq 0.1, \quad \|u(t)\|_\infty \leq 0.1, \quad t \in [0, T].$$

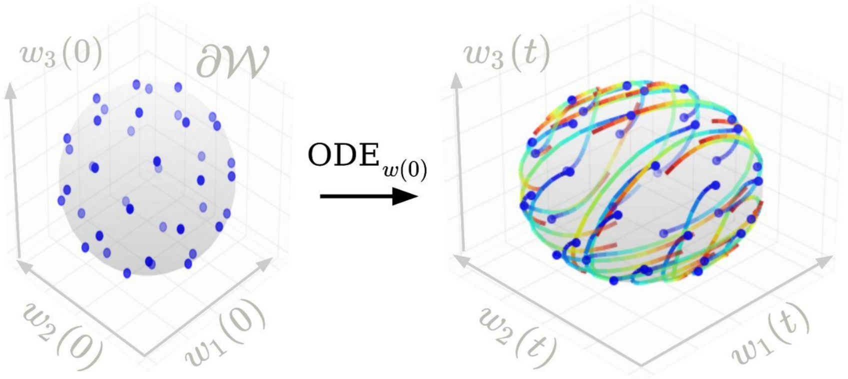


- Replanning at ~10Hz with 100 samples
- Robust constraints satisfaction

Conclusion & Outlook

$$\dot{x}(t) = f(x(t)) + w(t)$$

$$\boxed{\mathrm{H}\left(\mathcal{X}_t\right) = \mathrm{H}\left(F(\partial\mathcal{W}, t)\right)}$$



- Structural information about the convex hulls of reachable sets
- Informs the design of more efficient estimation algorithms
- Extensions
  - Uncertain x^0, more general dynamics & uncertainty sets
  - Tighter error bounds leveraging the boundary structure of $\mathrm{H}\left(\mathcal{X}_t\right)$
- Future work
  - **Algorithms to analyze $\mathrm{ODE}_{w(0)}$ beyond sampling methods**
  - Applications to robust control

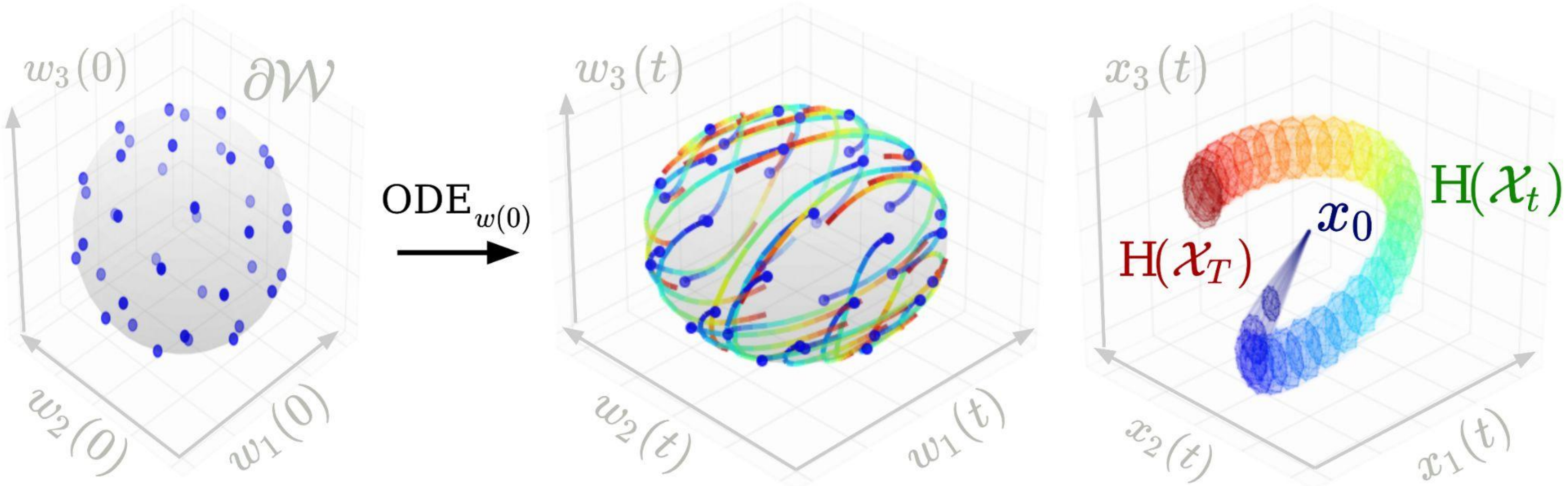
Acknowledgements

NASA

TOYOTA
RESEARCH INSTITUTE

Code

https://github.com/StanfordASL/convex_hull_reachability



Exact Characterization of the Convex Hulls of Reachable Sets

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International Online Seminar on Interval Methods in Control Engineering

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