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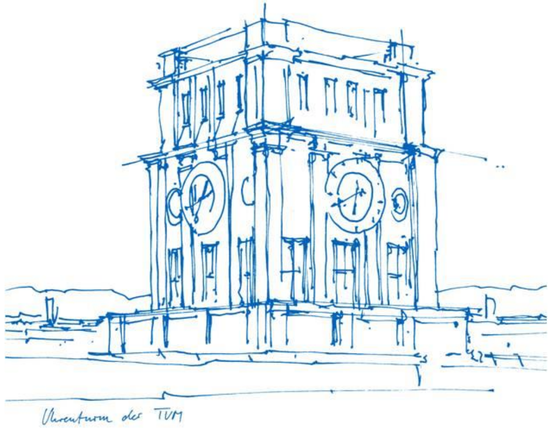

Reachability Analysis for Logical Systems Using Logical Zonotopes and their Polynomial Extension

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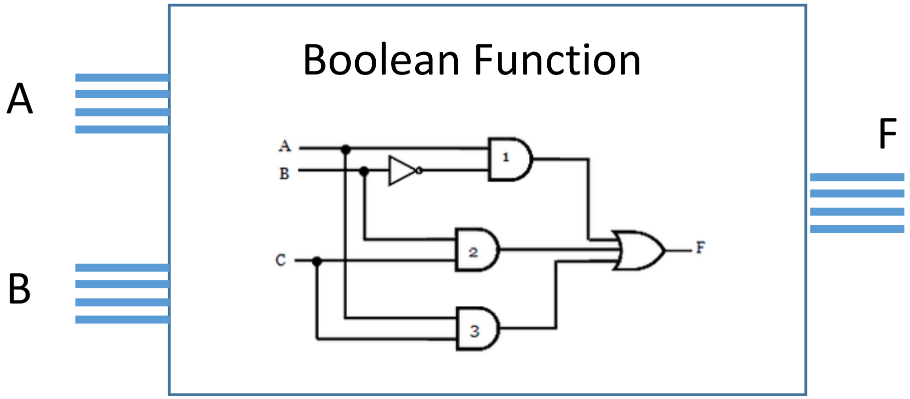
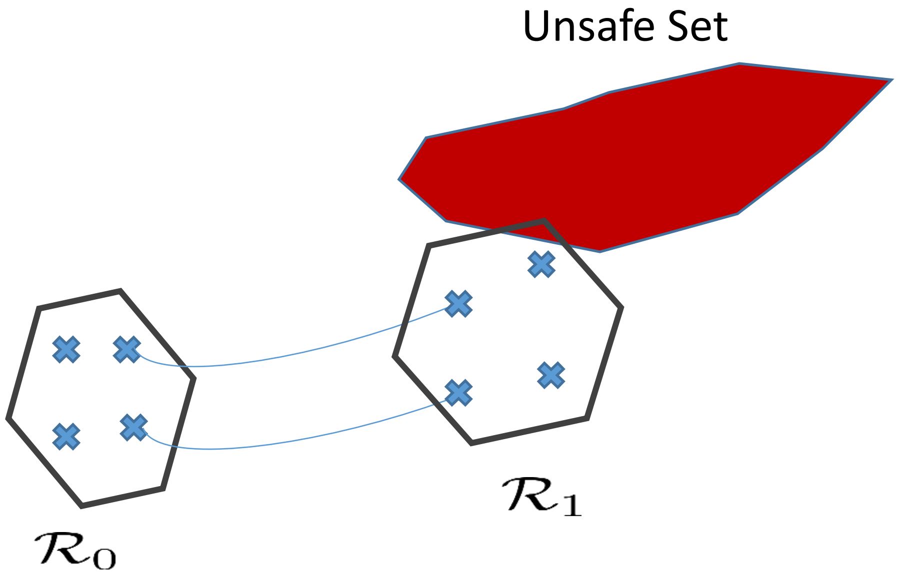
International Online Seminar on Interval Methods in Control Engineering

Safety Guarantees through Reachability Analysis

- We aim to guarantee that F does not go to unsafe set given set of inputs





Uncertainty in Logic

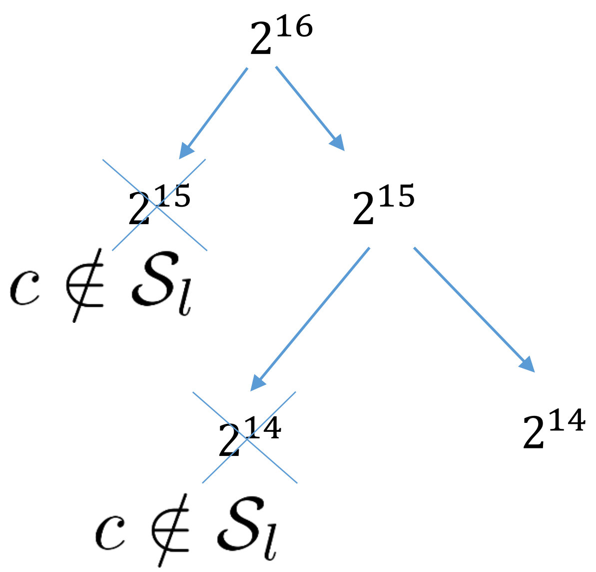
$$P_1 = \neg(P_2 \lor P_4) \land A_1 \ldots$$

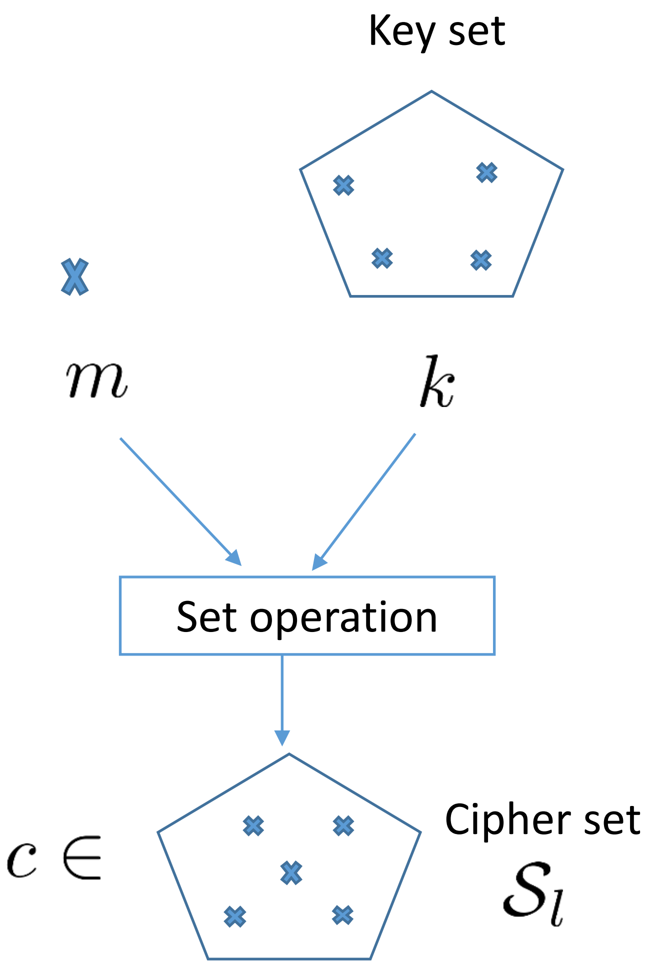
$$P_2 = 0 \qquad P_2 = 1$$

How can we handle such uncertainty on a large scale?

Further Motivation in Cryptography

- Chosen Plaintext-Ciphertext attack: We have a message m and its ciphertext c and we aim to find the 16-bits key





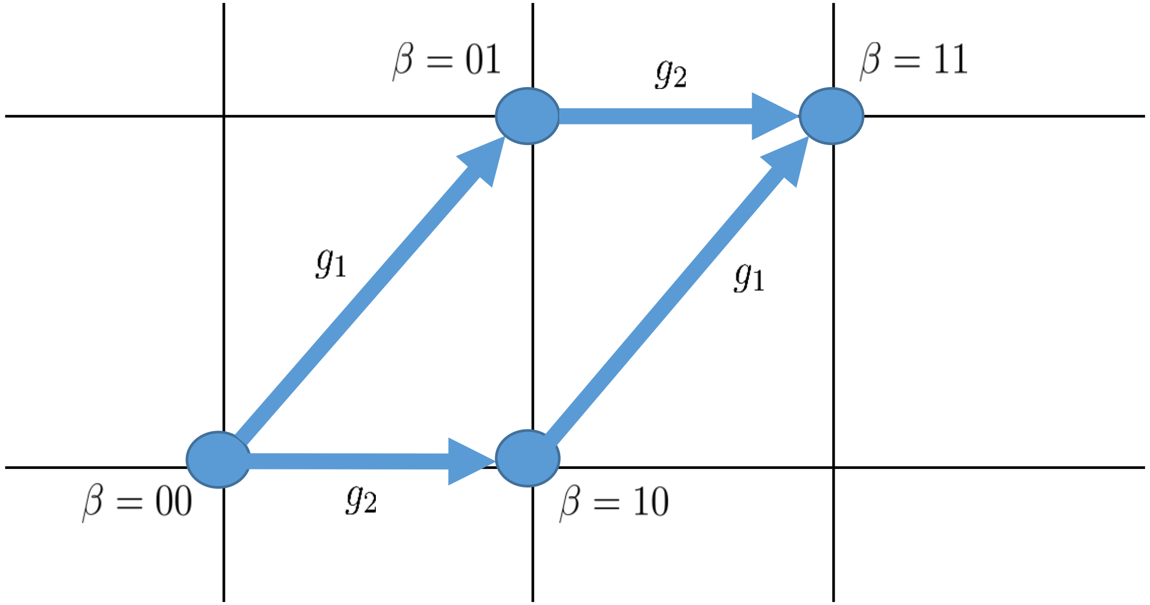
Logical Zonotope

A **logical zonotope** $\mathcal{L} = \langle c, G \rangle$ is a set

$$\mathcal{L} = \left\{ x \in \mathbb{B}^n \;\middle|\; x = c \bigoplus_{i=1}^{\gamma} g_i \beta_i,\ \beta_i \in \{0,1\} \right\}.$$

where $c \in \mathbb{B}^n$ is a point and $G = [g_1 \quad \dots \quad g_\gamma] \in \mathbb{B}^{n \times \gamma}$ the generator vectors

Logical zonotopes can represent up to 2^γ binary vectors with γ generators



Logical Zonotopes Examples

$$\mathcal{L} = \left\{ x \in \mathbb{B}^n \;\middle|\; x = c \bigoplus_{i=1}^{\gamma} g_i \beta_i,\ \beta_i \in \{0,1\} \right\}.$$

One generator

$$\mathcal{L}_1 = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Two generators

$$\mathcal{L}_2 = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

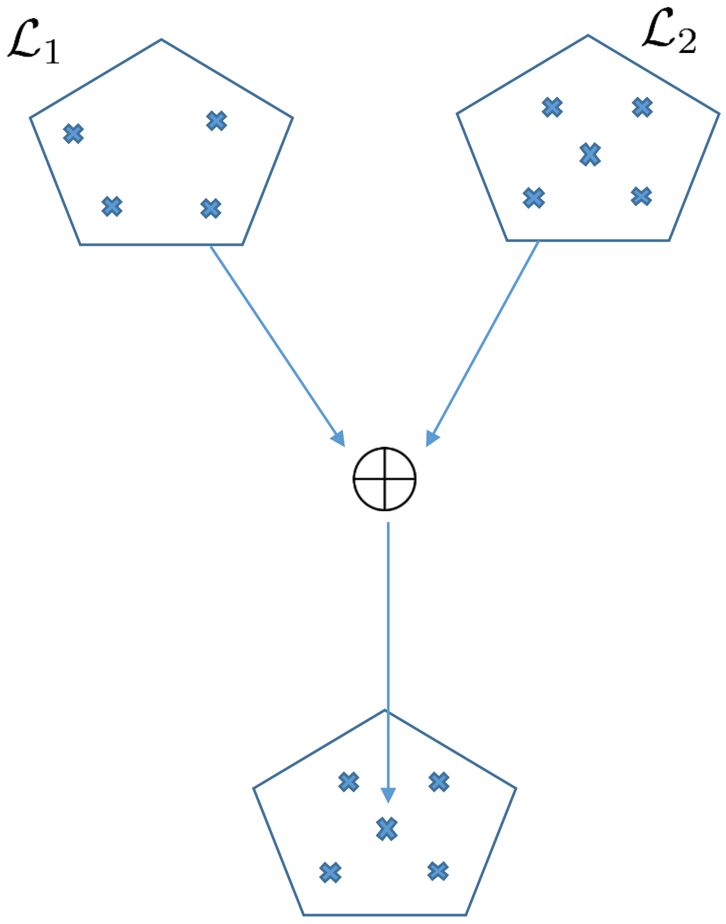
Minkowski XOR

- Definition

$$\mathcal{L}_1 \oplus \mathcal{L}_2 = \{z_1 \oplus z_2 | z_1 \in \mathcal{L}_1, z_2 \in \mathcal{L}_2\}$$

- Computation

$$\mathcal{L}_1 \oplus \mathcal{L}_2 = \left\langle c_1 \oplus c_2, \left[G_1, G_2\right] \right\rangle$$



Example - Minkowski XOR

A	B	A xor B
0	0	0
0	1	1
1	0	1
1	1	0

$$\mathcal{L}_1 = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \qquad \mathcal{L}_2 = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle$$

Logical zonotope

$$\mathcal{L}_1 \oplus \mathcal{L}_2 = \left\langle c_1 \oplus c_2, [G_1, G_2] \right\rangle$$

$$\mathcal{L}_1 \oplus \mathcal{L}_2 = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle$$

$$= \langle \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle$$

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Pointwise

$$\mathcal{L}_1 \rightarrow P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \mathcal{L}_2 \rightarrow P_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$P_1 \oplus P_2 \rightarrow \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

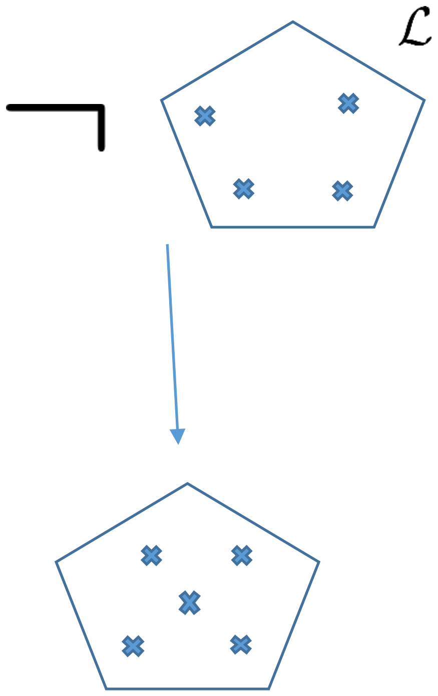
Minkowski NOT

- Definition

$$\neg\mathcal{L} = \{\neg z | z \in \mathcal{L}\}$$

- Computation

$$\neg\mathcal{L} = \left\langle c \oplus 1, G \right\rangle$$



Example - Minkowski NOT

$$\mathcal{L}_1 = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$$

Logical zonotope

$$\begin{aligned} \neg\mathcal{L} &= \langle c \oplus 1, G \rangle \\ &= \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus 1, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \\ &= \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle \end{aligned}$$

↓

$$\neg\mathcal{L}_1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Pointwise

$$\mathcal{L}_1 \rightarrow P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

↓

$$\neg P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Can we do more than Minkowski XOR and NOT?

Minkowski AND

- Definition

$$\mathcal{L}_1\mathcal{L}_2 = \{z_1 z_2 | z_1 \in \mathcal{L}_1, z_2 \in \mathcal{L}_2\}$$

- Computations

$$\mathcal{L}_1\mathcal{L}_2 \subseteq \langle c_1 c_2, \Big[c_1 g_{2,1}, \dots, c_1 g_{2,\gamma_2}, c_2 g_{1,1}, \dots, c_2 g_{1,\gamma_1}, g_{1,1} g_{2,1}, g_{1,1} g_{2,2}, \dots, g_{1,\gamma_1} g_{2,\gamma_2}\Big]\rangle$$

Minkowski NAND

- Definition

$$\mathcal{L}_1 \barwedge \mathcal{L}_2 = \{z_1 \barwedge z_2 | z_1 \in \mathcal{L}_1, z_2 \in \mathcal{L}_2\}$$

- Computation

$$\mathcal{L}_1 \barwedge \mathcal{L}_2 = \neg(\mathcal{L}_1 \mathcal{L}_2)$$

- NAND is a universal gate. Thus, we are able to do other logical operations:

OR $\quad \mathcal{L}_1 \vee \mathcal{L}_2 = (\mathcal{L}_1 \barwedge \mathcal{L}_1) \barwedge (\mathcal{L}_2 \barwedge \mathcal{L}_2)$

NOR $\quad \mathcal{L}_1 \not\vee \mathcal{L}_2 = \neg(\mathcal{L}_1 \vee \mathcal{L}_2)$

Reachability Analysis

Consider a system with a logical function

$$x(k+1) = f\big(x(k), u(k)\big)$$

Theorem *Given a logical function* $f : \mathbb{B}^{n_x} \times \mathbb{B}^{n_u} \rightarrow \mathbb{B}^{n_x}$ *and starting from initial set* $\hat{\mathcal{R}}_0$ *where* $x(0) \in \hat{\mathcal{R}}_0$*, then the reachable region computed as*

$$\hat{\mathcal{R}}_{k+1} = f\big(\hat{\mathcal{R}}_k, \mathcal{U}_k\big)$$

using logical zonotopes operations over-approximates the exact reachable set, i.e., $\hat{\mathcal{R}}_{k+1} \supseteq \mathcal{R}_{k+1}$.

Semi-tensor Product

Given two matrices $M \in \mathbb{B}^{m\times n}$ and $N \in \mathbb{B}^{p\times q}$, the semi-tensor product is defined as:

$$M \ltimes N = (M \otimes I_{s_1})(N \otimes I_{s_2}),$$

with s as the least common multiple of n and p, $s_1 = s/n$, and $s_2 = s/p$

$\otimes$: Kronecker product

Semi Tensor Product between Logical Zonotopes

- Definition

$$\mathcal{L}_1 \ltimes \mathcal{L}_2 = \{z_1 \ltimes z_2 | z_1 \in \mathcal{L}_1, z_2 \in \mathcal{L}_2\}$$

- Computation

$$\mathcal{L}_1 \ltimes \mathcal{L}_2 \subseteq \left\langle c_1 \ltimes c_2, G_{\ltimes} \right\rangle$$

where

$$G_{\ltimes} = \Big[c_1 \ltimes g_{2,1}, \dots, c_1 \ltimes g_{2,\gamma_1}, g_{1,1} \ltimes c_2, \dots, g_{1,\gamma_1} \ltimes c_2$$
$$g_{1,1} \ltimes g_{2,1}, \dots, g_{1,\gamma_1} \ltimes g_{2,\gamma_2}\Big]$$

Can we have an exact ANDing?

ANDing Problem

- ANDing Proof

$$\exists \hat{\beta}_1 : z_1 = c_1 \bigoplus_{i=1}^{\gamma_1} g_{1,i}\hat{\beta}_{1,i} \qquad \exists \hat{\beta}_2 : z_2 = c_2 \bigoplus_{i=1}^{\gamma_2} g_{2,i}\hat{\beta}_{2,i}$$

$$\longrightarrow \quad z_1 z_2 = c_1 c_2 \bigoplus_{i=1}^{\gamma_2} c_1 g_{2,i}\hat{\beta}_{2,i} \bigoplus_{i=1}^{\gamma_1} c_2 g_{1,i}\hat{\beta}_{1,i} \bigoplus_{i=1,j=1}^{\gamma_1,\gamma_2} g_{1,i} g_{2,j}\hat{\beta}_{1,i}\hat{\beta}_{2,j} \, .$$

- What if we allow for ANDing between factors?

$$\mathcal{P} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \alpha_1 \oplus \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \alpha_1 \alpha_2 \middle| \alpha_1, \alpha_2 \in \{0,1\} \right\}.$$

Polynomial Logical Zonotope

A **polynomial logical zonotope** $\mathcal{P} = \langle c, G, E \rangle$ is a set

$$\mathcal{P} = \left\{ x \in \mathbb{B}^n \;\middle|\; x = c \bigoplus_{i=1}^{h} \left(\prod_{k=1}^{p} \alpha_k^{E_{(k,i)}} \right) g_i, \alpha_k \in \{0,1\} \right\}.$$

- $c \in \mathbb{B}^n$ is a point
- $G = \begin{bmatrix} g_1 & \dots & g_q \end{bmatrix} \in \mathbb{B}^{n \times h}$ is a dependent generator matrix
- $E \in \mathbb{B}^{p \times h}$ is an exponent matrix

Example

- Consider the following polynomial logical zonotope

$$\bar{\mathcal{P}}_1 = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$$

- This translates to the following set

$$\bar{\mathcal{P}}_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \alpha_1 \oplus \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \alpha_1 \alpha_2 \middle| \alpha_1, \alpha_2 \in \{0, 1\} \right\}.$$

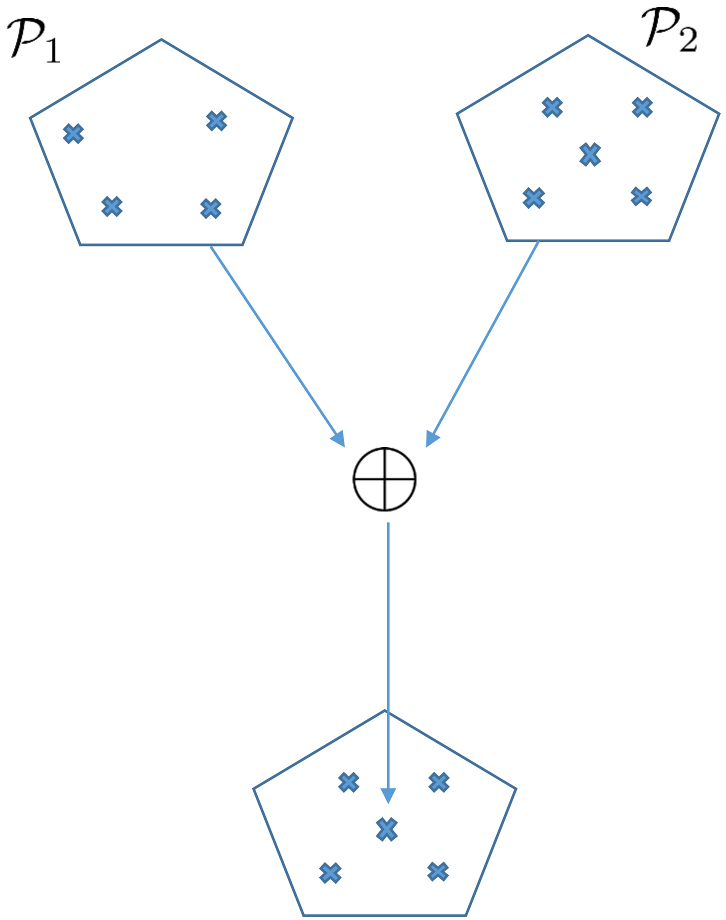
Minkowski XOR

- Definition

$$\mathcal{P}_1 \oplus \mathcal{P}_2 = \{z_1 \oplus z_2 | z_1 \in \mathcal{P}_1, z_2 \in \mathcal{P}_2\}$$

- Computation

$$\mathcal{P}_1 \oplus \mathcal{P}_2 = \left\langle c_1 \oplus c_2, [G_1, G_2], \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \right\rangle$$



Example - XOR

A	B	A xor B
0	0	0
0	1	1
1	0	1
1	1	0

$$\mathcal{P}_1 = \langle 0, 1, 1 \rangle$$

Polynomial Logical zonotope

$$\mathcal{P}_1 \oplus \mathcal{P}_2 = \left\langle c_1 \oplus c_1, [G_1, G_2], \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \right\rangle$$

$$\mathcal{P}_1 \oplus \mathcal{P}_1 = \left\langle 0, [1, 1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$$

$$\mathcal{P}_1 \oplus \mathcal{P}_1 \rightarrow \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Pointwise

$$\mathcal{P}_1 \oplus \mathcal{P}_1 \rightarrow 0$$

Dependency Problem

- Dependent factors

$$\alpha g_1 \oplus \alpha g_2 = \alpha(g_1 \oplus g_2)$$

- Independent factors

$$\alpha_1 g_1 \oplus \alpha_2 g_2 = \begin{bmatrix} g_1 & g_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

- Inspired by the traditional polynomial zonotopes, we need to identify the factors with ids

Niklas Kochdumper and Matthias Althoff. Sparse polynomial zonotopes: A novel set representation for reachability analysis. IEEE Transactions on Automatic Control, 66(9):4043–4058, 202

Polynomial Logical Zonotope

A **polynomial logical zonotope** $\mathcal{P} = \langle c, G, E, id \rangle$ is a set

$$\mathcal{P} = \left\{ x \in \mathbb{B}^n \;\middle|\; x = c \bigoplus_{i=1}^{h} \left(\prod_{k=1}^{p} \alpha_k^{E_{(k,i)}} \right) g_i, \alpha_k \in \{0,1\} \right\}.$$

- $c \in \mathbb{B}^n$ is a point
- $G = \begin{bmatrix} g_1 & \dots & g_q \end{bmatrix} \in \mathbb{B}^{n \times h}$ is a dependent generator matrix
- $E \in \mathbb{B}^{p \times h}$ is an exponent matrix
- Id vector for identifying the dependent factors

Merge IDs

$$\bar{\mathcal{P}}_1 = \left\langle \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 2\end{bmatrix} \right\rangle$$

$$\downarrow$$

$$\mathcal{P}_1 = \left\langle \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 2 & 3\end{bmatrix} \right\rangle$$

$$\bar{\mathcal{P}}_2 = \left\langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 3\end{bmatrix} \right\rangle$$

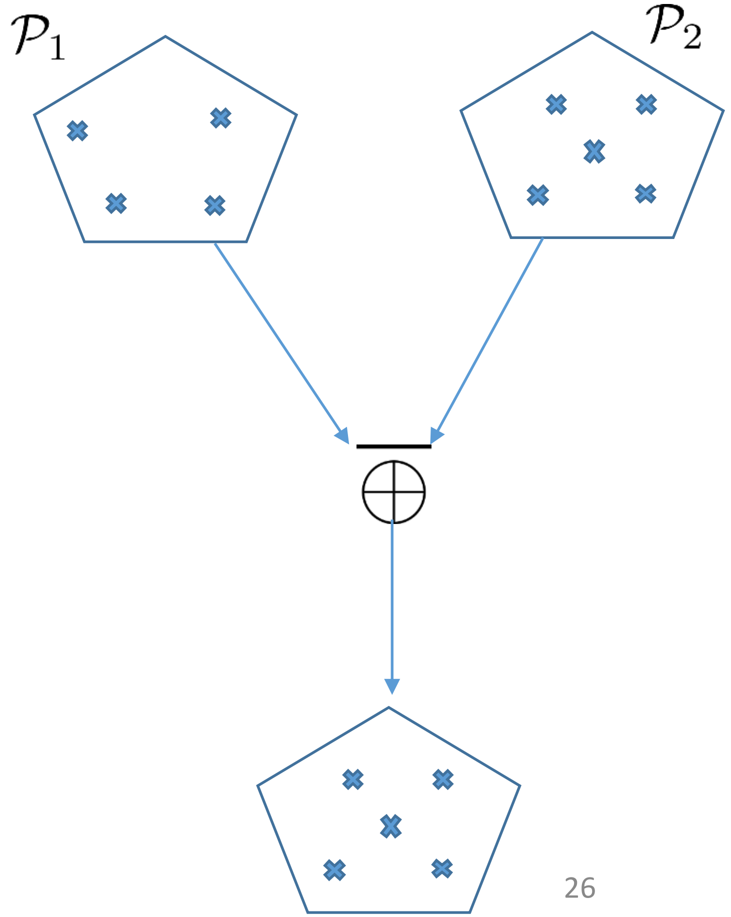
$$\downarrow$$

$$\mathcal{P}_2 = \left\langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 2 & 3\end{bmatrix} \right\rangle$$

Exact XOR

- Given two polynomial logical zonotopes $\mathcal{P}_1 = \langle c_1, G_1, E_1, id \rangle$ and $\mathcal{P}_2 = \langle c_2, G_2, E_2, id \rangle$ with a common identifier vector id, the exact XOR is computed as:

$$\mathcal{P}_1 \overline{\oplus} \mathcal{P}_2 = \left\langle c_1 \oplus c_2, \left[G_1, G_2\right], \left[E_1, E_2\right], id \right\rangle$$

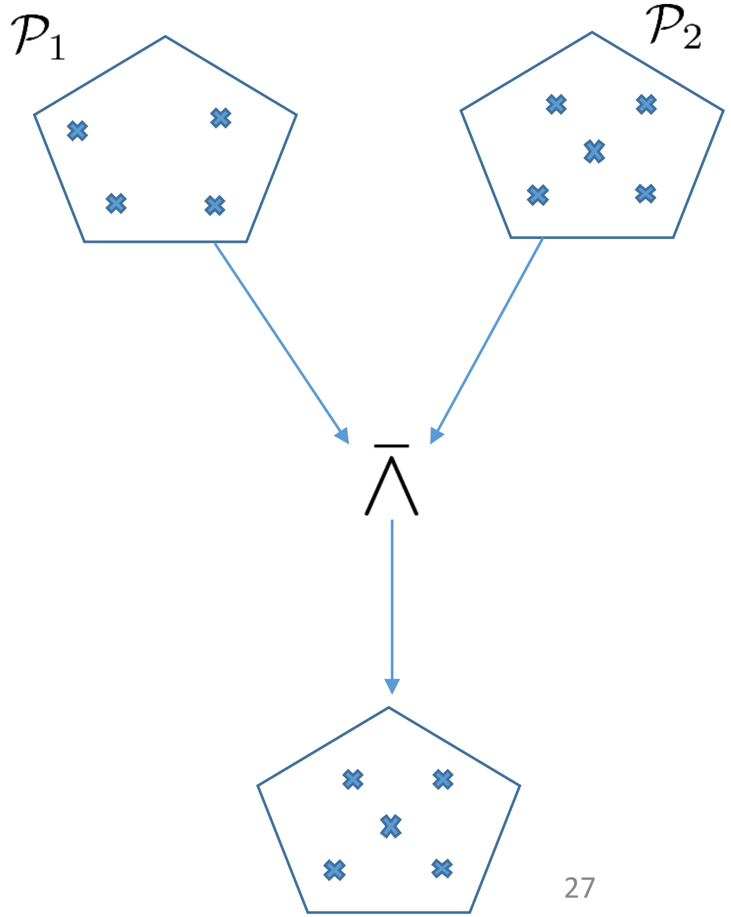


Exact AND

- Given two polynomial logical zonotopes $\mathcal{P}_1 = \langle c_1, G_1, E_1, id \rangle$ and $\mathcal{P}_2 = \langle c_2, G_2, E_2, id \rangle$

with a common identifier vector id, the exact AND is computed and lead to the following $\mathcal{P}_{\bar{\wedge}} = \langle c_{\bar{\wedge}}, G_{\bar{\wedge}}, E_{\bar{\wedge}}, id \rangle$ where

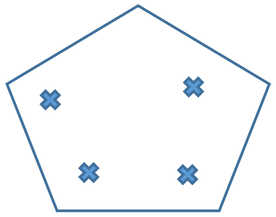
$$
\begin{aligned}
c_{\bar{\wedge}} =& c_1 c_2, \\
G_{\bar{\wedge}} =& \Big[c_1 g_{2,1}, \dots, c_1 g_{2,h_2}, c_2 g_{1,1}, \dots, c_2 g_{1,h_1}, \\
& g_{1,1} g_{2,1}, \dots, g_{1,h_1} g_{2,h_2} \Big], \\
E_{\bar{\wedge}} =& \Big[E_{2,(.,1)}, \dots, E_{2,(.,h_2)}, E_{1,(.,1)}, \dots, E_{1,(.,h_1)}, \\
& \max\big(E_{1,(.,1)}, E_{2,(.,1)}\big), \dots, \max\big(E_{1,(.,h_1)}, E_{2,(.,h_2)}\big) \Big]
\end{aligned}
$$



Enclose Points by a Logical Zonotope

Given a list $\mathcal{S}_p$ of p binary vectors, the logical zonotope $\mathcal{L}_p = \langle c_p, G_p \rangle$ with is given by $\mathcal{S}_{p,i} \in \mathcal{L}_p, \forall i = \{1, \dots, p\}$

$$c_p = \mathcal{S}_{p,1},$$
$$g_{p,i-1} = \mathcal{S}_{p,i} \oplus c_p, \ \forall i = \{2, \dots, p\}.$$



β_3	β_2	β_1
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

$$c_p = \mathcal{S}_{p,1}$$
$$g_{p,1} \oplus c_p = \mathcal{S}_{p,2} \oplus c_p \oplus c_p$$
$$g_{p,2} \oplus c_p = \mathcal{S}_{p,3} \oplus c_p \oplus c_p$$
$$g_{p,3} \oplus c_p = \mathcal{S}_{p,4} \oplus c_p \oplus c_p$$

Reduce the Number of Generators

Algorithm 1: Function `reduce` to decrease the number of generators of a logical zonotope.

Input: A logical zonotope $\mathcal{L} = \langle c_{\mathcal{L}}, G_{\mathcal{L}} \rangle$ with large number $\gamma_{\mathcal{L}}$ of generators.

Output: A logical zonotope $\mathcal{L}_r = \langle c_{\mathcal{L}_r}, G_{\mathcal{L}_r} \rangle$ with $\gamma_{\mathcal{L}_r} \leq \gamma_{\mathcal{L}}$ generators.

$B_{\mathcal{L}}=$ `evaluate`$(\mathcal{L})$ // Compute a list $B_{\mathcal{L}}$ of all binary vectors contained in $\mathcal{L}$.

for $i = 1 : \gamma_{\mathcal{L}}$ **do**

    $B_{\mathcal{L}_r}=$ `evaluate`$(\mathcal{L} \setminus g_{\mathcal{L}}^{(i)})$ // Compute a list $B_{\mathcal{L}_r}$ of all binary vectors contained in $\mathcal{L}$ without the generator $g_{\mathcal{L}}^{(i)}$.

    if *isequal*$(B_{\mathcal{L}}, B_{\mathcal{L}_r})$ **then**

        $g_{\mathcal{L}} =$ `removeGenerator`$(g_{\mathcal{L}}^{(i)})$

$\mathcal{L}_r = \langle c_{\mathcal{L}}, G_{\mathcal{L}} \rangle$

Binary Decision Diagram (BDD)

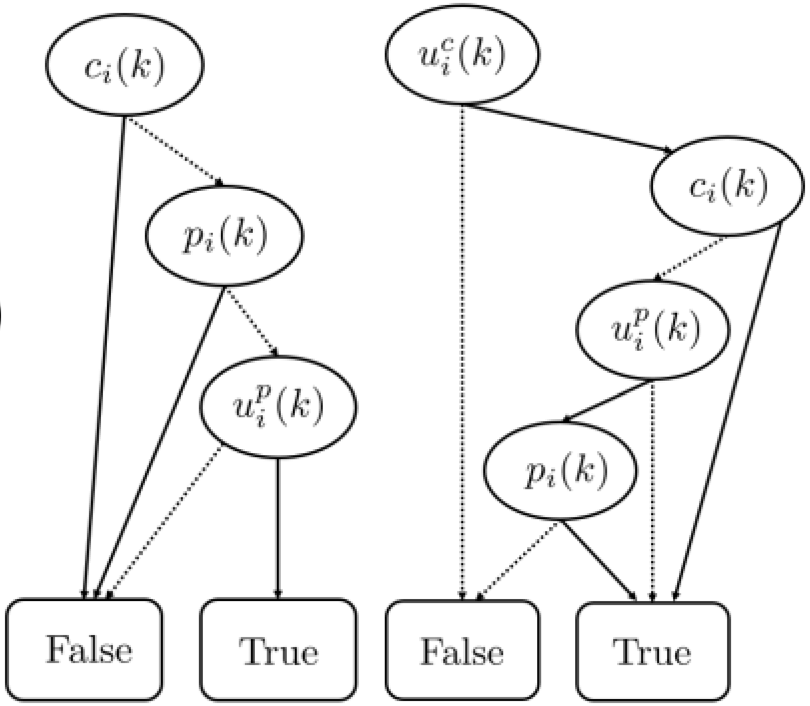
- Introduced by Randal E. Bryant in mid-80s
- A data structure that is used to represent a Boolean function
- Given a proper variable ordering, BDDs can evaluate Boolean functions with linear complexity in the number of variables

Alan John Hu. 1996. *Techniques for efficient formal verification using binary decision diagrams*. Stanford university

Reduced Binary Decision Diagram

$$p_i(k+1) = u_i^p(k)\neg p_i(k)\neg c_i(k)$$

$$c_i(k+1) = \neg p_i(k+1)(u_i^c(k) \vee (\neg p_i(k) p_i(k+1)))$$



(a) $p_i(k+1)$ **(b)** $c_i(k+1)$

Boolean Control Network (BCN)

h_i : Logical function

- We consider the BCN with the following dynamics

$$x_1(k+1) = h_1(x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k)),$$
$$\vdots$$
$$x_n(k+1) = h_n(x_1(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k)),$$

Fangfei Li and Yang Tang. 2017. Robust Reachability of Boolean Control Networks. IEEE/ACM Transactions on Computational Biology and Bioinformatics

Exact Reachable Set of BCN

- There exits a unique matrix, named structural matrix L such that

$$x(k+1) = L \ltimes u(k) \ltimes x(k)$$

where $x = \ltimes_{i=1}^{n} x_i$ and $u = \ltimes_{j=1}^{m} u_i$

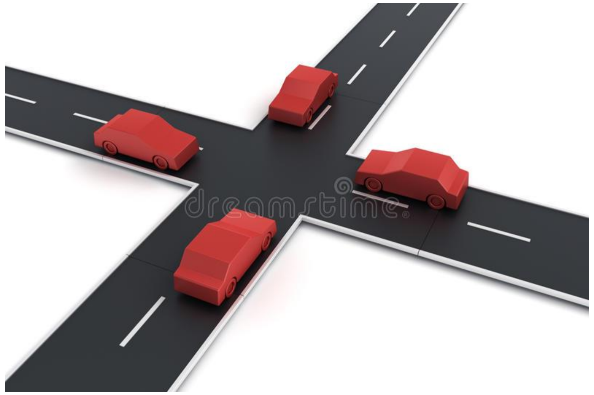
- Given a set of initial states $\mathcal{X}_0 \subset \mathbb{Z}^n$ and a set of possible inputs $\mathcal{U} \subset \mathbb{Z}^m$, the exact reachable set

$$\mathcal{R}_N = \{x(N) \in \mathbb{B}^n \mid \forall k \in \{0, \ldots, N-1\} : \\ x(k+1) = L \ltimes u(k) \ltimes x(k), x(0) \in \mathcal{X}_0, u(k) \in \mathcal{U}\}$$

Intersection Crossing Protocol

$$p_i(k+1) = u_i^p(k)\neg p_i(k)\neg c_i(k)$$
$$c_i(k+1) = \neg p_i(k+1)(u_i^c(k) \vee (\neg p_i(k)p_i(k+1)))$$

$$u_1^p(k) \in \{0,1\},\ u_2^p(k) \in \{0,1\},\ u_1^c(k) \in \{0,1\},\ u_2^c(k) \in \{0,1\},\ k = 0,\ldots,N$$



Intersection Crossing Protocol – 4 Cars

Table 1: Execution Time (seconds) and number of points in each set (size) for verifying an intersection crossing protocol.

	Zonotope		Poly. Zonotope		BDD		BCN	
Steps N	Time	Size	Time	Size	Time	Size	Time	Size
5	0.05	16	0.15	13	1.17	14	3.40	14
10	0.06	16	0.18	14	3.32	14	7.75	14
50	0.15	16	0.25	14	19.87	14	48.40	14
100	0.26	16	0.45	14	39.78	14	104.91	14
1000	1.84	16	2.84	14	406.60	14	1142.10	14

$$p_i(k+1) = u_i^p(k)\neg p_i(k)\neg c_i(k)$$

$$c_i(k+1) = \neg p_i(k+1)(u_i^c(k) \vee (\neg p_i(k)p_i(k+1)))$$

High-Dimensional Boolean Function

We initially assign sets of 10 possible values to $B_1(0), B_2(0), B_3(0)$

$$B_1(k+1) = U_1(k) \vee (B_2(k) \odot B_1(k)), \tag{1}$$

$$B_2(k+1) = B_2(k) \odot (B_1(k) \wedge U_2(k)), \tag{2}$$

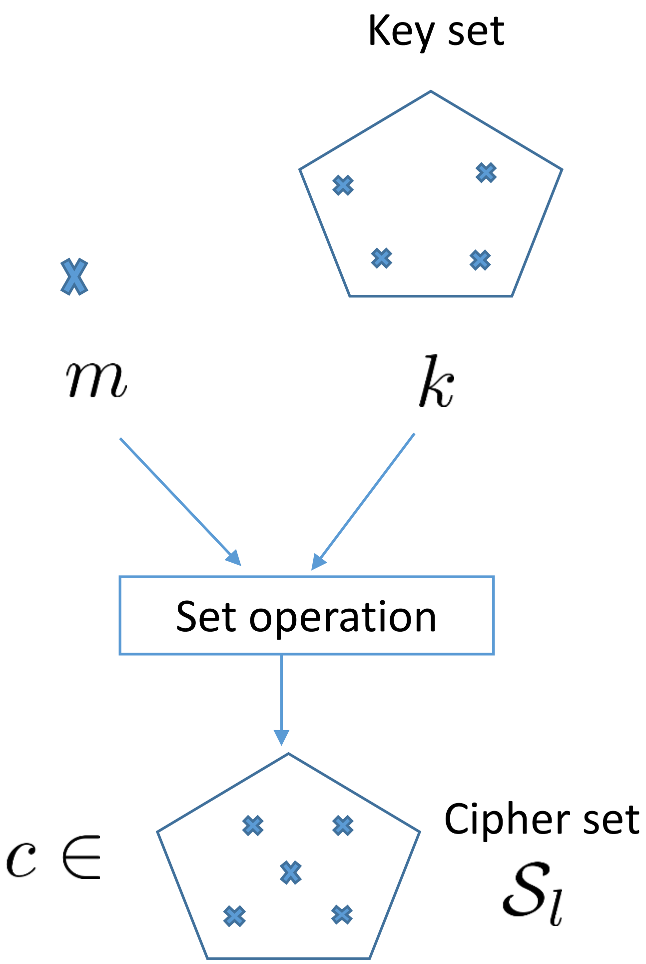
$$B_3(k+1) = B_3(k) \mathbin{\wedge\!\!\!\!\sim} (U_2(k) \odot U_3(k)). \tag{3}$$

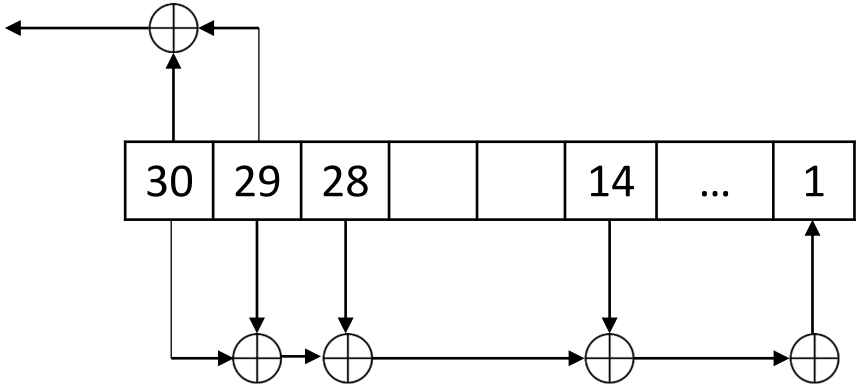
High-Dimensional Boolean Function

Table 2: Execution Time (seconds) for reachability analysis of a high-dimensional Boolean function (*estimated times).

	Zonotope		Poly. Zonotope		BDD	
Steps N	Time	Size	Time	Size	Time	Size
2	0.04	768	0.05	211	0.34	211
3	0.05	896	0.06	580	1.86×10^5	580
4	0.06	896	0.07	580	2.44×10^6*	-
5	0.07	896	0.56	580	$> 10^6$*	-

Key Search


Key set
m
k
Set operation
$c \in$
Cipher set
$\mathcal{S}_l$


30
29
28
14
...
1

Execution Time

Table 3: Execution Time (seconds) of exhaustive key search (*estimated times).

Key Size	Zonotope	Traditional Search
30	1.97	$1.18 \times 10^{6}*$
60	4.76	$1.26 \times 10^{15}*$
120	7.95	$1.46 \times 10^{33}*$

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1. “Logical zonotopes: A Set Representation for the Formal Verification of Boolean Functions” A Alanwar, FJ Jiang, S Amin, KH Johansson
2. “Polynomial Logical Zonotopes: A Set Representation for Reachability Analysis of Logical Systems” A Alanwar, FJ Jiang, KH Johansson

Conclusions

- Logical zonotope set representation
- Polynomial logical zonotope set representation
- Logical operations on generators instead of iterating over points
- Applications of logical zonotopes

https://github.com/aalanwar/Logical-Zonotope

https://sites.google.com/view/amr-alanwar/


THANKS