Set Inversion and Box Contraction on Lie groups using interval analysis

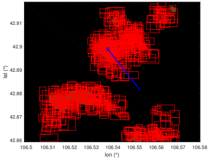
Nicolas Merlinge

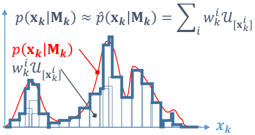
International Online Seminar on Interval Methods in Control Engineering

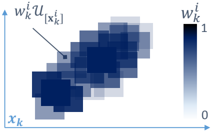
April 15th, 2024

Research interest

- Ph.D. thesis (2015-2018): State estimation by bounded kernels (probabilities and intervals)







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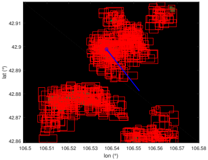

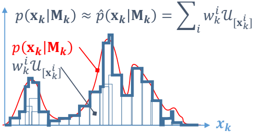

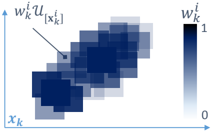

- Ph.d. theses supervision:
  - (collab.) C. Palmier: Adaptive Particle Filters for underwater navigation (2018-2021)
  - C. Chahbazian: Particle filters on Lie groups (2020-2023)
  - E. Lopes: Monte Carlo Markov Chains on Lie groups (2023-2026)
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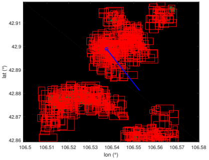

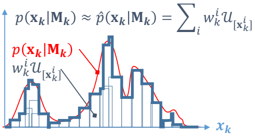

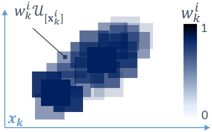

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- Interval analysis on Lie groups (application to attitude estimation)

Problem statement

Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$.

Set inversion (find all x that satisfy $f(x) \in [y]$):

$$\mathcal{S} = \left\{ x \in [x] \in \mathbb{IR}^n \;\middle|\; f(x) \in [y] \in \mathbb{IR}^m \right\} \tag{1}$$

where $\mathbb{IR}^d$ is the set of boxes of dimension d in $\mathbb{R}^d$.

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- Successfully solved using interval analysis (e.g. SIVIA[1], Constraint Satisfaction Problem[1]);

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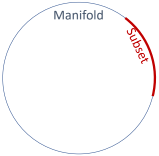
- Is it possible to solve (1) and (2) when $f : \mathcal{G} \to \mathcal{H}$ applies on non-Euclidean manifolds (e.g. rotation matrices) ?

Possible approaches of interval analysis on manifolds



- Partitioning the manifold itself

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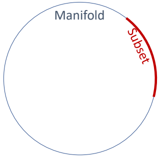

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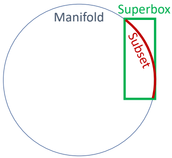

- Enclosing the manifold subset by a superbox

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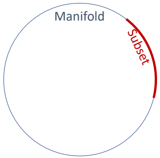

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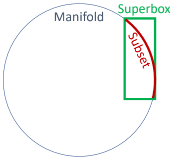
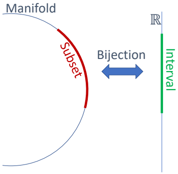
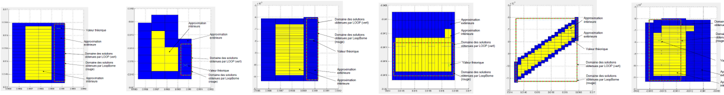


- Define a bijection with the Euclidean space



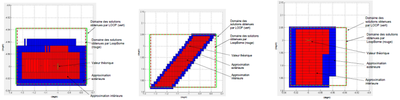
Interval analysis using Lie groups geometry

- Interval and Set membership dynamical propagation using Lie symmetries[1]
- Set inversion and contractors for quaternion estimation[2] (enclosed in a "super-set" of $\mathbb{R}^4$)



$q_0 - q_1$ $q_0 - q_2$ $q_0 - q_3$ $q_1 - q_2$ $q_1 - q_3$ $q_2 - q_3$

- ...and using the bijection of the quaternion space with Euler angles[2, 3]



$\varphi - \theta$ $\varphi - \psi$ $\theta - \psi$

1. Damers, J., Jaulin, L., Rohou, S. (2022). Lie symmetries applied to interval integration. Automatica, 144, 110502.
2. Nguyen, H. V. (2011). Estimation d'attitude et diagnostic d'une centrale d'attitude par des outils ensemblistes (Doctoral dissertation, Grenoble).
3. Nguyen, H. V., Berbra, C., Lesecq, S., Gentil, S., Barraud, A., Godin, C. (2009, June). Diagnosis of an inertial measurement unit based on set membership estimation. In 2009 17th Mediterranean Conference on Control and Automation (pp. 211-216). IEEE.

Bissectable Abstract Domains in $\mathbb{C}$

- In not-ordered sets or manifolds (e.g. $\mathbb{C}$, angles sets), it is not always possible to define a Moore family (stable set by intersection, e.g. $\mathbb{IR}$);
- Possibility to extend the concepts of interval analysis (e.g. intersection, bissection) to domains belinging to a Lattice stable by inclusion[1].
- Application to circular pavings (e.g. large angular domains):

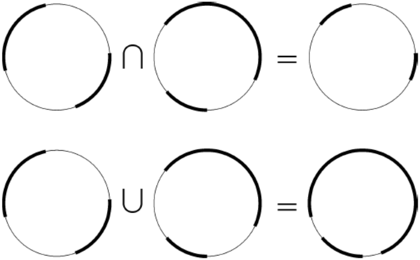
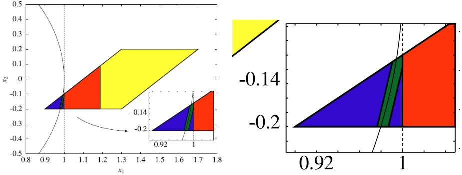


Figure 1: Intersection and union of two circular pavings

1. Jaulin, L., Desrochers, B., Massé, D. (2016). Bisectable Abstract Domains for the Resolution of Equations Involving Complex Numbers. Reliable Computing, 23(1), 35-46.

Set membership estimation in Lie algebra

- Constrained zonotopes using invariant properties[1]. Application to quaternions estimation enclosed in a "superset" of $\mathbb{R}^4$



- Constrained polytopes[2] as a "superset" enclosing $SO(3) \times \mathbb{R}^3$ for attitude and gyro bias estimation
- 3D ellipsoids in the Lie algebra $\mathfrak{so}(3)$ of SO(3) for a specific measurement equation[3]

1. Rego, B. S., Scott, J. K., Raimondo, D. M., Raffo, G. V. (2021). Set-valued state estimation of nonlinear discrete-time systems with nonlinear invariants based on constrained zonotopes. Automatica, 129, 109638.
2. Brás, S., Rosa, P., Silvestre, C., Oliveira, P. (2013). Global attitude and gyro bias estimation based on set-valued observers. Systems and Control Letters, 62(10), 937-942.
3. Sanyal, A. K., Lee, T., Leok, M., McClamroch, N. H. (2008). Global optimal attitude estimation using uncertainty ellipsoids. Systems and Control Letters, 57(3), 236-245.

Observers and filters on Lie groups

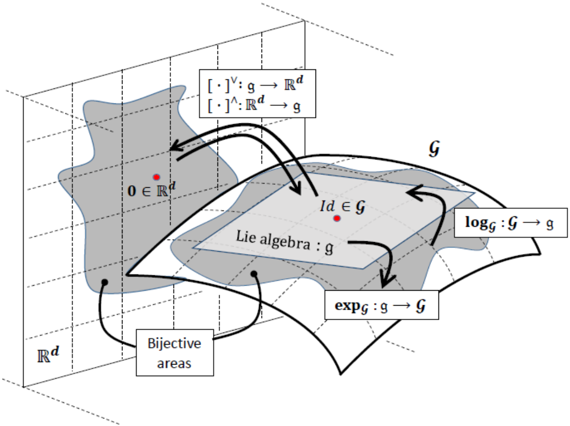
- Invariant observers[1, 2]
- Invariant Kalman Filter[3], Invariant UKF[4], Lie group EKF[5, 8]
- Invariant Rao-Backwellized Particle Filter[6], Particle filters in Lie group[7, 8]
- And many other contributions

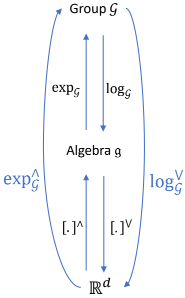
Euclidean Gaussian density ($x \in \mathbb{R}^d$)	(left) Lie group Gaussian density ($X \in \mathcal{G}$)
$p(x) = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} e^{-\frac{1}{2}\|x-\mu\|^2_\Sigma}$	$p(X) = \frac{1}{\sqrt{(2\pi)^d \det\left(\Phi^L_{\mathcal{G}}(X,\mu)\Sigma\Phi^L_{\mathcal{G}}(X,\mu)^T\right)}} e^{-\frac{1}{2}\left\|\log^\vee_{\mathcal{G}}(\mu^{-1}X)\right\|^2_\Sigma}$

Example: the Gaussian density translated into Lie groups (concentrated density[5]).

1. Barrau, A., Bonnabel, S. (2018). Linear observation systems on groups (I). working paper or preprint.
2. Mahony, R., Hamel, T., Trumpf, J. (2020). Equivariant systems theory and observer design. arXiv preprint arXiv:2006.08276.
3. Barrau, A., Bonnabel, S. (2018). Invariant kalman filtering. Annual Review of Control, Robotics, and Autonomous Systems, 1, 237-257.
4. Brossard, M., Bonnabel, S., Condomines, J. P. (2017, September). Unscented Kalman filtering on Lie groups. In 2017 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS) (pp. 2485-2491). IEEE.
5. Bourmaud, G., Mégret, R., Giremus, A., Berthoumieu, Y. (2013, September). Discrete extended Kalman filter on Lie groups. In 21st European Signal Processing Conference (EUSIPCO 2013) (pp. 1-5). IEEE.
6. Barrau, A., Bonnabel, S. (2014, December). Invariant particle filtering with application to localization. In 53rd IEEE Conference on Decision and Control (pp. 5599-5605). IEEE.
7. Zhang, C., Taghvaei, A., Mehta, P. G. (2017). Feedback particle filter on riemannian manifolds and matrix lie groups. IEEE Transactions on Automatic Control, 63(8), 2465-2480.
8. Chahbazian, C. (2023). Particle Filtering on Lie Groups: Application to Navigation (Doctoral dissertation, Université Paris-Saclay).

Lie groups properties: the exponential mapping (bijection exp/log)





1. Hilgert, J., Neeb, K. H. (2011). Structure and geometry of Lie groups. Springer Science and Business Media.
2. Chahbazian, C. (2023). Particle Filtering on Lie Groups: Application to Navigation (Doctoral dissertation, Université Paris-Saclay).

"Errors" in Lie groups

Let $\mathcal{G}$ be a Lie group endowed with a "product" internal binary operation.

1. Two ways to compute an error between two elements ($A, B \in \mathcal{G}$):

Side	Generic notation	Example with $(\mathbb{R}, +)$
(Left)	$B^{-1}A$	$-B + A$
(Right)	AB^{-1}	$A - B$

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2. *Log-Euclidean error*: mapping to the Euclidean space.

Side	Generic notation	Example with $(\mathbb{R}, +)$
(Left)	$\log_{\mathcal{G}}^{\vee}(B^{-1}A)$	$-B + A$
(Right)	$\log_{\mathcal{G}}^{\vee}(AB^{-1})$	$A - B$

Lie group boxes

A Lie group box can be defined by an *origin* $\mu_X \in \mathcal{G}$ and an *algebra domain* $[x] \in \mathbb{IR}^n$:

$$[X] \equiv \langle \mu_X, [x] \rangle \in \mathbb{I}_\bullet \mathcal{G} \tag{3}$$

where $\bullet$ is L or R.

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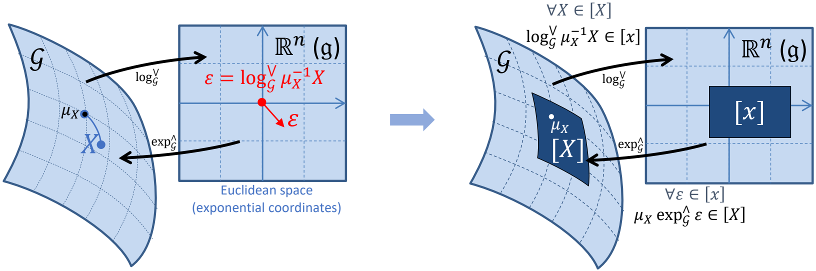

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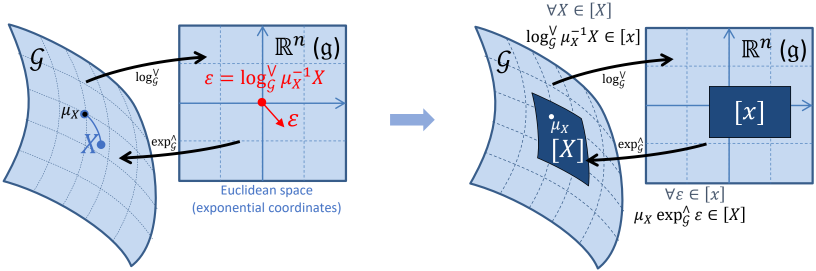


That is to say:

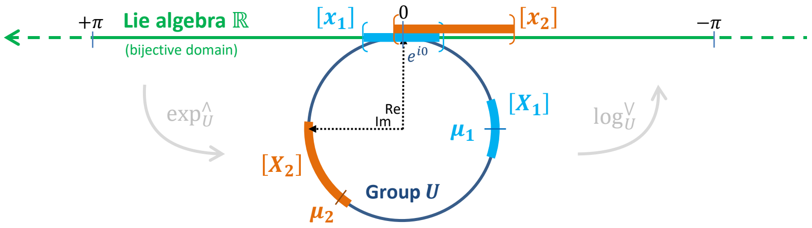
$$\begin{aligned} \textbf{Left:} \quad & [X] = \mu_X \exp_{\mathcal{G}}^{\wedge}([x]) \\ \textbf{Right:} \quad & [X] = \exp_{\mathcal{G}}^{\wedge}([x])\mu_X \end{aligned} \tag{4}$$

Merlinge, N. (2024). Set Inversion and Box Contraction on Lie groups using interval analysis, accepted to Automatica, to appear soon.

Example

Consider the unit circle group $(U, \cdot)$:

$$U \triangleq \{X \in \mathbb{C}^* \mid |X| = 1\} \tag{5}$$



- $\exp_U$ and $\log_U$ are the usual complex exponential and logarithm functions
- $\exp_U$ is bijective on $(-\pi, \pi]$
- The multiplication $\cdot$ is commutative

Width, volume

Definition

The width of a Lie group box $[X] \equiv \langle \mu_X, [x] \rangle \in \mathbb{I}_\bullet \mathcal{G}$ is defined by:

$$w_{\mathcal{G}}([X]) \triangleq w([x]). \tag{6}$$

Definition

The volume of a Lie group box $[X] \equiv \langle \mu_X, [x] \rangle \in \mathbb{I}_\bullet \mathcal{G}$ is defined by:

$$V_{\mathcal{G}}([X]) \triangleq \int_{[X]} d_H X = \int_{[x]} |\det \Phi_{\mathcal{G}}(\varepsilon)| d\varepsilon. \tag{7}$$

Inclusion, intersection

Property

Consider two Lie group boxes $[X] \equiv \langle \mu, [x] \rangle$ and $[Y] \equiv \langle \mu, [y] \rangle$ in $\mathbb{I}_{\bullet}\mathcal{G}$ sharing the same origin $\mu \in \mathcal{G}$. Then, the inclusion in the algebra domains is equivalent to the inclusion in the group:

$$[x] \subset [y] \Leftrightarrow [X] \subset [Y]. \tag{8}$$

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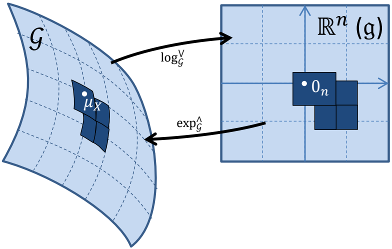
$$[X] \cap [Y] \equiv \langle \mu, [x] \cap [y] \rangle \,. \tag{9}$$

Lie group subpaving

A Lie group regular subpaving of a subset $\mathcal{A} \subset \mathcal{G}$ can be defined as follows:

Definition

$$\{[X_i]\}_{i\in[1,N]}^{\mathcal{A}} \triangleq \left\{ [X_i] \in \mathbb{I}_\bullet\mathcal{G} \;\middle|\; \begin{array}{l} \bigcup_{i=1}^{N}[X_i] = \mathcal{A} \\ \forall i,j \in [1,N], i \neq j, \\ \quad V_{\mathcal{G}}([X_i] \cap [X_j]) = 0 \end{array} \right\}. \tag{10}$$



Property

A given regular Euclidean subpaving $\{[x_i]\}_{i\in[1,N]}^{\mathcal{B}}$ on a Euclidean domain $\mathcal{B} \subset \mathbb{R}^n$ can be mapped to a unique subpaving on a Lie group $\mathcal{G}$ around a given reference point $\mu_X \in \mathcal{G}$.

Inclusion functions

A Lie group inclusion function can be defined as follows ($\bullet$ can be L or R):

Definition

Let $f : \mathcal{G} \to \mathcal{H}$. A *Lie group inclusion function* $[f] : \mathbb{I}_\bullet \mathcal{G} \to \mathbb{I}_\bullet \mathcal{H}$ can be defined by:

$$[f]([X]) \triangleq \left[\left\{ f(X) \in \mathcal{H} \;\middle|\; X \in [X] \right\} \right]_{\mathcal{H}}^{\bullet}, \tag{11}$$

where $[\ldots]_{\mathcal{H}}^{\bullet}$ is the wrapping operator from a subset of $\mathcal{H}$ to the box set $\mathbb{I}_\bullet \mathcal{H}$.

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Equivalently, by choosing $\mu_Y \in \mathcal{H}$:

$$[f](\langle \mu_X, [x] \rangle) \equiv \langle \mu_Y, [y] \rangle \equiv \left\langle \mu_Y, [f^\bullet_{\mu_X, \mu_Y}]([x]) \right\rangle. \tag{12}$$

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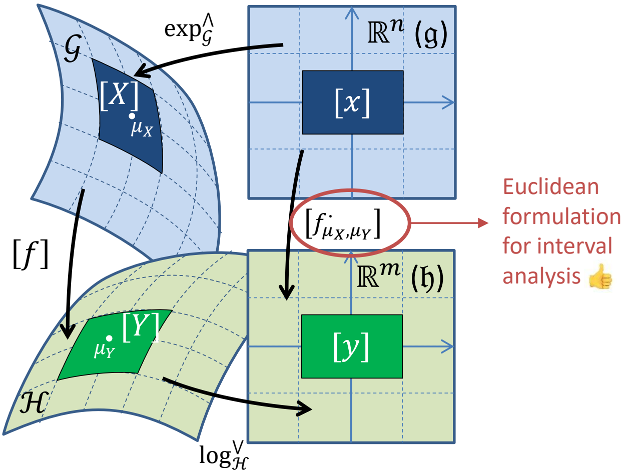
Property

Equivalent Euclidean inclusion function: $[f^\bullet_{\mu_X,\mu_Y}] : \mathbb{IR}^n \to \mathbb{IR}^m$:

$$[f^L_{\mu_X,\mu_Y}]([x]) \triangleq \left[\left\{ \log^\vee_{\mathcal{H}}(\mu_Y^{-1} f(\mu_X \exp^\wedge_{\mathcal{G}}(\varepsilon))) \;\middle|\; \varepsilon \in [x] \right\}\right] \tag{13}$$

$$[f^R_{\mu_X,\mu_Y}]([x]) \triangleq \left[\left\{ \log^\vee_{\mathcal{H}}(f(\exp^\wedge_{\mathcal{G}}(\varepsilon)\mu_X)\mu_Y^{-1}) \;\middle|\; \varepsilon \in [x] \right\}\right]. \tag{14}$$

Inclusion functions



Set inversion on Lie groups

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Euclidean equivalent problem

Problem (15) is thus equivalent to the following problem:

$$\mathcal{S}^{\mathcal{G}} = \left\{ \varepsilon \in [x] \;\middle|\; f^{\bullet}_{\mu_X,\mu_Y}(\varepsilon) \in [y] \right\}. \tag{16}$$

Set inversion on Lie groups

Definition

The set inversion problem on Lie groups is defined as follows ($[X] \in \mathbb{I}_\bullet\mathcal{G}$, $[Y] \in \mathbb{I}_\bullet\mathcal{H}$):

$$\mathcal{S}^{\mathcal{G}} = \left\{ X \in [X] \;\middle|\; f(X) \in [Y] \right\}. \tag{15}$$

Euclidean equivalent problem

Problem (15) is thus equivalent to the following problem:

$$\mathcal{S}^{\mathcal{G}} = \left\{ \varepsilon \in [x] \;\middle|\; f^{\bullet}_{\mu_X,\mu_Y}(\varepsilon) \in [y] \right\}. \tag{16}$$

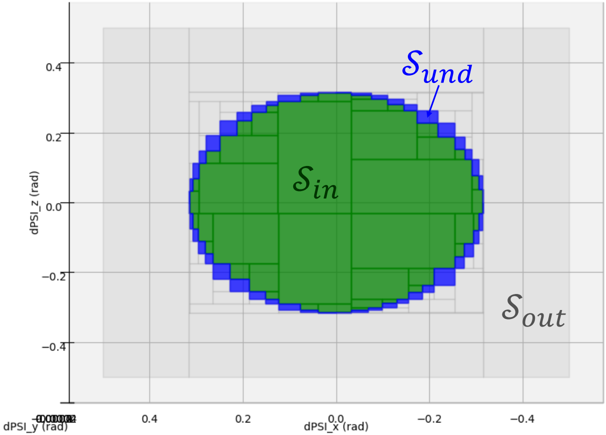
Theorem

Consider a function $f : \mathcal{G} \to \mathcal{H}$ admitting a convergent Lie group inclusion function and an output domain $[Y] \subset \mathcal{H}$ of which the algebra domain $[y]$ is a full compact set. Then, Problem (15) can be solved in the Euclidean space of the Lie algebra $\mathfrak{g}$ using SIVIA. The resulting undetermined domain $\mathcal{S}^{\mathcal{G}}_{und} \subset \mathcal{G}$ has a finite volume in the group. Moreover, each box of the regular subpaving that covers $\mathcal{S}^{\mathcal{G}}_{und}$ has a bounded volume:

$$\forall [U_i] \in \mathcal{S}^{\mathcal{G}}_{und}, V_{\mathcal{G}}([U_i]) < \sup_{[u_i]}(|\det \Phi_{\mathcal{G}}|)\epsilon^n < \infty \tag{17}$$

where $\epsilon > 0$ such that $\forall [U_i] \in \mathcal{S}^{\mathcal{G}}_{und}, w_{\mathcal{G}}([U_i]) < \epsilon$.

Set inversion on Lie groups (SIVIA output)



Solution in $\mathbb{R}^n$ (isomorphic to $\mathfrak{g}$)

In the Euclidean space:

- $\mathcal{S}_{und}$ is a paving of boxes $[s^i_{und}]$ such that $\forall i$ $w([s^i_{und}]) < \epsilon$
- $\mathcal{S}_{und}$ corresponds to a Lie group paving $\mathcal{S}^{\mathcal{G}}_{und}$ that has a bounded volume in the group

Contractor on Lie groups

Definition

The Lie group box contraction problem to be solved is defined by:

$$\mathcal{C}_{\mathcal{G}}([X]) \triangleq \left[\left\{X \in [X] \in \mathbb{I}_{\bullet}\mathcal{G} \;\middle|\; f(X) \in [Y] \in \mathbb{I}_{\bullet}\mathcal{H}\right\}\right]^{\bullet}_{\mathcal{G}} \tag{18}$$

where $\mathcal{C}_{\mathcal{G}}$ is a Lie group contractor and $\bullet$ is L or R.

Theorem

Given a function $f : \mathcal{G} \rightarrow \mathcal{H}$ admitting a Lie group inclusion function and an output domain $[Y] \subset \mathcal{H}$ of which the algebra domain $[y]$ is compact, Problem (18) can be solved in the Euclidean space of the Lie algebra $\mathfrak{g}$ using a conventional box contractor $\mathcal{C}$ and provides a solution of finite volume in the group. Therefore, the Lie group contractor can be defined by:

$$\mathcal{C}_{\mathcal{G}}\left(\langle \mu_X, [x] \rangle\right) \equiv \langle \mu_X, \mathcal{C}([x]) \rangle \tag{19}$$

where $\mathcal{C}([x]) = \left[\left\{\varepsilon \in [x] \;\middle|\; f^{\bullet}_{\mu_X,\mu_Y}(\varepsilon) \in [y]\right\}\right]$.

Contractor properties

Definition

(used in the proof) A Lie group contractor has the following properties:

$$\begin{array}{ll} \mathcal{C}_{\mathcal{G}}([X]) \subset [X] & \text{(contractance)} \\ [X] \cap \mathcal{S}^{\mathcal{G}} \subset \mathcal{C}_{\mathcal{G}}([X]) & \text{(correctness)} \end{array} \tag{20}$$

where $\mathcal{S}^{\mathcal{G}}$ is the exact solution set (15).

Property

A Lie group contractor $\mathcal{C}_{\mathcal{G}}$ is locally monotonic if the Euclidean contractor $\mathcal{C}$ is monotonic. The local monotonicity is defined for any pair of Lie group boxes of $\mathbb{I}_{\bullet}\mathcal{G}$ sharing the same origin $\mu \in \mathcal{G}$:

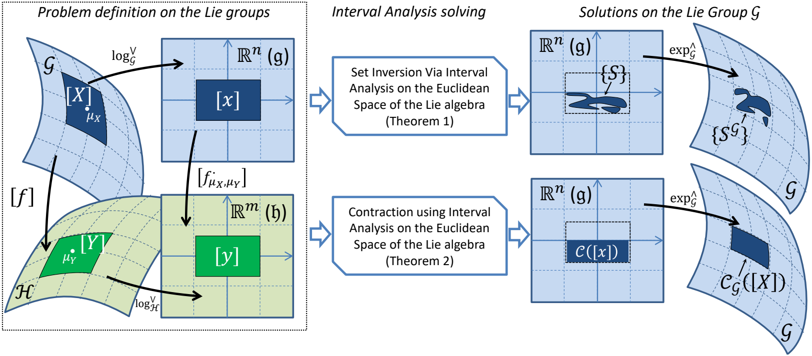
$$\begin{array}{l} \forall [X_1] \equiv \langle \mu, [x_1] \rangle \\ \forall [X_2] \equiv \langle \mu, [x_2] \rangle \end{array}, [X_1] \subset [X_2] \Rightarrow \mathcal{C}_{\mathcal{G}}([X_1]) \subset \mathcal{C}_{\mathcal{G}}([X_2]). \tag{21}$$

Property

A Lie group contractor $\mathcal{C}_{\mathcal{G}}$ is idempotent if the Euclidean contractor $\mathcal{C}$ is idempotent, i.e.:

$$\mathcal{C}_{\mathcal{G}} \circ \mathcal{C}_{\mathcal{G}}([X]) = \mathcal{C}_{\mathcal{G}}([X]). \tag{22}$$

Summary



1. Merlinge, N. (2024). Set Inversion and Box Contraction on Lie groups using interval analysis, accepted to Automatica, to appear soon.

Examples with rotation matrices (Special Orthogonal group SO(3))

- Definition of the rotation matrices SO(3) group:

$$\mathrm{SO}(3) = \left\{ R \in \mathbb{R}^{3\times 3} \;\middle|\; RR^T = I_3, \det R = 1 \right\} \tag{23}$$

- E.g. state attitude of a drone/spacecraft $R_{b\to E}$, body frame to Earth frame.
- Lie algebra $\mathfrak{so}(3)$: $[\varepsilon]_\times = \begin{bmatrix} 0 & -d\Psi_3 & d\Psi_2 \\ d\Psi_3 & 0 & -d\Psi_1 \\ -d\Psi_2 & d\Psi_1 & 0 \end{bmatrix}$
- Exponential coordinates: $\varepsilon = \begin{bmatrix} d\Psi_1 & d\Psi_2 & d\Psi_3 \end{bmatrix}^T \in \mathbb{R}^3$
- Exponential and logarithm mappings:

$$\exp^{\wedge}_{\mathrm{SO}(3)}(\varepsilon) = \begin{cases} I_3 + \frac{\sin\|\varepsilon\|}{\|\varepsilon\|}[\varepsilon]_\times + \frac{1-\cos\|\varepsilon\|}{\|\varepsilon\|^2}[\varepsilon]^2_\times & \textit{if } \|\varepsilon\| > 0 \\ I_3 & \textit{else} \end{cases} \tag{24}$$

$$\log^{\vee}_{\mathrm{SO}(3)}(R) = \begin{cases} \frac{\alpha}{2\sin\alpha} u & \textit{if } \alpha \neq 0 \\ 0_3 & \textit{else} \end{cases} \tag{25}$$

where $\alpha = \arccos \frac{\mathrm{tr}(R)-1}{2}$ and $[u]_\times = R - R^T$.

Problem 1: Ball on SO(3) (left case)

Define $R_0 \in \mathrm{SO}(3)$ a state attitude matrix.
We search for all attitude matrices for which the rotation norm is below a threshold.

$$f: \begin{array}{l} \mathrm{SO}(3) \to \mathbb{R} \\ X \mapsto \| \log^{\vee}_{\mathrm{SO}(3)}(R_0^T X) \|^2. \end{array} \quad (26)$$

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$$f : \begin{array}{l} \mathrm{SO}(3) \to \mathbb{R} \\ X \mapsto \| \log^{\vee}_{\mathrm{SO}(3)}(R_0^T X)\|^2. \end{array} \tag{26}$$

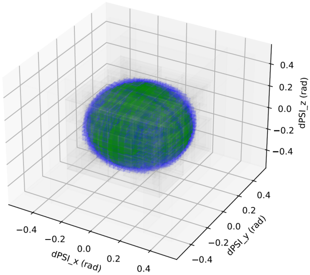
Euclidean equivalent function (by injecting $X = R_0 \exp^{\wedge}_{\mathrm{SO}(3)} \varepsilon$):

$$[f^{\bullet}_{R_0,0}]([x]) = \left[\left\{\varepsilon^T \varepsilon \;\middle|\; \varepsilon \in [x]\right\}\right]. \tag{27}$$

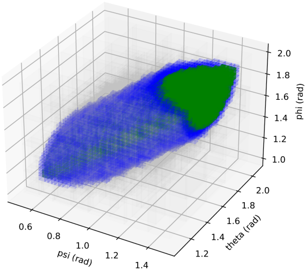
(the right case is identical)

Problem 1: set inversion in the Lie group (left case) and with Euler angles

State uncertainty	$\pm$ 500 mrad
Output uncertainty	$[0, 0.1]$ mrad^2
State attitude R_0	$\psi = 1.0$ rad, $\theta = \pi/2$ rad, $\varphi = 1.5$ rad
SIVIA accuracy	$\epsilon = 0.05$



Lie algebra $\mathfrak{so}(3)$

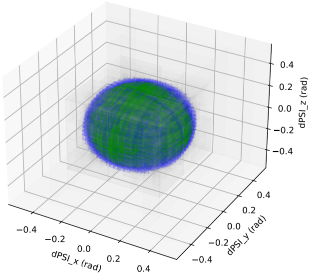


Euler angles

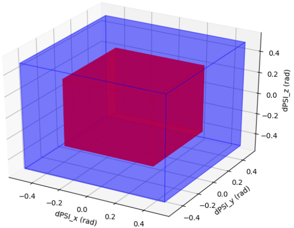
The code uses the codac library: *Rohou, Simon, Desrochers, Benoit, et al. (2022). The Codac library – Constraint-programming for robotics.* *http://codac.io.*

Problem 1: box contraction in the group (left case)

State uncertainty	$\pm$ 500 mrad
Measurement uncertainty	$[0, 100]$ mrad^2
State attitude R_0	$\psi = 1.0$ rad, $\theta = \pi/2$ rad, $\varphi = 1.5$ rad



Set inversion (SIVIA)



Contractor (Forward-Backward)

Problem 1: Monte-Carlo results (100 runs)

Results with SIVIA:

ϵ (rad)	Framework	time (ms)	V_{und}	N_{boxes}
	SO(3) (left)	25	0.029	2,744.0
0.05	SO(3) (right)	25	0.029	2,744.0
	$\mathbb{R}^3$ (Euler)	11,348	0.143	10,278.9

Results for the contractor:

Test case	Framework	time (ms)	r_{cont}	r_{lost}
	SO(3) (left)	$\ll 1$	0.75	0.0
1	SO(3) (right)	$\ll 1$	0.75	0.0
	$\mathbb{R}^3$ (Euler)	11	0.0	-

Criteria:

- CPU time (ms)
- V_{und}: Total volume of the Euclidean *undetermined subpaving* of SIVIA output.
- N_{boxes}: Total number of boxes obtained by SIVIA for the three output subpavings.
- r_{cont}: Contraction rate of the contractor defined by $1 - V(\mathcal{C}([x]))/V([x])$.
- r_{lost}: Efficiency with respect to the box hull $[\mathcal{S}]$ of the solution set obtained with SIVIA, $1 - V([\mathcal{S}])/V(\mathcal{C}([x]))$.

Problem 1b: Another norm (left case)

Define $R_0 \in SO(3)$ a state attitude matrix. We search for all attitude matrices for which the squared Frobenius norm is greater than a threshold.

$$f : \begin{array}{l} SO(3) \to \mathbb{R} \\ X \mapsto \mathrm{tr}(R_0^T X). \end{array} \tag{28}$$

Problem 1b: Another norm (left case)

Define $R_0 \in SO(3)$ a state attitude matrix. We search for all attitude matrices for which the squared Frobenius norm is greater than a threshold.

$$f: \begin{array}{l} SO(3) \to \mathbb{R} \\ X \mapsto \mathrm{tr}(R_0^T X). \end{array} \quad (28)$$

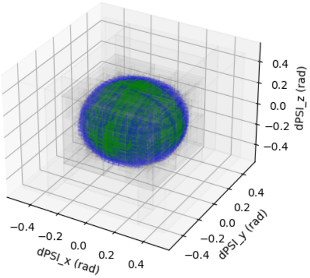
Euclidean equivalent function (by injecting $X = R_0 \exp^{\wedge}_{SO(3)} \varepsilon$):

$$[f^{\bullet}_{R_0,0}]([x]) = \left[\left\{ 1 + 2\cos \|\varepsilon\| \;\middle|\; \varepsilon \in [x] \right\} \right]. \quad (29)$$

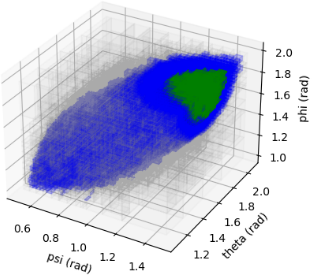
(the right case is identical)

Problem 1b: set inversion in the Lie group (left case) and with Euler angles

State uncertainty	$\pm$ 500 mrad
Measurement uncertainty	$[2.9, 3.0]$
State attitude R_0	$\psi = 1.0$ rad, $\theta = \pi/2$ rad, $\varphi = 1.5$ rad
SIVIA accuracy	$\epsilon = 0.05$



Lie algebra $\mathfrak{so}(3)$



Euler angles

Problem 1b: Monte-Carlo results (100 runs)

Results with SIVIA:

ϵ (rad)	Framework	time (ms)	V_{und}	N_{boxes}
	SO(3) (left)	26	0.030	2,762.0
0.05	SO(3) (right)	26	0.030	2,762.0
	$\mathbb{R}^3$ (Euler)	2,308	0.353	22,102.4

Results for the contractor:

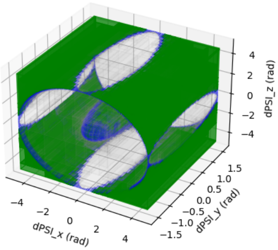
Test case	Framework	time (ms)	r_{cont}	r_{lost}
	SO(3) (left)	$\ll 1$	0.74	0.0
1b	SO(3) (right)	$\ll 1$	0.74	0.0
	$\mathbb{R}^3$ (Euler)	0.71	0.0	-

Criteria:

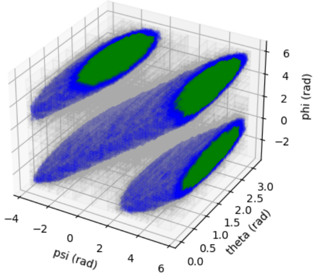
- CPU time (ms)
- V_{und}: Total volume of the Euclidean *undetermined subpaving* of SIVIA output.
- N_{boxes}: Total number of boxes obtained by SIVIA for the three output subpavings.
- r_{cont}: Contraction rate of the contractor defined by $1 - V(\mathcal{C}([x]))/V([x])$.
- r_{lost}: Efficiency with respect to the box hull $[\mathcal{S}]$ of the solution set obtained with SIVIA, $1 - V([\mathcal{S}])/V(\mathcal{C}([x]))$.

Problem 1b: what happens if we go into the surjective domain of $\exp^{\wedge}_{SO(3)}$?

State uncertainty	$\pm \frac{3\pi}{2}$ rad (truncated on *dPSI$_y$* for readability)
Output uncertainty	$[1.5, 3.0]$
State attitude μ_X	$\psi = 1.0$ rad, $\theta = \pi/2$ rad, $\varphi = 1.5$ rad
SIVIA accuracy	$\epsilon = 0.2$

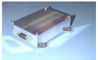


Lie algebra $\mathfrak{so}(3)$



Euler angles

Problem 2: Magnetic field measurement for satellite attitude estimation.



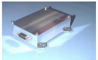
Magnetometers

Define a tri-axis magnetometer aboard a spacecraft:

$$f : \begin{array}{l} SO(3) \to \mathbb{R}^3 \\ X \mapsto X^T B_0 \end{array} \tag{30}$$

where $B_0 \in \mathbb{R}^3$ is the local magnetic field model in a fixed Earth frame and $X \in [X]$ a left or right Lie group box.

Problem 2: Magnetic field measurement for satellite attitude estimation.



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Define a tri-axis magnetometer aboard a spacecraft:

$$f: \begin{array}{l} SO(3) \to \mathbb{R}^3 \\ X \mapsto X^T B_0 \end{array} \tag{30}$$

where $B_0 \in \mathbb{R}^3$ is the local magnetic field model in a fixed Earth frame and $X \in [X]$ a left or right Lie group box. The equivalent Euclidean formulation is obtained by defining $X = R_0 \exp^{\wedge}_{SO(3)} \varepsilon$ (left) and $X = \exp^{\wedge}_{SO(3)} \varepsilon R_0$ (right):

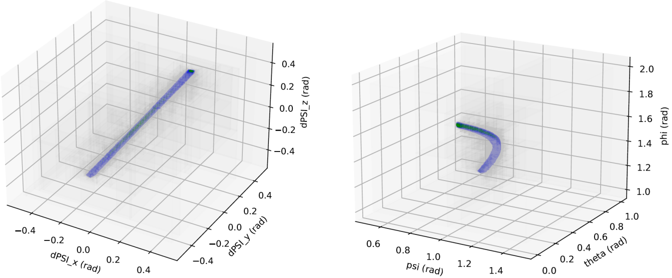
$$\begin{aligned} [f^L_{\mu_X,\tilde{B}}]([x]) &= \left[\left\{\exp^{\wedge}_{SO(3)}(-\varepsilon)R_0^T B_0 - \tilde{B} \;\middle|\; \varepsilon \in [x]\right\}\right] \\ [f^R_{\mu_X,\tilde{B}}]([x]) &= \left[\left\{R_0^T \exp^{\wedge}_{SO(3)}(-\varepsilon)B_0 - \tilde{B} \;\middle|\; \varepsilon \in [x]\right\}\right]. \end{aligned} \tag{31}$$

For implementation purpose, the following Taylor expansion can be used for $\exp^{\wedge}_{SO(3)}$ (assumption: "small" angle errors):

$$\exp^{\wedge}_{SO(3)}(\varepsilon) = I_3 + [\varepsilon]_\times + \frac{1}{2}[\varepsilon]^2_\times + o([\varepsilon]^2_\times) \tag{32}$$

Problem 2: set inversion in the Lie group (left case) and with Euler angles

State uncertainty	$\pm$ 500 mrad
Measurement uncertainty	$\pm$ 0.6 μT
State attitude μ_X	$\psi = 1.0$ rad, $\theta = 0.5$ rad, $\varphi = 1.5$ rad
SIVIA accuracy	$\epsilon = 0.01$



Lie group (SO(3) left) Euler angles

Problem 2: Monte-Carlo results (100 runs)

Results with SIVIA:

ϵ (rad)	Framework	time (ms)	V_{und}	N_{boxes}
	SO(3) (left)	378	$4.74\ 10^{-4}$	8,339.2
0.01	SO(3) (right)	766	$6.40\ 10^{-4}$	9,744.5
	$\mathbb{R}^3$ (Euler)	1,467	$10.1\ 10^{-4}$	12,459.3

Results for the contractor:

Test case	Framework	time (ms)	r_{cont}	r_{lost}
	SO(3) (left)	1	0.61	0.78
2	SO(3) (right)	1	0.36	0.87
	$\mathbb{R}^3$ (Euler)	1	0.37	0.84

Criteria:

- CPU time (ms)
- V_{und}: Total volume of the Euclidean *undetermined subpaving* of SIVIA output.
- N_{boxes}: Total number of boxes obtained by SIVIA for the three output subpavings.
- r_{cont}: Contraction rate of the contractor defined by $1 - V(\mathcal{C}([x]))/V([x])$.
- r_{lost}: Efficiency with respect to the box hull $[\mathcal{S}]$ of the solution set obtained with SIVIA, $1 - V([\mathcal{S}])/V(\mathcal{C}([x]))$.

Conclusion and perspective

To conclude:

- Theoretical framework using the exp/log mappings to deal with interval analysis when the inputs/outputs belong to Lie groups
- Interesting parametrization for bounded attitude estimation problems
- Left and right formulations are not equivalent

Perspectives:

- Dealing with cases where the search domain is outside the bijective domain of $\exp_{\mathcal{G}}^{\wedge}$
- Tackling dynamical propagation of Lie group boxes

Conclusion and perspective

Thank you !