International Online Seminar on Interval Methods in Control Engineering, March 15, 2024

A Review of Set Representation, Partition and Propagation Techniques for Set-Theoretic Methods in Control

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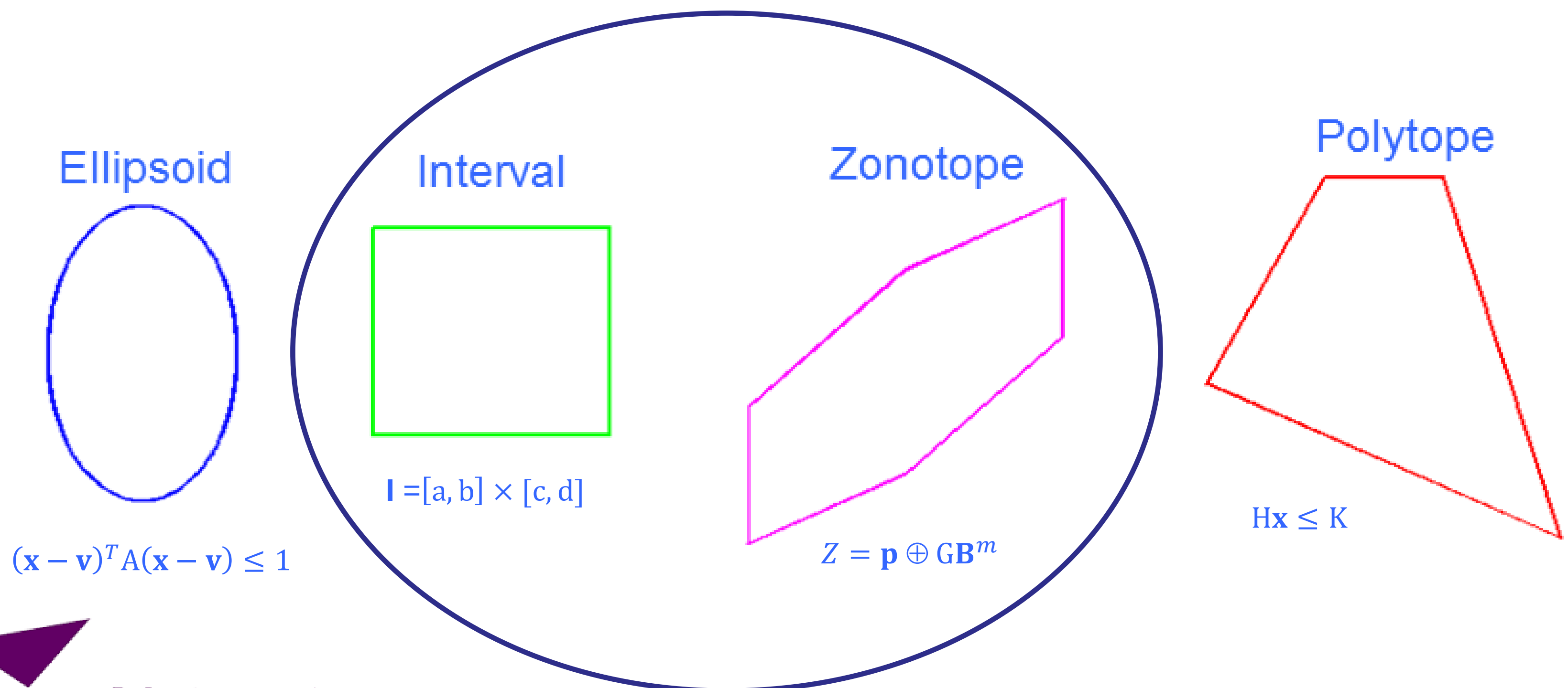
Background

- Set-theoretic methods refer to those methods involving various sets and set operations for representing uncertainties, checking the properties of certain domains, propagating system states and uncertainties, and others.



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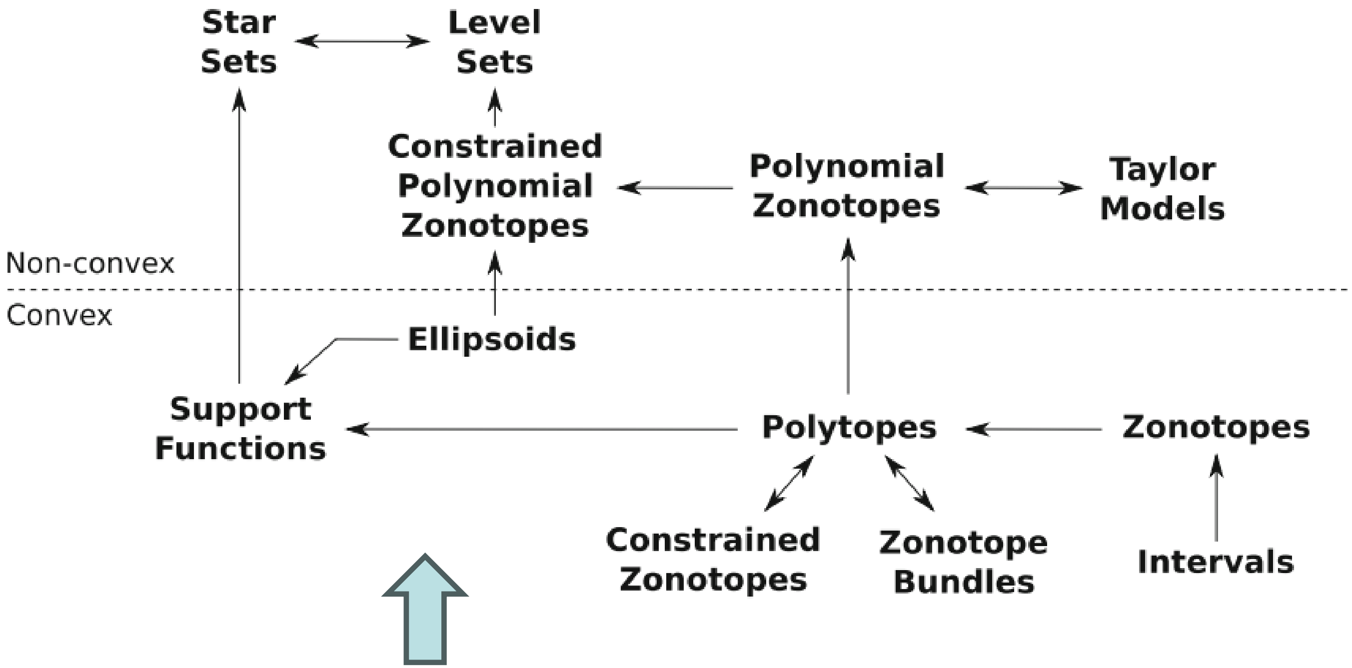

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Niklas Kochdumper and Matthias Althoff. Constrained polynomial zonotopes. *Acta Informatica*, Vol. 60, No. 3, pp. 279-316, 2023.

Interval Analysis – Set Representation

- Interval analysis uses a lower bound and an upper bound to represent a set and the shape of the set is fixed as an interval or a box.

[a, b]

[c, d]

[a, b] x [c, d]

[a, b] x [c, d] x [e, f]

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Interval Analysis – Set Partition

- An interval or a box can be partitioned into two subsets through bisection-and-selection for finer set computation.

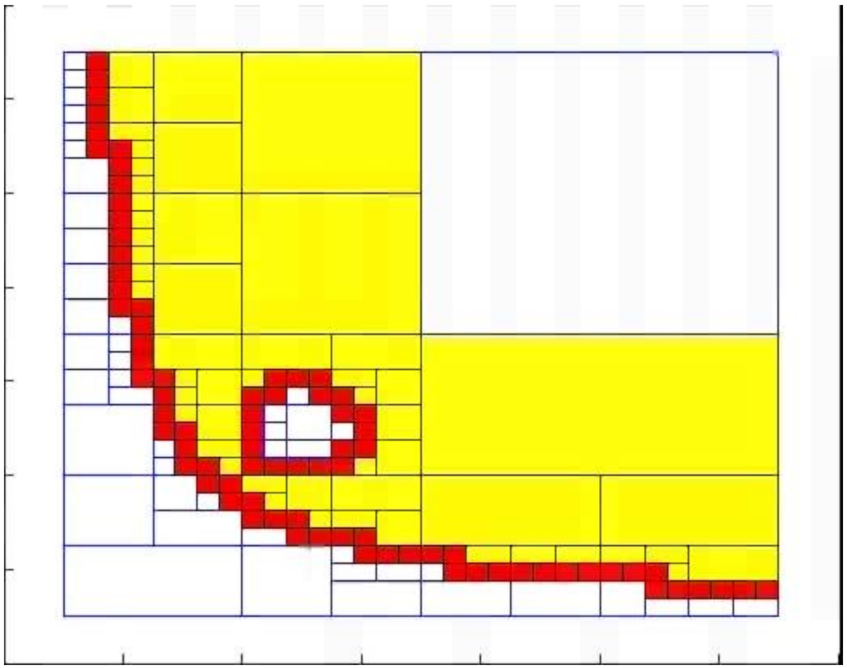
$$G(s) = \frac{1}{s^3 + (p_1 + p_2 + 2)s^2 + (p_1 + p_2 + 2)s + 2p_1p_2 + 6p_1 + 6p_2 + 3}$$

$p_1 \in [-3,9]$ and $p_2 \in [-3,9]$

The Routh vector for stability:

$$\mathbf{r}(\mathbf{p}) = \begin{Bmatrix} p_1 + p_2 + 2 \\ (p_1 - 1)^2 + (p_2 - 1)^2 - 1 \\ 2(p_1 + 3)(p_2 + 3) - 15 \end{Bmatrix}$$

Luc Jaulin, Michel Kieffer, Oilvier Didrit and Eric Walter. Applied Interval Analysis with Examples in Parameter and State Estimation, Robust Control and Robotics. Springer London, 2001.



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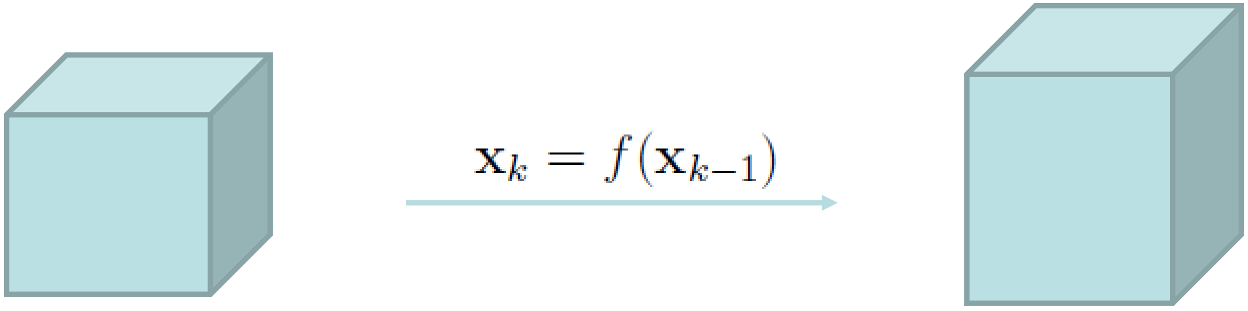
Interval Analysis – Set Propagation

- Set propagation of intervals or boxes for a function can be implemented through interval arithmetic:
  - Replace each occurrence of every variable with the corresponding interval variable
  - Execute all operations via interval arithmetic
  - Compute ranges of standard functions



$\mathbf{x}_{k-1}$

$$\mathbf{x}_k = f(\mathbf{x}_{k-1})$$

$\mathbf{x}_k$

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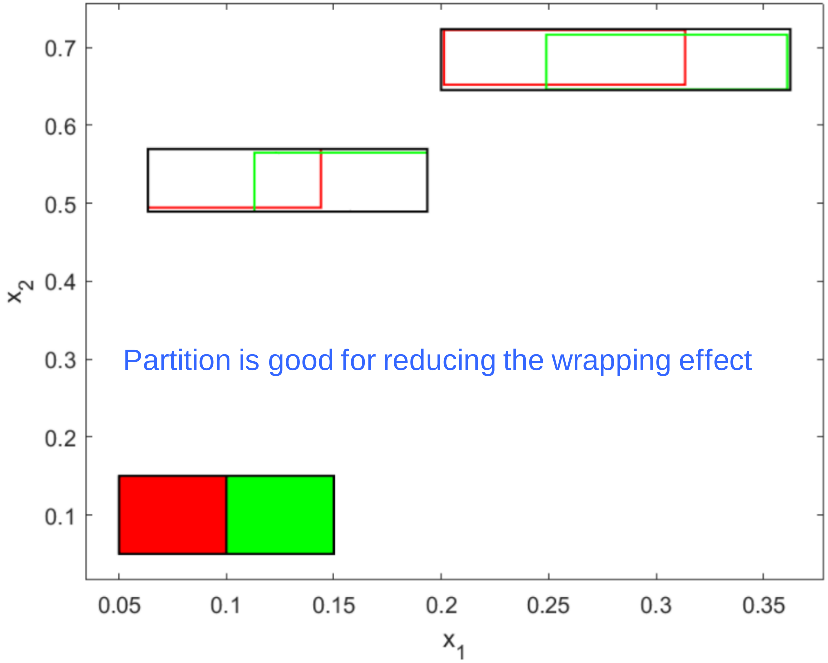
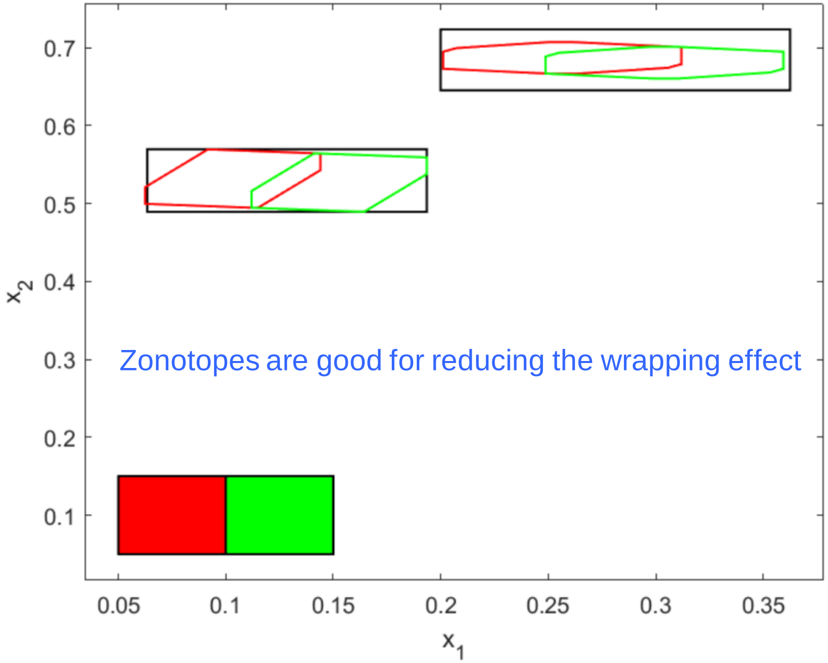
Interval Analysis – Some Observations

- Set propagation of intervals or boxes for a function via interval arithmetic usually results in a natural inclusion function with some wrapping effect.





$$\begin{cases} x_1(k+1) = 0.99x_1(k) + \delta(k)x_2(k), \delta(k) \in [0.28, 0.3] \\ x_2(k+1) = -0.1x_1(k) + \frac{0.5x_2(k)}{1+x_2^2(k)} + \omega(k), \omega(k) \in [0.48, 0.5] \end{cases}$$

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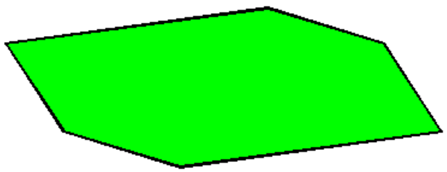
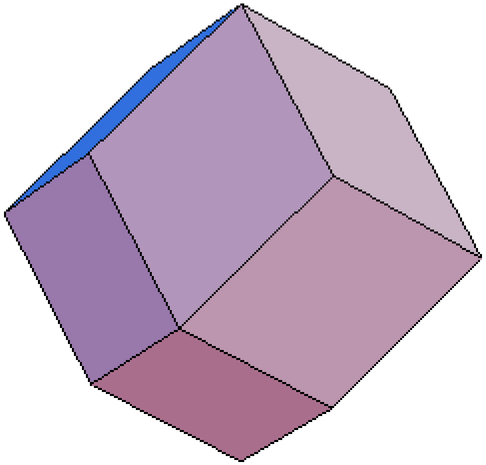
Zonotope Geometry – Set Representation

- Zonotopes are a special type of polytopes that can be represented by an equality equation involved with unitary intervals.

$$\mathbf{p} \oplus H\mathbf{B}^m = \{\mathbf{p} + H\mathbf{z} | \mathbf{z} \in \mathbf{B}^m\}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} [-1,1] \\ [-1,1] \\ [-1,1] \end{bmatrix}$$





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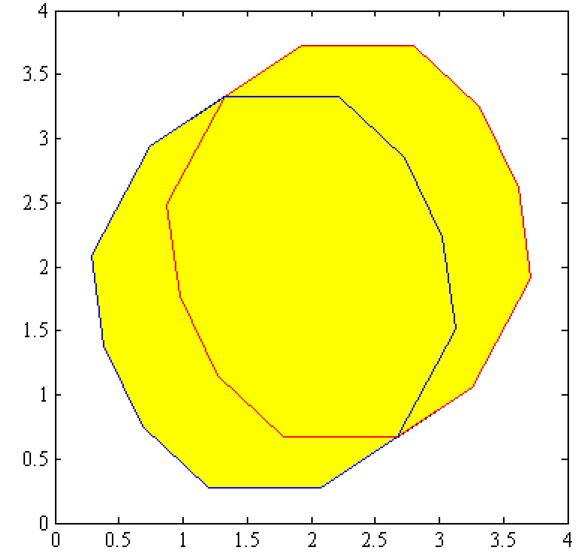
Zonotope Geometry – Set Partition

- Similar to the bisection of an interval or a box, a zonotope can also be bisected into two small zonotopes:

$$L\mathcal{Z} = (\mathbf{p} - \frac{\mathbf{h}_k}{2}) \oplus [\mathbf{h}_1 \cdots \frac{\mathbf{h}_k}{2} \cdots \mathbf{h}_m]\mathbf{B}^m$$

$$R\mathcal{Z} = (\mathbf{p} + \frac{\mathbf{h}_k}{2}) \oplus [\mathbf{h}_1 \cdots \frac{\mathbf{h}_k}{2} \cdots \mathbf{h}_m]\mathbf{B}^m$$

Jian Wan, Josep Vehi and Ningsu Luo. A Numerical Approach to Design Control Invariant Sets for Constrained Nonlinear Discrete-time Systems with Guaranteed Optimality. *Journal of Global Optimization*, Vol. 44, No. 3, pp. 395-407, 2009.



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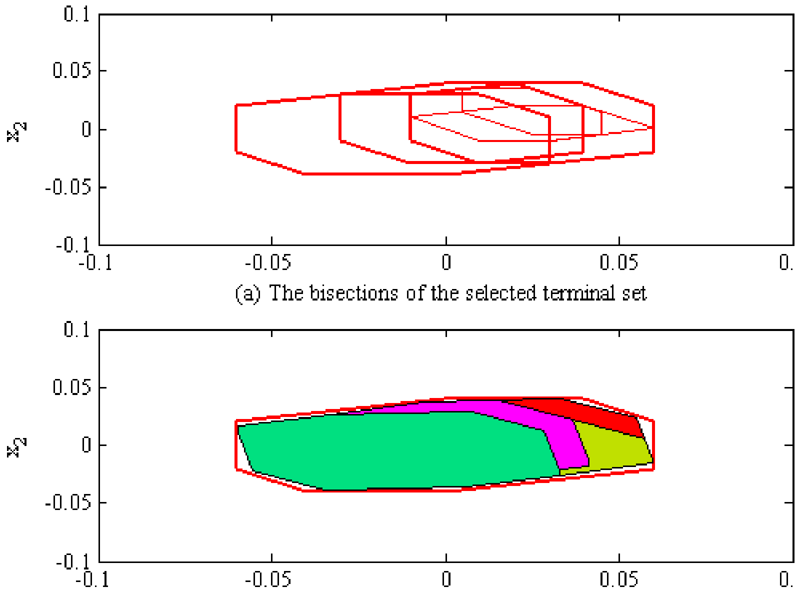

(a) The bisections of the selected terminal set

(b) The evolution of every sub-zonotope of the bisected terminal set

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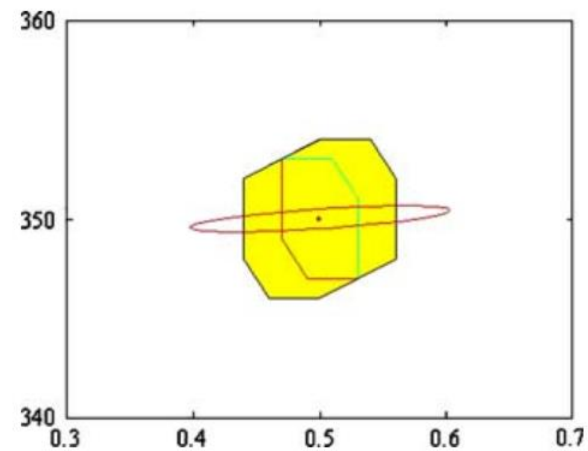

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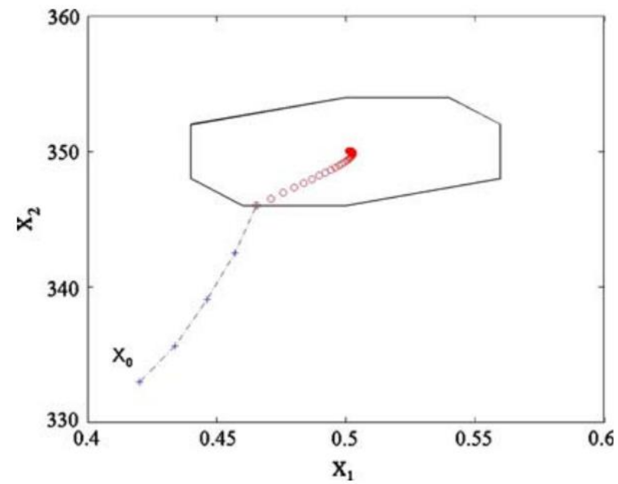

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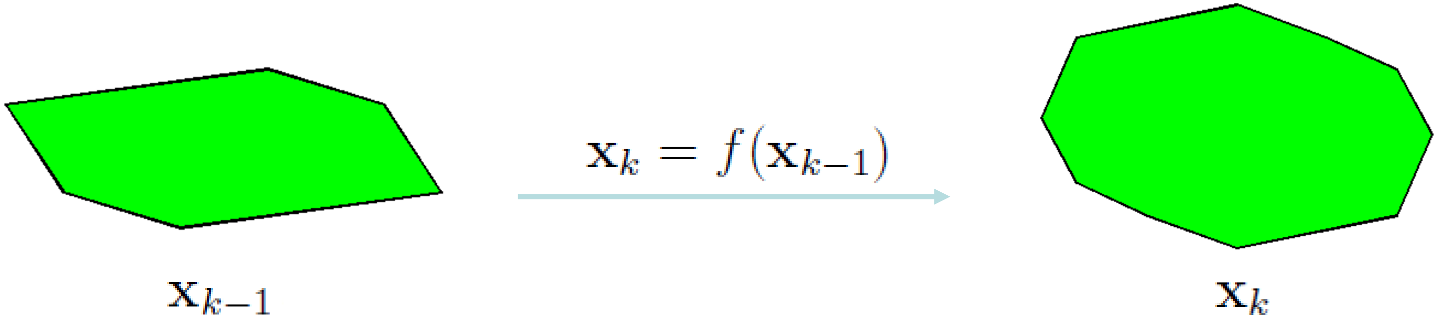
Zonotope Geometry – Set Propagation

- Set propagation of a zonotope for a nonlinear systems is implemented by the centred inclusion function via the mean-value theorem.

$$\mathbf{F}_c(\mathcal{Z}) = \mathbf{f}(\mathbf{p}) + \nabla_{\mathbf{x}}\mathbf{f}(\mathbb{X})(\mathcal{Z} - \mathbf{p})$$



Wolfgang Kühn. Rigorously computed orbits of dynamical systems without the wrapping effect. *Computing*, Vol. 61, No. 1, pp. 47-67, 1998.

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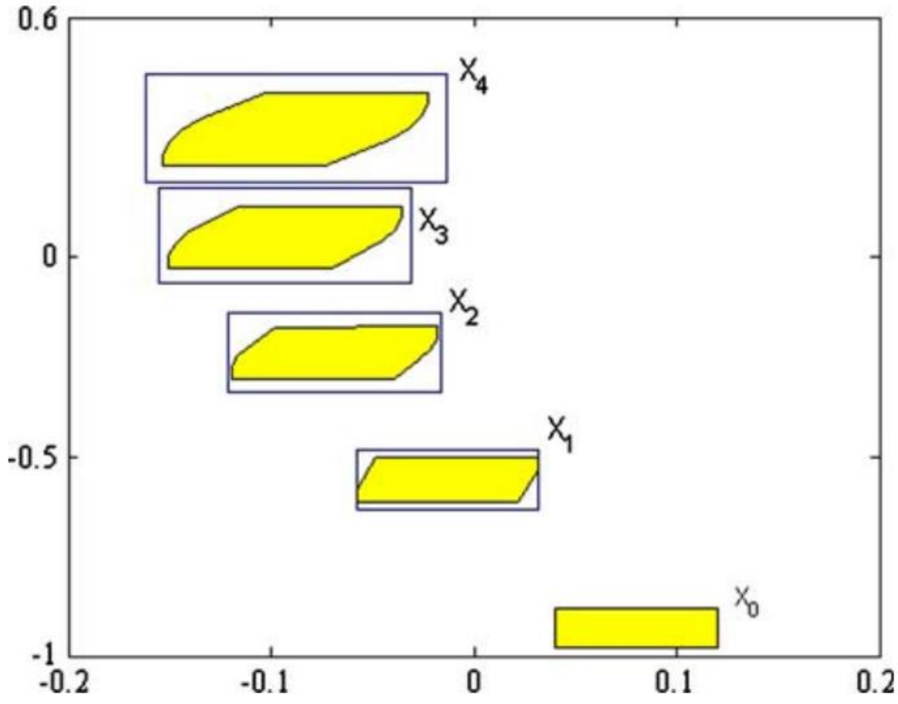
$$\mathbf{F}_c(\mathcal{Z}) = \mathbf{f}(\mathbf{p}) + \nabla_{\mathbf{x}}\mathbf{f}(\mathbb{X})(\mathcal{Z} - \mathbf{p})$$

$$\begin{cases} x_1(k+1) = x_1(k) + 0.1x_2(k) \\ x_2(k+1) = x_2(k) + 0.1[x_1^2(k) + x_2^2(k) + u(k)] \end{cases}$$

$X_0 = [0.04, 1.12] \times [-0.98, -0.88]$

$$X_0 = \begin{bmatrix} 0.08 \\ -0.93 \end{bmatrix} + \begin{bmatrix} 0.04 & 0 \\ 0 & 0.05 \end{bmatrix} \mathbf{B}^2$$



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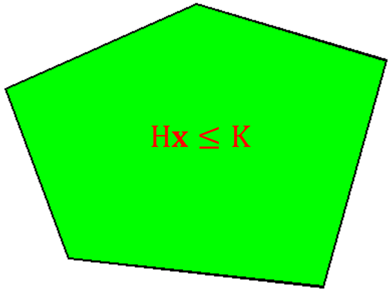
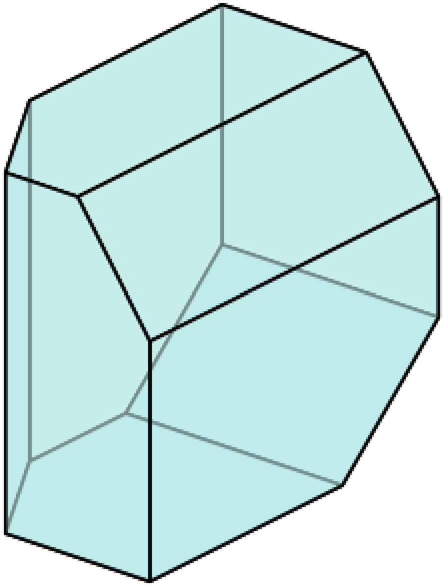
Polytope Geometry – Set Representation

- Polytopes are convex sets that can be represented by linear inequality constraints $H\mathbf{x} \leq K$ or the vertices of the polytope.





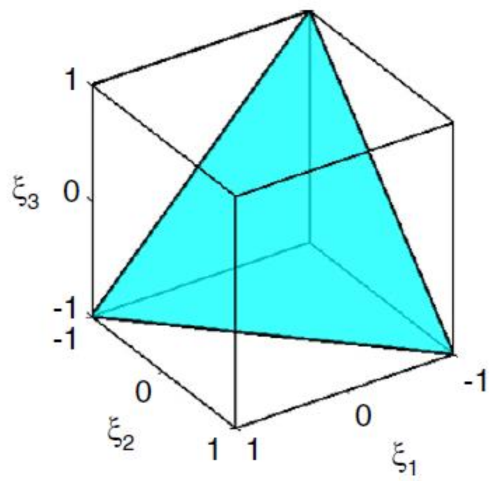
Polytope Geometry – Set Representation via Constrained Zonotope

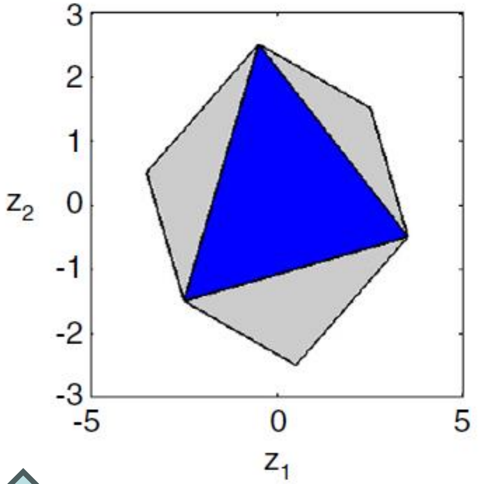
- A polytope can also be represented as a constrained zonotope:

$$Z = \{\mathbf{G}\boldsymbol{\xi} + \mathbf{c} : \|\boldsymbol{\xi}\|_\infty \leq 1,\ \mathbf{A}\boldsymbol{\xi} = \mathbf{b}\}$$

$$Z = \left\{ \begin{bmatrix} 1.5 & -1.5 & 0.5 \\ 1 & 0.5 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, -1 \right\}$$

$$\mathbf{1}^{\mathrm{T}}\boldsymbol{\xi} = -1$$





Joseph K. Scott, Davide M. Raimondo, Giuseppe Roberto Marseglia and Richard D. Braatz. Constrained zonotopes: A new tool for set-based estimation and fault detection, Automatica, Vol. 69, pp. 126-136, 2016.

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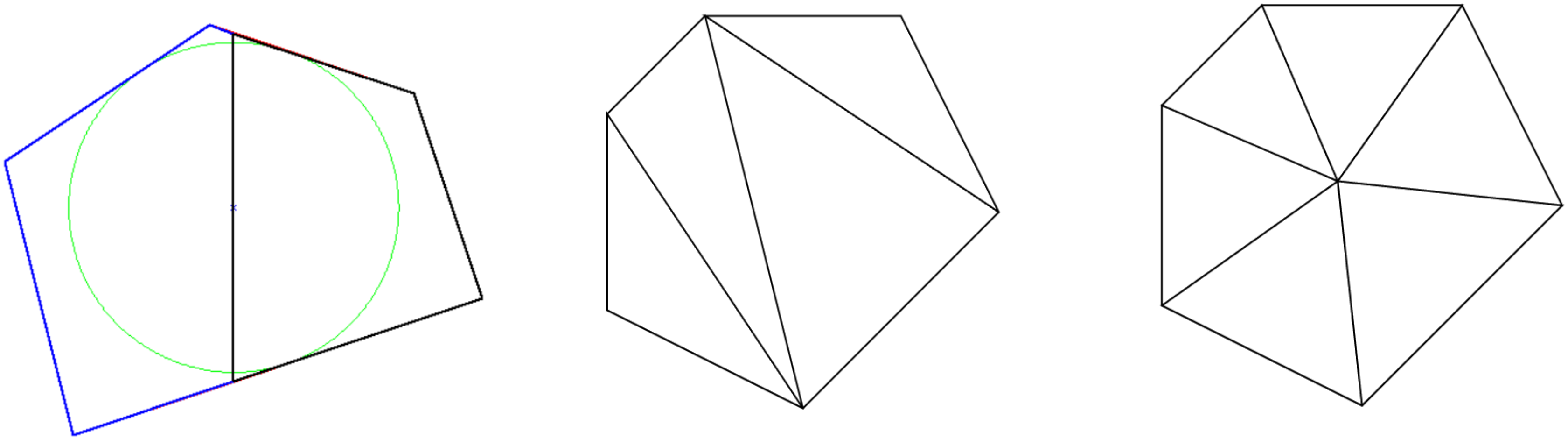
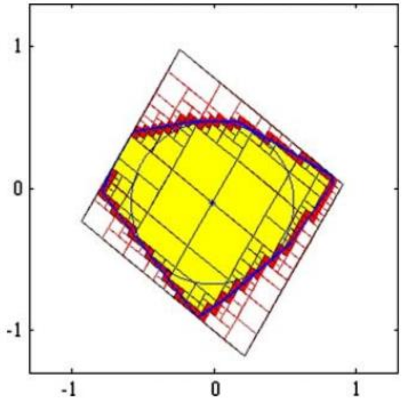
Polytope Geometry – Set Partition

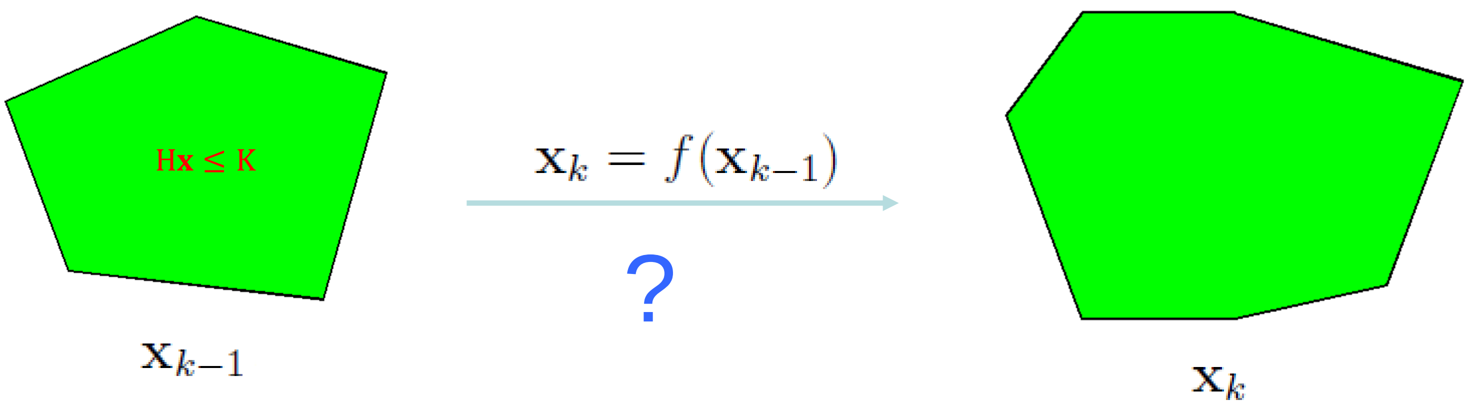
- A polytope can be partitioned through its Chebyshev centre or through Delaunay triangulation for keeping all edges intact.





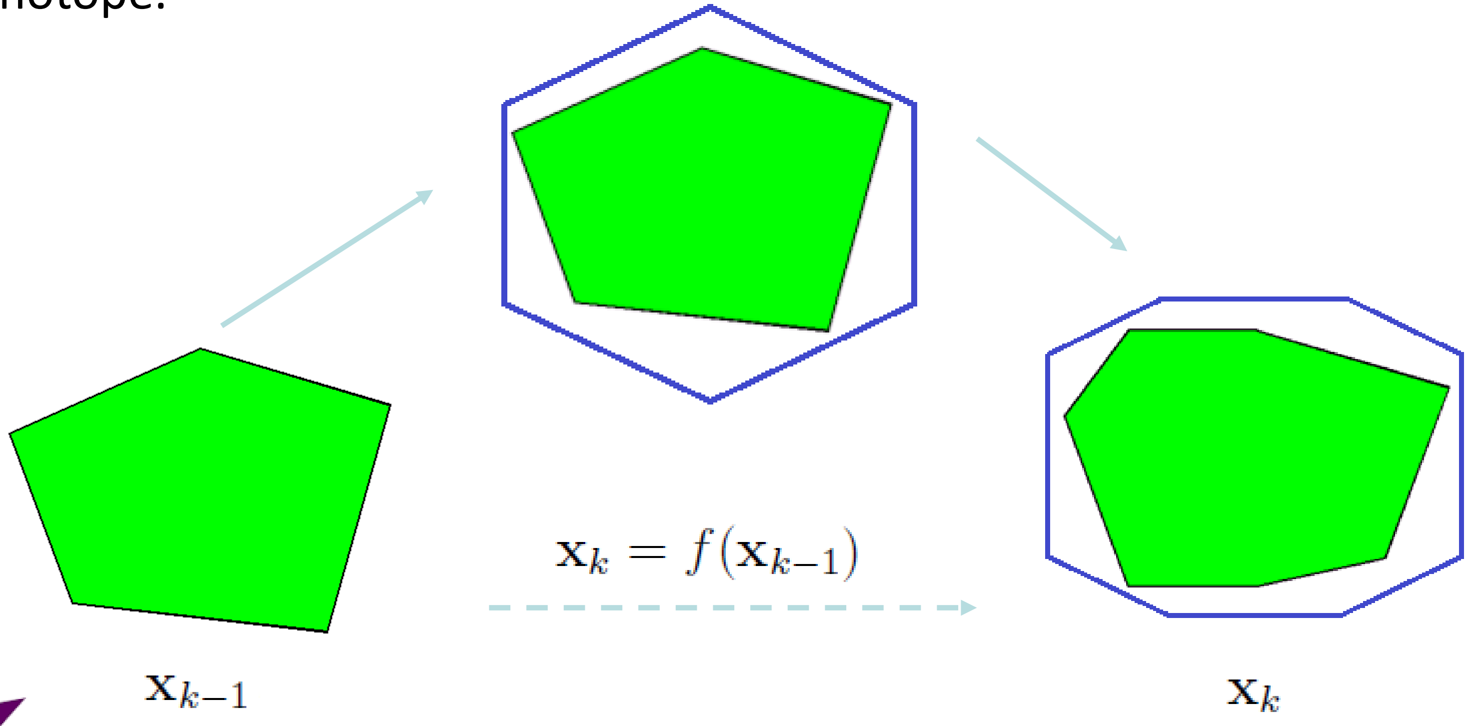
Polytope Geometry – Set Propagation

- Set propagation of a polytope for a nonlinear system is very challenging due to its mathematical format involving linear inequality constraints.
- There still lack of direct polytopic set propagation methods for nonlinear systems.



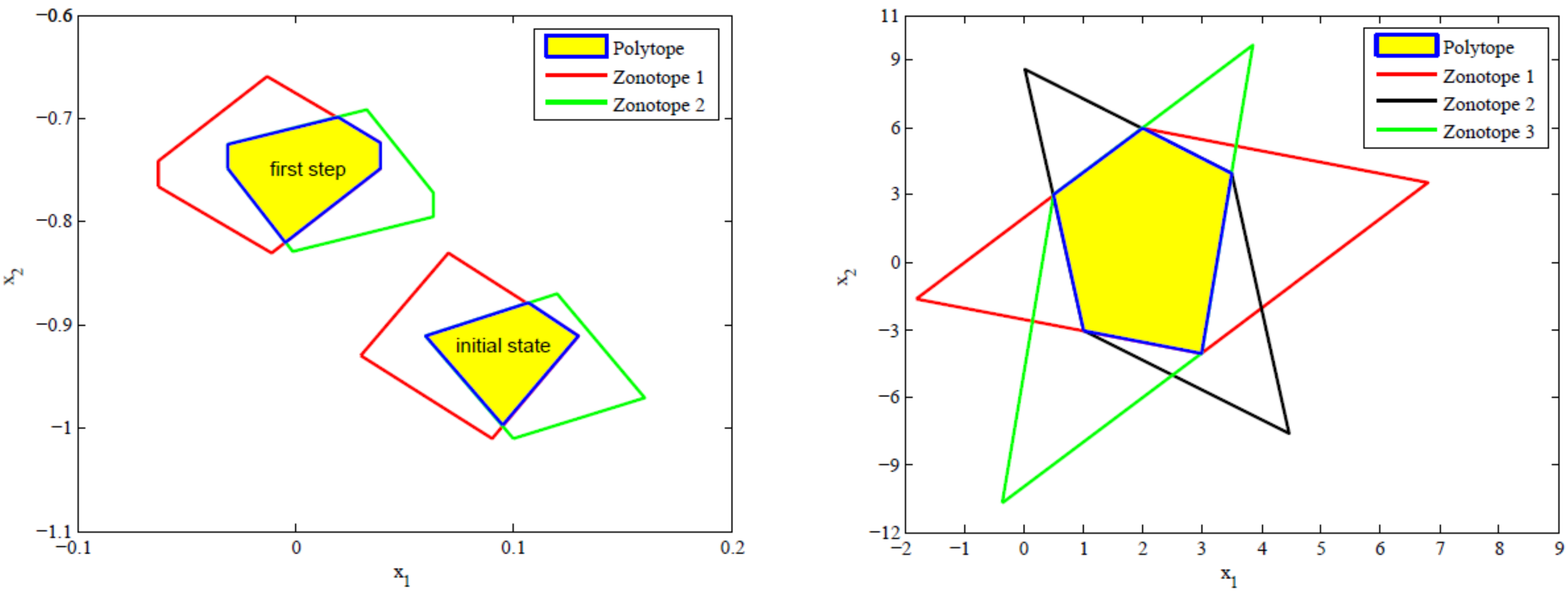
Polytope Geometry – Set Propagation via Zonotopes

- A common practice of polytopic set propagation is to approximate the polytope by a single zonotope at first and then to propagate this single zonotope.



Polytope Geometry – Set Propagation via Zonotopes

- Representing a polytope exactly by the intersection of zonotopes or zonotope bundles, set propagation of a polytope for a nonlinear system can be implemented indirectly via zonotopes.



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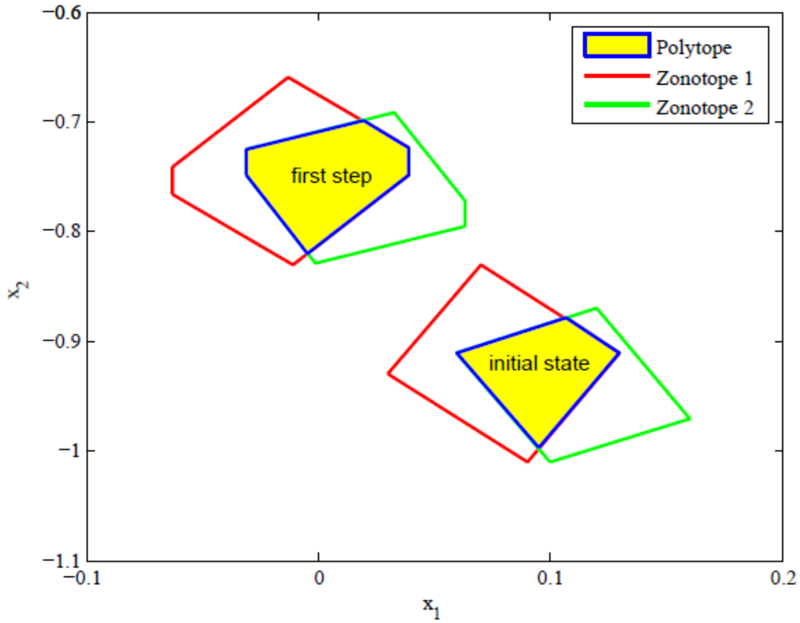

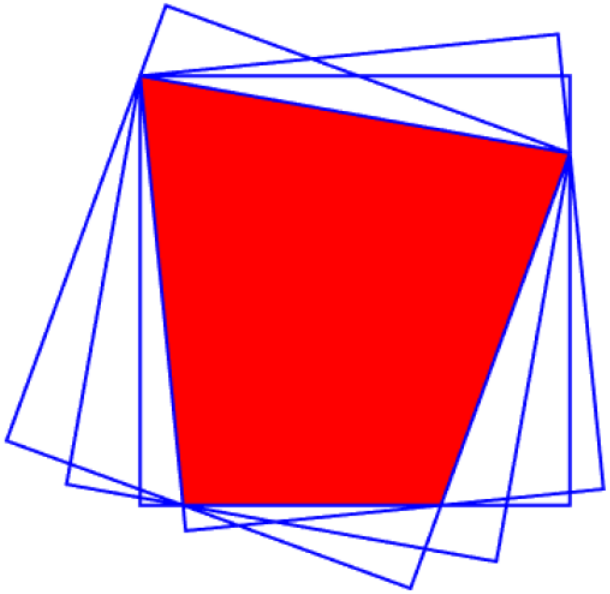

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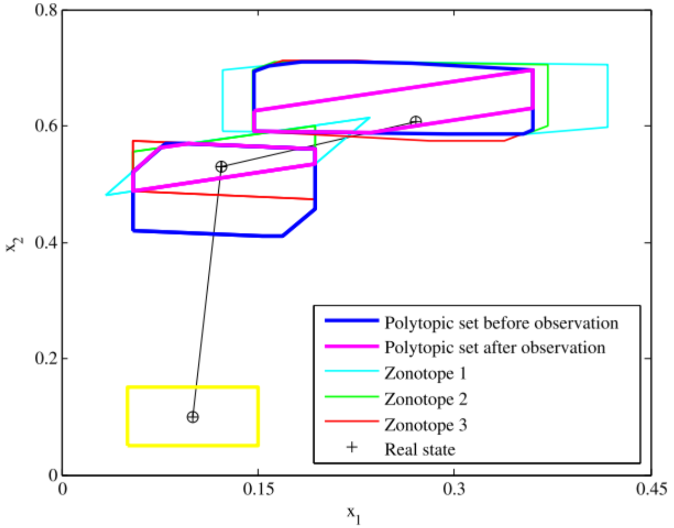

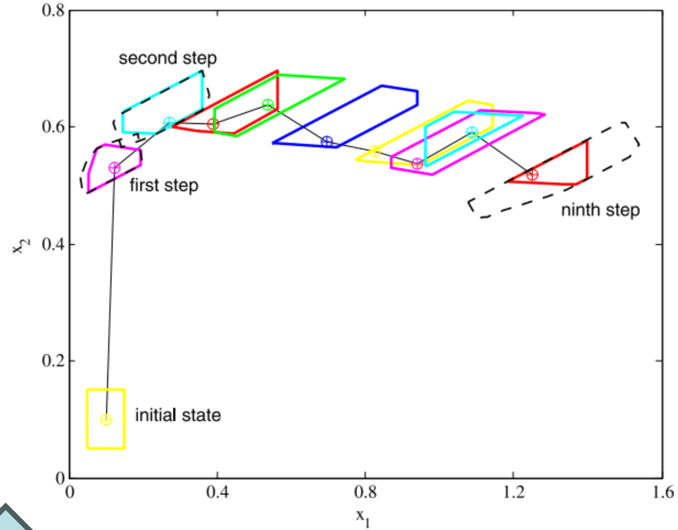

Jian Wan, Sanjay Sharma and Robert Sutton. Guaranteed state estimation for nonlinear discrete-time systems via indirectly implemented polytopic set computation. IEEE Transactions on Automatic Control, Vol. 63, No. 12, pp. 4317-4322, 2018.

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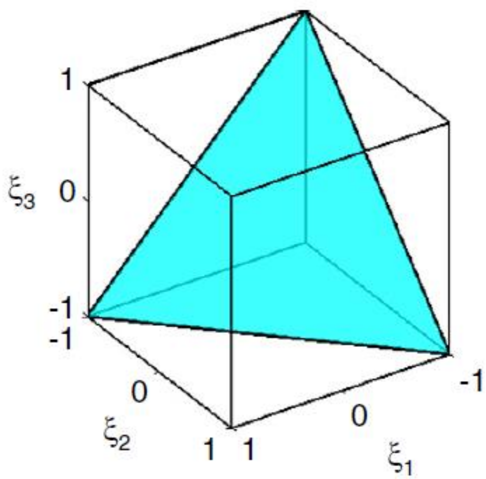
Q1: Can a polytope be represented by a union of zonotopes instead?

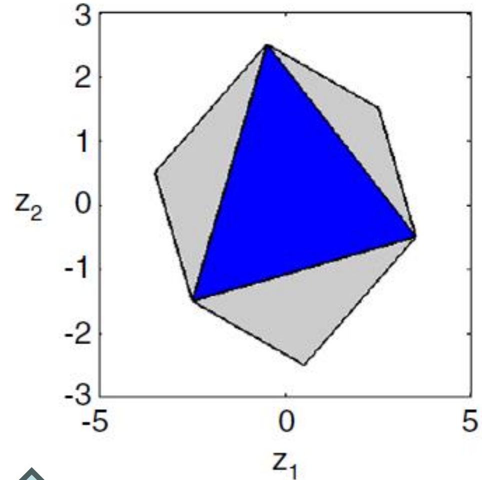
Polytope Geometry – Set Propagation via Constrained Zonotopes

- Representing a polytope by a constrained zonotope, set propagation of a polytope for a nonlinear system can be also be implemented indirectly via constrained zonotopes.

$$Z = \left\{ \begin{bmatrix} 1.5 & -1.5 & 0.5 \\ 1 & 0.5 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, -1 \right\}$$

$$\mathbf{1}^{\mathrm{T}}\xi = -1$$





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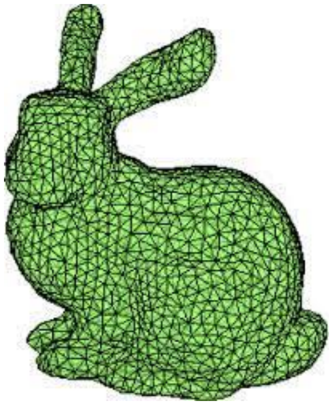
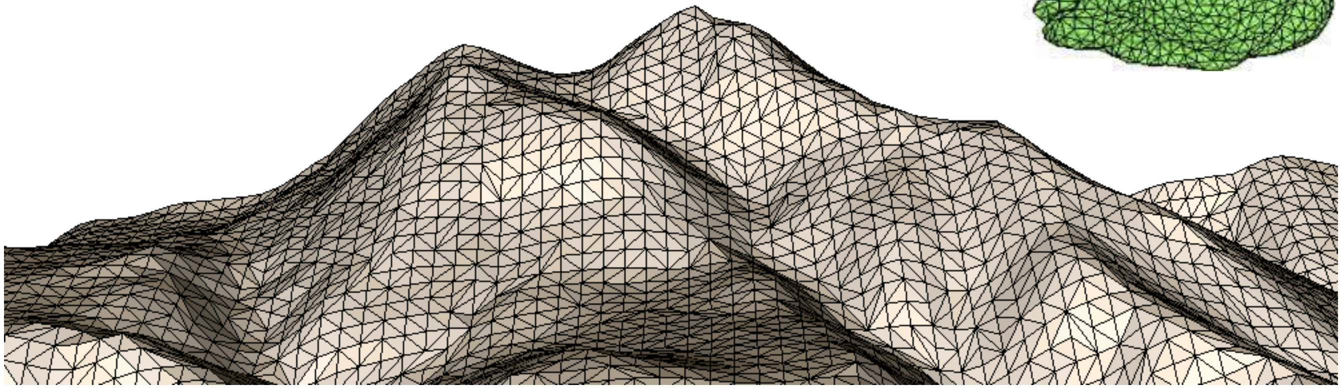
Q2: Can a polytope be partitioned into triangles or tetrahedrons?

Reducing the Wrapping Effect via Delaunay Triangulation

- Delaunay triangulation is an important topic in computational geometry and it provides an efficient way to organize distributed data points in a triangular mesh.
  - Computer graphics
  - Terrain representation
  - Finite element analysis
  - Others





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Reducing the Wrapping Effect via Delaunay Triangulation

- Delaunay triangulation's connection to set-theoretic method has not been recognised or explored thoroughly in the literature in terms of:
    - Set partition
    - Set representation
    - Set propagation

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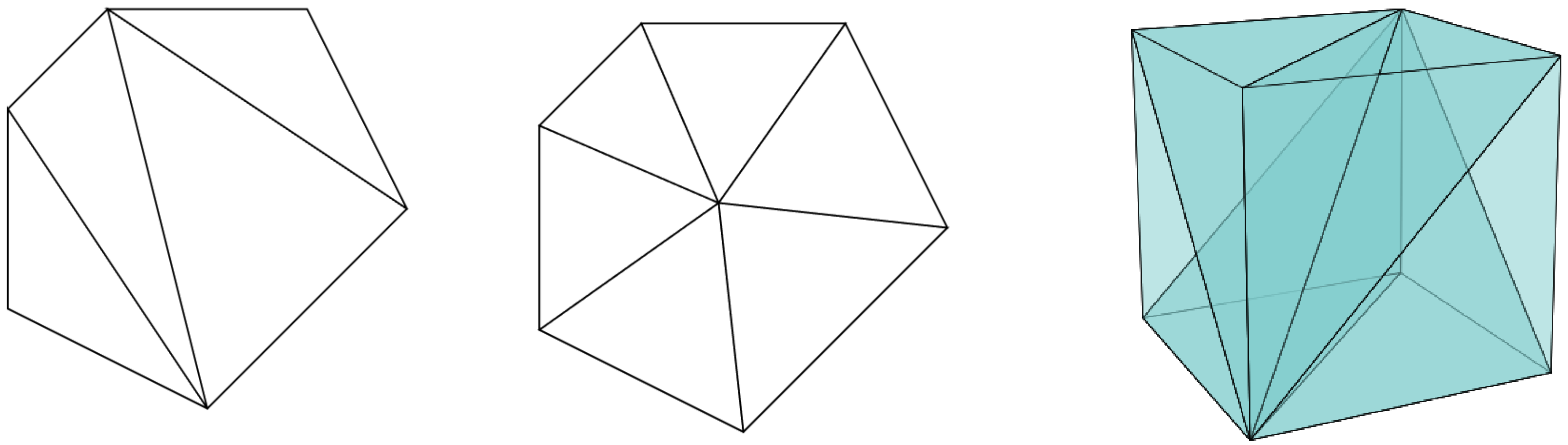
Reducing the Wrapping Effect via Delaunay Triangulation

- Set partition via Delaunay triangulation

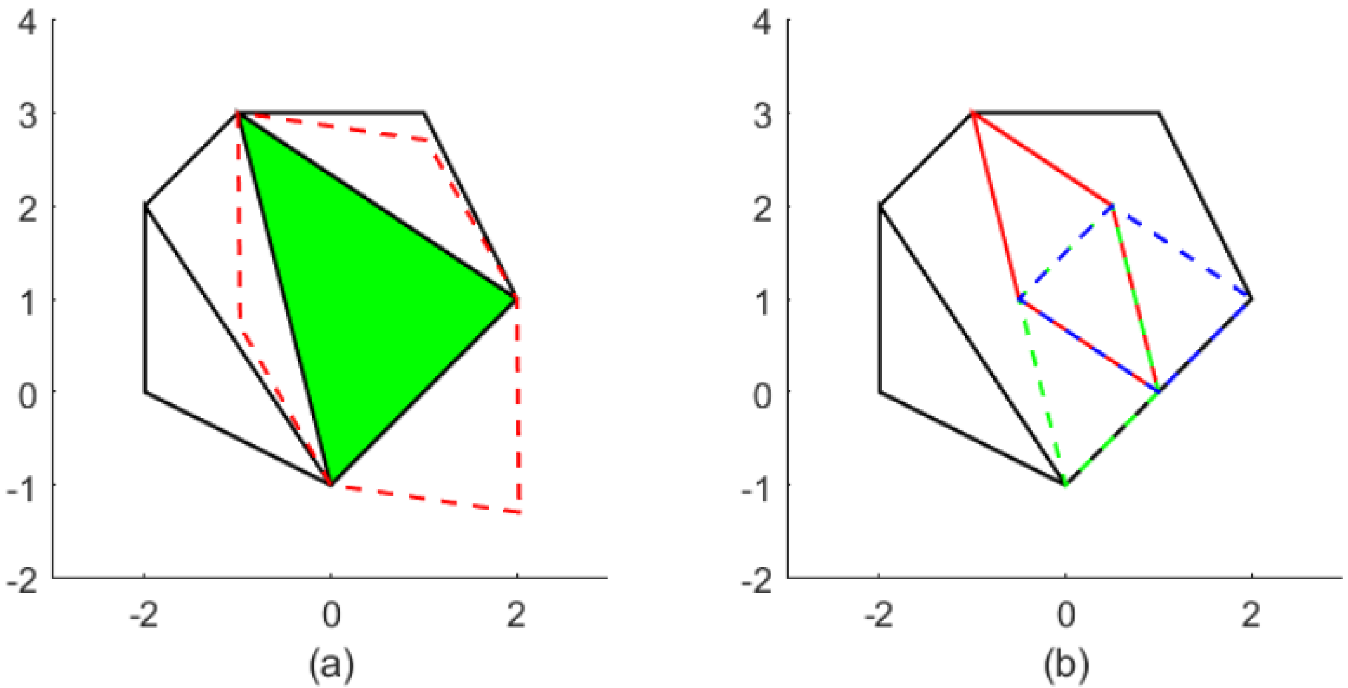


Q2: Can a polytope be partitioned into triangles or tetrahedrons? Yes

Reducing the Wrapping Effect via Delaunay Triangulation

- Set representation after Delaunay triangulation

$$z = \left\{ \begin{bmatrix} -0.4868 & 1.0132 & 0.0132 \\ 0.8528 & -0.1472 & -1.1472 \end{bmatrix}, \begin{bmatrix} 0.5132 \\ 0.8528 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, -1 \right\}$$



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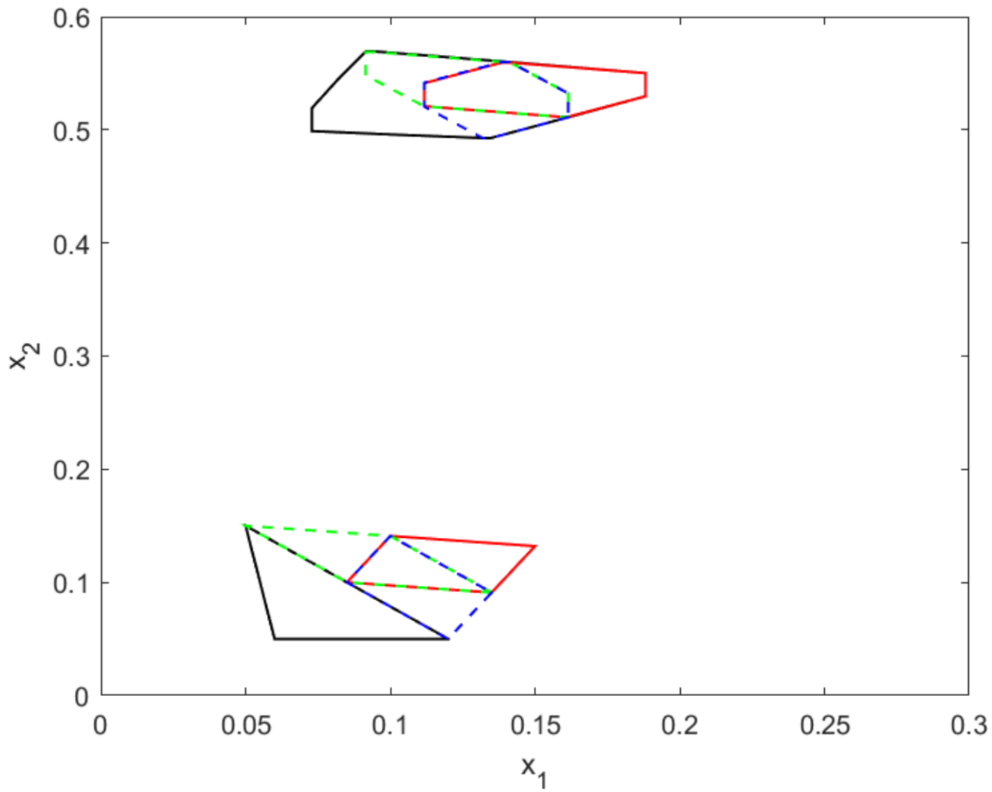
Reducing the Wrapping Effect via Delaunay Triangulation

- Set propagation after Delaunay triangulation



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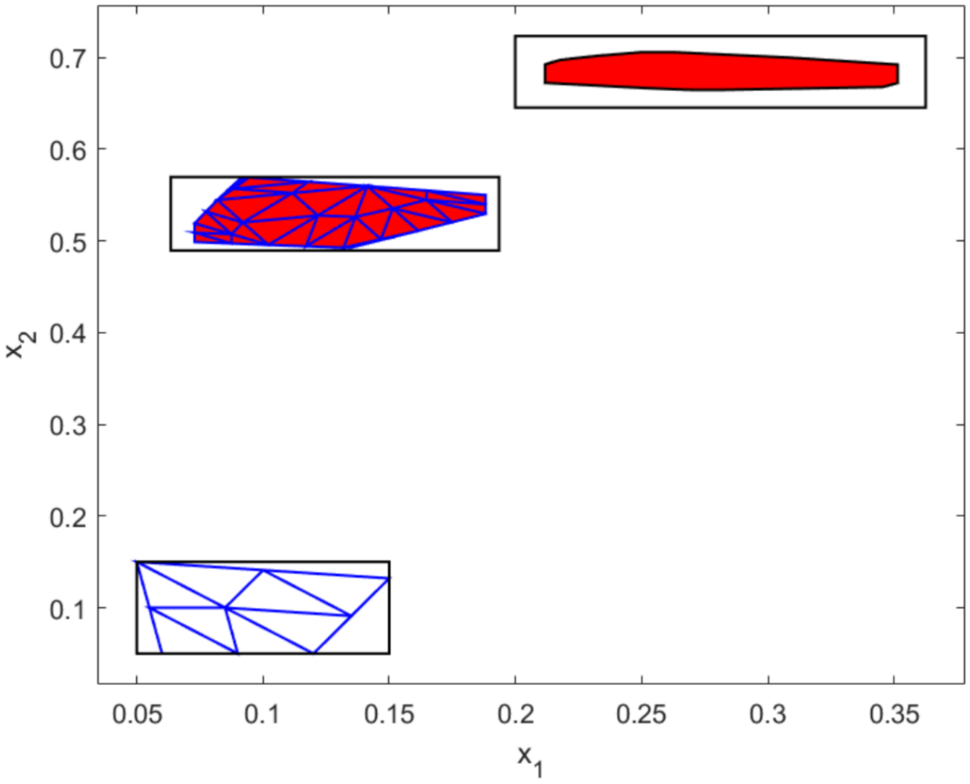
Reducing the Wrapping Effect via Delaunay Triangulation

- Set propagation after Delaunay triangulation



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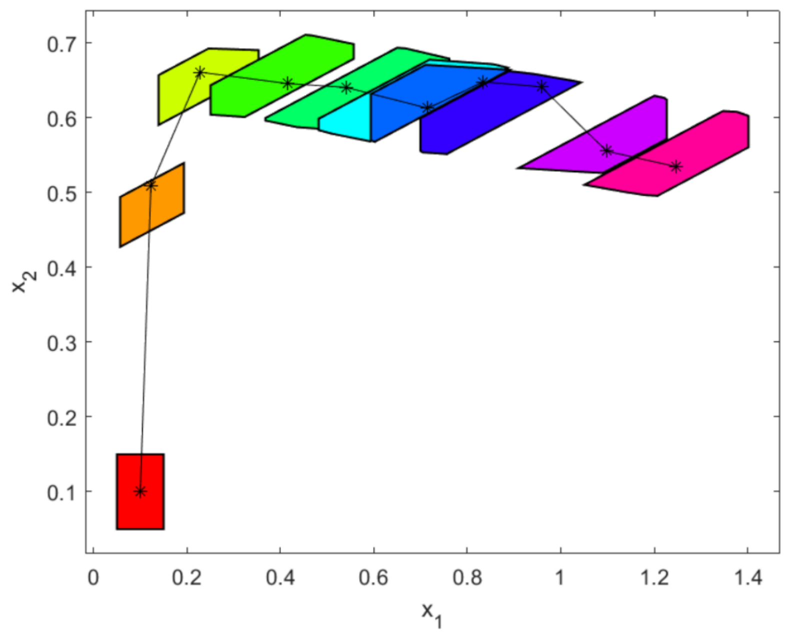
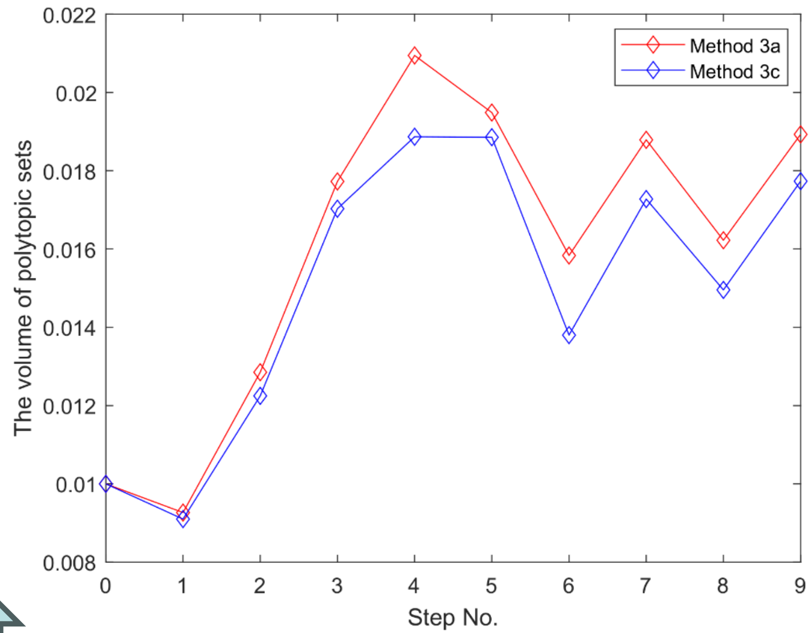
Basic set units have been extended from intervals or boxes to triangles or tetrahedrons.

Reducing the Wrapping Effect via Delaunay Triangulation

- Set propagation after Delaunay triangulation





Jian Wan and Luc Jaulin. Reducing the wrapping effect of set computation via Delaunay triangulation for guaranteed state estimation of nonlinear discrete-time systems. *Computing*, DOI: 10.1007/s00607-024-01275-0, 2024.

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Conclusions and Future Work

- Delaunay triangulation could potentially play a transformative role in set-theoretic methods for successful applications to control in terms of set invariance and state estimation for nonlinear systems:
  - Represent triangles by constrained zonotopes
  - Design polytopic control invariant sets
  - Estimate system states for nonlinear systems
  - Establish triangles or tetrahedrons as basic set units like intervals or boxes
  - …

A Review of Set Representation, Partition and Propagation Techniques for Set-Theoretic Methods in Control

Thank you very much!

Any Question?

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